

**ARITHMETIC OF INFINITE PRODUCTS AND
ROGERS-RAMANUJAN CONTINUED FRACTIONS**

DAEYEOUL KIM*, JA KYUNG KOO**, AND YILMAZ SIMSEK***

Reprinted from the
Communications of the Korean Mathematical Society
Vol. 22, No. 3, July 2007

ARITHMETIC OF INFINITE PRODUCTS AND ROGERS-RAMANUJAN CONTINUED FRACTIONS

DAEYEOUL KIM*, JA KYUNG KOO**, AND YILMAZ SIMSEK***

ABSTRACT. Let k be an imaginary quadratic field, \mathfrak{h} the complex upper half plane, and let $\tau \in \mathfrak{h} \cap k$, $q = e^{\pi i \tau}$. We find a lot of algebraic properties derived from theta functions, and by using this we explore some new algebraic numbers from Rogers-Ramanujan continued fractions.

1. Introduction

Let

$$f(a, b) = 1 + \sum_{m=1}^{\infty} (ab)^{m(m-1)/2} (a^m + b^m) = \sum_{m=-\infty}^{\infty} a^{m(m+1)/2} b^{m(m-1)/2}$$

be the theta function for $a, b \in \mathbb{C}$ with $|ab| < 1$, which was defined by Ramanujan ([5], [20]). If we set $a = qe^{2iz}$, $b = qe^{-2iz}$ for $z \in \mathbb{C}$ and $\tau \in \mathfrak{h}$ then $f(a, b) = \theta_3(z, \tau)$, where $\theta_3(z, \tau)$ is one of the classical theta series in its standard notation ([30, p.464]). Berndt proved many interesting formulas for the above theta functions in [4], [5], [6], of which we list the following properties for later use:

$$([5, \text{p.35 }]) \quad f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}$$

$$([5, \text{p.46 }]) \quad f(a, b) + f(-a, -b) = 2f(a^3b, ab^3),$$

where $(a; b)_{\infty} = \prod_{m=0}^{\infty} (1 - ab^m)$. If $ab = cd$, then

$$(1.0) \quad f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)f(ad, bc)$$

$$(1.0^*) \quad f(a, b)f(c, d) - f(-a, -b)f(-c, -d)$$

Received November 13, 2006.

2000 *Mathematics Subject Classification.* 11Jxx, 11R04, 11F11.

Key words and phrases. transcendental number, algebraic number, theta series, Rogers-Ramanujan continued fraction.

*Research supported in part by the Basic Research Program of the Korea Science & Engineering Foundation (R08-2003-10439-0).

**This work was supported in part by the second stage of BK 21 Project.

***He was supported by the Scientific Research Project Administration Akdeniz University.

$$([5, \text{p.45}]) \quad = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right).$$

Using Berndt's idea, we explore some algebraic properties on the values of those theta functions(Lemma 2.2, Theorem 2.4).

In [17], we found that $q^a \prod_{m=1}^{\infty} (1 - q^{nm-t})(1 - q^{nm-(n-t)})$ are algebraic numbers, where (a, n, t) are the following:

$$\begin{aligned} & \left(-\frac{1}{12}, 2, 1\right), \left(-\frac{1}{12}, 3, 1\right), \left(-\frac{1}{24}, 4, 1\right), \left(\frac{1}{60}, 5, 1\right), \left(-\frac{11}{60}, 5, 2\right), \\ & \left(\frac{1}{12}, 6, 1\right), \left(\frac{13}{84}, 7, 1\right), \left(-\frac{11}{84}, 7, 2\right), \left(-\frac{23}{84}, 7, 3\right), \\ & \left(\frac{11}{48}, 8, 1\right), \left(-\frac{13}{48}, 8, 3\right), \left(\frac{11}{36}, 9, 1\right), \left(-\frac{1}{36}, 9, 2\right), \left(-\frac{13}{36}, 9, 4\right), \\ & \left(\frac{23}{60}, 10, 1\right), \left(-\frac{13}{60}, 10, 3\right), \left(\frac{13}{24}, 12, 1\right), \left(-\frac{11}{24}, 12, 5\right), \left(\frac{59}{84}, 14, 1\right), \\ & \left(-\frac{1}{84}, 14, 3\right), \left(-\frac{37}{84}, 14, 5\right), \left(\frac{37}{36}, 18, 1\right), \left(-\frac{11}{36}, 18, 5\right), \left(-\frac{23}{36}, 18, 7\right). \end{aligned}$$

In the examples 2.6~2.9 of Section 2, we further show that $q^a \prod_{m=1}^{\infty} (1 - q^{nm-t})(1 - q^{nm-(n-t)})$ are algebraic numbers for the cases (a, n, t) :

$$\begin{aligned} & \left(\frac{47}{60}, 15, 1\right), \left(\frac{23}{60}, 15, 2\right), \left(-\frac{13}{60}, 15, 4\right), \left(-\frac{37}{60}, 15, 7\right), \\ & \left(\frac{107}{84}, 21, 1\right), \left(\frac{71}{84}, 21, 2\right), \left(\frac{1}{84}, 21, 4\right), \left(-\frac{13}{84}, 21, 5\right), \\ & \left(-\frac{61}{84}, 21, 8\right), \left(-\frac{73}{84}, 21, 10\right), \left(\frac{121}{60}, 30, 1\right), \left(-\frac{11}{60}, 30, 7\right), \\ & \left(-\frac{59}{60}, 30, 11\right), \left(-\frac{71}{60}, 30, 13\right), \left(\frac{253}{84}, 42, 1\right), \left(\frac{109}{84}, 42, 5\right), \\ & \left(-\frac{47}{84}, 42, 11\right), \left(-\frac{83}{84}, 42, 13\right), \left(-\frac{131}{84}, 42, 17\right), \left(-\frac{143}{84}, 42, 19\right). \end{aligned}$$

Ramanujan's lost notebook [21] has many important results on the Rogers-Ramanujan continued fraction, and some of these have been proved by several people such as Andrews [1], [2], Berndt and Chan [9], Berndt, Chan and Zhang [10], Huang [12], Kang [14], [15], and Son [28], [29]. In Section 3, we find some algebraic numbers from Rogers-Ramanujan continued fractions and we also consider the algebraic properties for certain infinite products(Theorem 3.1 and Corollary 3.2).

On the other hand, Borwein and Zhou showed that if a is an integer greater than one, r and s are positive rational numbers such that $1 + a^m r - a^{2m} s \neq 0$ for all integers $m \geq 0$, then $\sum_{m=0}^{\infty} \frac{1}{1 + a^m r - a^{2m} s}$ is irrational and is not a Liouville number ([11]). When $\alpha = \frac{1 - \sqrt{5}}{2}$ and $\tau \in \mathfrak{h} \cap k$, if we replace \sum , a by \prod , q and set $s = 1$ in the above, we can show that $\prod_{m=0}^{\infty} \frac{1}{1 + q^m \alpha + q^{2m}}$ is

a transcendental number, and hence $\prod_{m=0}^{\infty} \frac{1}{q^{\frac{1}{12}}(1+q^m\alpha+q^{2m})}$ is an algebraic number ([Theorem 3.1, (b)]).

2. Properties of theta functions

Let $\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ and $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$. According to Berndt [5, p.49] there are two identities

$$(2.1) \quad \phi(q) = \phi(q^{n^2}) + \sum_{r=1}^{n-1} q^{r^2} f(q^{n(n-2r)}, q^{n(n+2r)})$$

and

$$(2.2) \quad \psi(q) = \frac{1}{2} \sum_{r=0}^{n-1} q^{r(r-1)/2} f(q^{n(n-2r+1)/2}, q^{n(n+2r-1)/2}),$$

where n is a positive integer.

Now, let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $b \pmod d$ and $|\alpha|$ be the determinant of α , and set

$$\phi_{\alpha}(\tau) := |\alpha|^{12} \frac{\Delta(\alpha \begin{pmatrix} \tau \\ 1 \end{pmatrix})}{\Delta(\begin{pmatrix} \tau \\ 1 \end{pmatrix})} = |\alpha|^{12} d^{-12} \frac{\Delta(\alpha\tau)}{\Delta(\tau)}$$

where $\Delta(\tau) = (2\pi)^{12} q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^{24}$.

Then we recall from ([19]) that

$$(2.3) \quad \begin{aligned} &\text{for any } \tau \in k \cap \mathfrak{h} \text{ the value } \phi_{\alpha}(\tau) \\ &\text{is an algebraic integer, which divides } |\alpha|^{12}. \end{aligned}$$

Lemma 2.0. *Let $a, n \in \mathbb{Z}^+$, $1 \leq a \leq n - 1$ with $n \geq 2$. Then we have $\prod_{m=0}^{\infty} (1 - q^{n+a}q^{nm})(1 - q^{-a}q^{nm}) = -q^{-a} \prod_{m=1}^{\infty} (1 - q^{nm-a})(1 - q^{nm-(n-a)})$.*

Proof. It is immediate. □

Proposition 2.1 ([16]). *For $\tau \in k \cap \mathfrak{h}$,*

$$\begin{aligned} &\sqrt{2}q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^m), \quad q^{\frac{-1}{24}} \prod_{m=1}^{\infty} (1 - q^{2m-1}), \\ &q^{\frac{-1}{24}} \prod_{m=1}^{\infty} (1 + q^{2m-1}) \quad \text{and} \quad \frac{1}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^m)} \end{aligned}$$

are algebraic integers.

Lemma 2.2. *Let $\tau \in k \cap \mathfrak{h}$, $n, r \in \mathbb{Z}^+$ with $n \geq 2$.*

(a) *If n is an odd positive integer then*

$$\sum_{r=1}^{\frac{n-1}{2}} \left((-1)^{r^2} q^{\frac{r^2}{n} - \frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm-(n-2r)})(1 - q^{2nm-(n+2r)}) \right)$$

and

$$\prod_{r=1}^{\frac{n-1}{2}} \left(q^{\frac{r^2}{n} - \frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm - (n-2r)})(1 - q^{2nm - (n+2r)}) \right)$$

are algebraic numbers.

(b) If n is an integer then

$$\begin{aligned} & \sum_{r=0}^{\lfloor \frac{n}{2} + \frac{1}{4} \rfloor} q^{\frac{r(2r-1)}{n} + \frac{1}{8n} - \frac{n}{6}} \left(\prod_{m=1}^{\infty} (1 + q^{4nm - (2n-4r+1)})(1 + q^{4nm - (2n+4r-1)}) \right) \\ & + \sum_{r=\lfloor \frac{n}{2} + \frac{1}{4} \rfloor + 1}^{n-1} (-1)^r q^{\frac{r(2r-1)}{n} + 2n - 4r + 1 + \frac{1}{8n} - \frac{n}{6}} \\ & \cdot \left(\prod_{m=1}^{\infty} (1 + q^{4nm - (-2n+4r-1)})(1 + q^{4nm - (6n-4r+1)}) \right) \end{aligned}$$

is an algebraic number where $[a]$ denote the greatest integer less than or equal to a .

(c) If $n = 6g + 1$ or $6g - 1$ then

$$\sum_{k=1}^{\frac{n-1}{2}} (-1)^{g+k} q^{\frac{k(3k-n)}{2n}} \prod_{m=1}^{\infty} \left(\frac{(1 - q^{nm-2k})(1 - q^{nm-(n-2k)})}{(1 - q^{nm-k})(1 - q^{nm-(n-k)})} \right)$$

is also an algebraic number.

Proof. (a) Since n is an odd integer, so are $n(n-2r)$ and $n(n+2r)$. If we replace q by $-q$ in (2.1), we can deduce the formula

$$\phi(-q) = \phi(-q^{n^2}) + \sum_{r=1}^{n-1} (-1)^{r^2} q^{r^2} f(-q^{n(n-2r)}, -q^{n(n+2r)}),$$

and replacing q^n by q in (2.1) attain the formula

$$(2.4) \quad \phi(-q^{\frac{1}{n}}) = \phi(-q^n) + \sum_{r=1}^{n-1} (-1)^{r^2} q^{\frac{r^2}{n}} f(-q^{(n-2r)}, -q^{(n+2r)}).$$

Thus we derive by Lemma 2.0 and (2.4) that

$$\begin{aligned} & \prod_{m=1}^{\infty} (1 - q^{\frac{1}{n}(2m-1)})^2 (1 - q^{\frac{1}{n}(2m)}) - \prod_{m=1}^{\infty} (1 - q^{n(2m-1)})^2 (1 - q^{n(2m)}) \\ & = \sum_{r=1}^{\frac{n-1}{2}} (-1)^{r^2} q^{\frac{r^2}{n}} \prod_{m=1}^{\infty} (1 - q^{2nm - (n-2r)})(1 - q^{2nm - (n+2r)})(1 - q^{2nm}) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{r=\frac{n+1}{2}}^{n-1} (-1)^{r^2+1} q^{\frac{r^2}{2}} q^{n-2r} \prod_{m=1}^{\infty} (1 - q^{2nm-(n-2r)-2n}) \\
 & \cdot (1 - q^{2nm-(n+2r)+2n})(1 - q^{2nm}).
 \end{aligned}$$

If we set $\frac{r^2}{n} = \frac{(n+r)^2}{n} + n - 2r$, then we have

$$\begin{aligned}
 & \prod_{m=1}^{\infty} (1 - q^{\frac{1}{n}(2m-1)})^2 (1 - q^{\frac{1}{n}(2m)}) - \prod_{m=1}^{\infty} (1 - q^{n(2m-1)})^2 (1 - q^{n(2m)}) \\
 & = 2 \sum_{r=1}^{\frac{n-1}{2}} (-1)^{r^2} q^{\frac{r^2}{n}} \prod_{m=1}^{\infty} (1 - q^{2nm-(n-2r)})(1 - q^{2nm-(n+2r)})(1 - q^{2nm}).
 \end{aligned}$$

Hence, multiplying $\frac{1}{q^{\frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm})}$ on both sides in the above, we see that

$$\begin{aligned}
 (2.5) \quad & \left(q^{-\frac{1}{12n}} \prod_{m=1}^{\infty} (1 - q^{\frac{1}{n}(2m-1)})^2 \right) \left(\frac{q^{\frac{1}{12n}} (1 - q^{\frac{1}{n}(2m)})}{q^{\frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm})} \right) \\
 & - \left(q^{-\frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{n(2m-1)})^2 \right) \\
 & = 2 \sum_{r=1}^{\frac{n-1}{2}} (-1)^{r^2} q^{\frac{r^2}{n} - \frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm-(n-2r)})(1 - q^{2nm-(n+2r)})
 \end{aligned}$$

is an algebraic number by (2.3) and Proposition 2.1.

On the other hand, we know that $\sum_{r=1}^{\frac{n-1}{2}} (\frac{r^2}{n} - \frac{n}{12}) = \frac{n-1}{24}$ and hence

$$\begin{aligned}
 (2.6) \quad & \prod_{r=1}^{\frac{n-1}{2}} \left[q^{\frac{r^2}{n} - \frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm-(n-2r)})(1 - q^{2nm-(n+2r)}) \right] \\
 & = q^{\frac{n-1}{24}} \prod_{m=1}^{\infty} \left(\frac{1 - q^{2m-1}}{1 - q^{2mn-n}} \right).
 \end{aligned}$$

By Proposition 2.1, (2.5) and (2.6), we get the result.

(b) It follows from (2.2) when we replace q^n by q because the argument is quite similar to those adopted in (a).

(c) Let $n = 6g + 1$ be a positive integer. We see from [5, p.274] that

$$\begin{aligned}
 \prod_{m=1}^{\infty} \frac{(1 - q^{\frac{m}{n}})}{(1 - q^{nm})} & = (-1)^g q^{\frac{n^2-1}{24n}} + \sum_{k=1}^{\frac{n-1}{2}} (-1)^{k+g} \\
 & \cdot q^{(k-g)(3k-3g-1)/2n} \prod_{m=1}^{\infty} \frac{(1 - q^{nm-2k})(1 - q^{nm-(n-2k)})}{(1 - q^{nm-k})(1 - q^{nm-(n-k)})}.
 \end{aligned}$$

And, multiplying both sides by $q^{-\left(\frac{n^2-1}{24n}\right)}$ and using (2.3), we conclude that

$$\sum_{k=1}^{\frac{n-1}{2}} (-1)^{g+k} g^{\frac{k(3k-n)}{2n}} \prod_{m=1}^{\infty} \left(\frac{(1 - q^{nm-2k})(1 - q^{nm-(n-2k)})}{(1 - q^{nm-k})(1 - q^{nm-(n-k)})} \right)$$

is an algebraic number.

As for the case $n = 6g - 1$, we can get it in like manner. □

Example 2.3. Put $n = 5$ in Lemma 2.2 (a). Then we see that

$$S := -q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-3})(1 - q^{10m-7}) + q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9})$$

and

$$T := \left(-q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-3})(1 - q^{10m-7}) \right) \left(q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9}) \right)$$

are algebraic numbers. Thus we can find an algebraic equation $g(X) = X^2 - SX + T \in \overline{\mathbb{Q}}[X]$ satisfying $g\left(q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9})\right) = 0$. Hence,

$$q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-3})(1 - q^{10m-7})$$

and

$$q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9})$$

are algebraic numbers, too as in [17].

Theorem 2.4.

- (a) Let n, u, w be positive integers with $1 \leq u, w \leq 2n-1$, u even and w odd. If $q^{t_u} \prod_{m=1}^{\infty} (1 \pm q^{2nm-u})(1 \pm q^{2nm-(2n-u)})$ and $q^{t_w} \prod_{m=1}^{\infty} (1 \pm q^{2nm-w})(1 \pm q^{2nm-(2n-w)})$ are algebraic numbers for $t_u, t_w \in \overline{\mathbb{Q}}$, then $q^{t_u+t_w-\frac{n}{6}} \prod_{m=1}^{\infty} (1 - q^{4nm-(u+w)})(1 - q^{4nm-(4n-(u+w))})(1 - q^{4nm-(2n+u-w)})(1 - q^{4nm-(2n-u+w)})$ is an algebraic number with double signs in the same order.
- (b) Let n, u, w be odd positive integers with $1 \leq u, w \leq n-1$. If $q^{t_u} \prod_{m=1}^{\infty} (1 \pm q^{nm-u})(1 \pm q^{nm-(n-u)})$, $q^{t_w} \prod_{m=1}^{\infty} (1 \pm q^{nm-w})(1 \pm q^{nm-(n-w)})$ and $q^{\frac{t_u+t_w}{2}} \prod_{m=1}^{\infty} (1 - q^{nm-(\frac{u+v}{2})})(1 - q^{nm-(n-\frac{u+v}{2})})$ are algebraic numbers, then $q^{t_u+t_w-\frac{n}{2}-2t\frac{u+v}{2}} \prod_{m=1}^{\infty} (1 - q^{2nm-(n+u-w)})(1 - q^{2nm-(n-u+w)})$ is an algebraic number with double signs in the same order.
- (c) Let n, u be odd positive integers with $1 \leq u \leq n-1$. If $q^{t_u} \prod_{m=1}^{\infty} (1 - q^{nm-u})(1 - q^{nm-(n-u)})$ and $q^{\frac{t}{2}-t\frac{u}{2}} \prod_{m=1}^{\infty} (1 - q^{nm-u})(1 - q^{nm-(n-\frac{n-u}{2})})$

are algebraic numbers, then

$$q^{t_u - 2(t \frac{n-u}{2})} \prod_{m=1}^{\infty} (1 - q^{2nm-u})(1 - q^{2nm-(2n-u)})$$

is an algebraic number.

- (d) If $q^{-\frac{1}{2} + \frac{1}{4n} + \frac{n}{6}} \prod_{m=1}^{\infty} (1 - q^{2nm-1})(1 - q^{2nm-(2n-1)})$ and $q^{-\frac{1}{2} + \frac{1}{6n} + \frac{n}{4}} \prod_{m=1}^{\infty} (1 - q^{3nm-1})(1 - q^{3nm-(3n-1)})$ are algebraic numbers, then $q^{-\frac{1}{2} + \frac{1}{12n} + \frac{n}{2}} \prod_{m=1}^{\infty} (1 - q^{6nm-1})(1 - q^{6nm-(6n-1)})$ is an algebraic number.

- (e) Let n be an even positive integer. If

$$q^{\frac{2}{n} - \frac{n}{24}} \prod_{m=1}^{\infty} (1 \pm q^{nm - (\frac{n-4}{2})})(1 \pm q^{nm - (\frac{n+4}{2})}),$$

$$q^{-\frac{1}{2n} - \frac{n}{24}} \prod_{m=1}^{\infty} (1 \pm q^{nm - (\frac{n-2}{2})})(1 \pm q^{nm - (\frac{n+2}{2})}) \text{ and}$$

$$q^{-\frac{1}{2} + \frac{1}{2n} + \frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{nm-1})(1 - q^{nm-(n-1)})$$

are algebraic numbers, then $q^{-\frac{1}{2} + \frac{1}{4n} + \frac{n}{6}} \prod_{m=1}^{\infty} (1 - q^{2nm-1})(1 - q^{2nm-(2n-1)})$ and $q^{-\frac{3}{2} + \frac{9}{4n} + \frac{n}{6}} \prod_{m=1}^{\infty} (1 - q^{2nm-3})(1 - q^{2nm-(2n-3)})$ are algebraic numbers with double signs in the same order.

Proof. (a) By (1.0),

$$(2.7) \quad f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)f(ad, bc).$$

Let $a = q^u$, $b = q^{2n-u}$, $c = -q^w$, $d = -q^{2n-w}$,

$$A := \prod_{m=1}^{\infty} (1 + q^{2nm-u})(1 + q^{2nm-(2n-u)})(1 - q^{2nm-w})(1 - q^{2nm-(2n-w)}),$$

$$B := \prod_{m=1}^{\infty} (1 - q^{2nm-u})(1 - q^{2nm-(2n-u)})(1 + q^{2nm-w})(1 + q^{2nm-(2n-w)}),$$

and

$$C := 2 \prod_{m=1}^{\infty} (1 - q^{4nm-(u+w)})(1 - q^{4nm-(4n-(u+w))})$$

$$\cdot (1 - q^{4nm-(2n+u-w)})(1 - q^{4nm-(2n-u+w)}).$$

If we multiply through (2.7) by $\frac{q^{t_u+t_w-\frac{n}{6}}}{\prod_{m=1}^{\infty} (1 - q^{4nm})^2}$, we get by Proposition 2.1 and the assumption that

$$\frac{q^{t_u+t_w-\frac{n}{6}} A}{\prod_{m=1}^{\infty} (1 + q^{2nm})^2} + \frac{q^{t_u+t_w-\frac{n}{6}} B}{\prod_{m=1}^{\infty} (1 + q^{2nm})^2} = q^{t_u+t_w-\frac{n}{6}} C$$

is an algebraic number.

- (b) It goes over the same argument as in (a).

(c) It follows from the fact that

$$(2.8) \quad \begin{aligned} & \prod_{m=1}^{\infty} (1 - q^{2nm-u})(1 - q^{2nm-(2n-u)}) \\ &= \prod_{m=1}^{\infty} \frac{(1 - q^{nm-u})(1 - q^{nm-(n-u)})}{(1 - q^{2nm-(n-u)})(1 - q^{2nm-(n+u)})}. \end{aligned}$$

(d) For an integer $N \geq 6$ and an integer $r \not\equiv 0 \pmod{N}$, let $X_r(\tau)$ be the function defined by

$$X_r(\tau) = X_r(\tau, N) = e^{-2\pi i\tau(\frac{(r-1)(N-1)}{4N})} \prod_{s=0}^{N-1} \frac{K_{r,s}(\tau)}{K_{1,s}(\tau)},$$

where $K_{u,v}(\tau)$ are Klein forms of level N ([18]).

Ishida found in [13] that some polynomial $F(X, Y) \in \mathbb{Q}[X, Y]$ satisfies an affine singular model $F(X_2, X_3) = 0$ for the modular curve $X(N)$. From this and the fact that $q^{3(-\frac{1}{2} + \frac{1}{4n} + \frac{n}{6})} \prod_{m=1}^{\infty} (1 - q^{6nm-3})(1 - q^{26nm-(6n-3)})$ and $q^{2(-\frac{1}{2} + \frac{1}{6n} + \frac{n}{4})} \prod_{m=1}^{\infty} (1 - q^{6nm-2})(1 - q^{6nm-(6n-2)})$ are algebraic numbers, we can find an algebraic equation for

$$q^{-\frac{1}{2} + \frac{1}{12n} + \frac{n}{2}} \prod_{m=1}^{\infty} (1 - q^{6nm-1})(1 - q^{6nm-(6n-1)}).$$

Thus we get (d).

(e) Set $a = q^{\frac{n-4}{2}}$, $b = q^{\frac{n+4}{2}}$, $c = -q^{\frac{n-2}{2}}$ and $d = -q^{\frac{n+2}{2}}$ in (1.0*). Then we obtain that

$$\begin{aligned} & \prod_{m=1}^{\infty} \left(q^{\frac{2}{n} - \frac{n}{24}} (1 + q^{nm - (\frac{n-4}{2})}) (1 + q^{nm - (\frac{n+4}{2})}) \right) \\ & \quad \times \left(q^{\frac{1}{2n} - \frac{n}{24}} (1 - q^{nm - (\frac{n-2}{2})}) (1 - q^{nm - (\frac{n+2}{2})}) \right) \left(\frac{q^{-\frac{n}{12}}}{(1 + q^nm)^2} \right) \\ & - \prod_{m=1}^{\infty} \left(q^{\frac{2}{n} - \frac{n}{24}} (1 - q^{nm - (\frac{n-4}{2})}) (1 - q^{nm - (\frac{n+4}{2})}) \right) \\ & \quad \times \left(q^{\frac{1}{2n} - \frac{n}{24}} (1 + q^{nm - (\frac{n-2}{2})}) (1 + q^{nm - (\frac{n+2}{2})}) \right) \left(\frac{q^{-\frac{n}{12}}}{(1 + q^nm)^2} \right) \\ & = 2 \prod_{m=1}^{\infty} \left(q^{-\frac{1}{2} + \frac{1}{4n} + \frac{n}{6}} (1 - q^{2nm-1})(1 - q^{2nm-(n-1)}) \right) \\ & \quad \times \left(q^{-\frac{3}{2} + \frac{9}{4n} + \frac{n}{6}} (1 - q^{2nm-3})(1 - q^{2nm-(n-3)}) \right) \text{ is an algebraic number.} \end{aligned}$$

Combining this fact and the Ishida's equation for $F(X_2, X_3) = 0$, we readily get (e). □

Example 2.5. In [17], we showed that $q^{\frac{13}{84}} \prod_{m=1}^{\infty} (1 \pm q^{7m-1})(1 \pm q^{7m-6})$, $q^{\frac{-11}{84}} \prod_{m=1}^{\infty} (1 \pm q^{7m-2})(1 \pm q^{7m-5})$, $q^{\frac{-23}{84}} \prod_{m=1}^{\infty} (1 \pm q^{7m-3})(1 \pm q^{7m-4})$, $q^{\frac{26}{84}} \prod_{m=1}^{\infty} (1 \pm q^{14m-2})(1 \pm q^{14m-12})$ and $q^{-\frac{1}{84}} \prod_{m=1}^{\infty} (1 \pm q^{14m-3})(1 \pm q^{14m-11})$ are algebraic numbers with double signs in the same order. By Theorem 2.4, we further conclude that $q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1 - q^{28m-5})(1 - q^{28m-13})(1 - q^{28m-15})(1 - q^{28m-23})$ and $q^{-\frac{37}{84}} \prod_{m=1}^{\infty} (1 - q^{14m-5})(1 - q^{14m-9})$ are algebraic numbers.

Example 2.6. Let us consider some algebraic numbers from the case $n = 15$. In [17], we proved that

$$(2.9) \quad q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-1})(1 - q^{5m-4}) \quad \text{and} \quad q^{\frac{-11}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-2})(1 - q^{5m-3})$$

are algebraic numbers.

Meanwhile, it can be checked by Proposition 2.1 that

$$(2.10) \quad \begin{aligned} & \left(q^{\frac{47}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-1})(1 + q^{15m-14}) \right) \\ & \cdot \left(q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-2})(1 + q^{15m-13}) \right) \\ & \cdot \left(q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-4})(1 + q^{15m-11}) \right) \\ & \cdot \left(q^{-\frac{37}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-7})(1 + q^{15m-8}) \right) \\ & = \prod_{m=1}^{\infty} q^{\frac{1}{3}} \left(\frac{1 + q^m}{1 + q^{15m}} \right) \left(\frac{1 + q^{15m}}{1 + q^{3m}} \right) \left(\frac{1 + q^{15m}}{1 + q^{5m}} \right) \end{aligned}$$

and

$$(2.11) \quad \begin{aligned} & \left(q^{\frac{47}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-1})(1 + q^{15m-14}) \right) \\ & \cdot \left(q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-4})(1 + q^{15m-11}) \right) \\ & = \prod_{m=1}^{\infty} \frac{q^{\frac{1}{60}} (1 + q^{5m-1})(1 + q^{5m-4})}{q^{-\frac{33}{60}} (1 + q^{15m-6})(1 + q^{15m-9})} \end{aligned}$$

are algebraic numbers.

And, Rogers ([3], [22, p.332]) derived the identities

$$\begin{aligned}
 (2.12) \quad & \prod_{m=1}^{\infty} (1 + q^{15m-7})(1 + q^{15m-8})(1 - q^{15m}) \\
 & - q \prod_{m=1}^{\infty} (1 + q^{15m-2})(1 + q^{15m-13})(1 - q^{15m}) \\
 & = \prod_{m=1}^{\infty} \frac{(1 - q^{10m-4})(1 - q^{10m-6})(1 - q^{10m})}{1 + q^m}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.13) \quad & \prod_{m=1}^{\infty} (1 + q^{15m-4})(1 + q^{15m-11})(1 - q^{15m}) \\
 & - q \prod_{m=1}^{\infty} (1 + q^{15m-1})(1 + q^{15m-14})(1 - q^{15m}) \\
 & = \prod_{m=1}^{\infty} \frac{(1 - q^{10m-2})(1 - q^{10m-8})(1 - q^{10m})}{1 + q^m}.
 \end{aligned}$$

It then follows from (2.3), (2.9), (2.12) and (2.13) that

$$\begin{aligned}
 (2.14) \quad & q^{-\frac{37}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-7})(1 + q^{15m-8}) - q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-2})(1 + q^{15m-13}) \\
 & = \prod_{m=1}^{\infty} \left(q^{-\frac{22}{60}} (1 - q^{10m-4})(1 - q^{10m-6}) \right) \left(\frac{q^{-\frac{1}{24}}}{(1 + q^m)} \right) \frac{q^{\frac{10}{24}} (1 - q^{10m})}{q^{\frac{15}{24}} (1 - q^{15m})}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.15) \quad & q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-4})(1 + q^{15m-11}) - q^{\frac{47}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-1})(1 + q^{15m-14}) \\
 & = \prod_{m=1}^{\infty} \left(q^{\frac{2}{60}} (1 - q^{10m-2})(1 - q^{10m-8}) \right) \left(\frac{q^{-\frac{1}{24}}}{(1 + q^m)} \right) \frac{q^{\frac{10}{24}} (1 - q^{10m})}{q^{\frac{15}{24}} (1 - q^{15m})}
 \end{aligned}$$

are algebraic numbers. Hence, there exists a polynomial $g(X) \in \overline{\mathbb{Q}}[X]$ satisfying $g(q^{-\frac{37}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-7})(1 + q^{15m-8})) = 0$. Therefore, we conclude that

$$\begin{aligned}
 & q^{\frac{47}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-1})(1 + q^{15m-14}), \\
 & q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-2})(1 + q^{15m-13}),
 \end{aligned}$$

$$q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-4})(1 + q^{15m-11})$$

and

$$q^{-\frac{37}{60}} \prod_{m=1}^{\infty} (1 + q^{15m-7})(1 + q^{15m-8})$$

are algebraic numbers.

Example 2.7. Now, we consider the case $n = 30$. In [3] and [23, p.333] we see that

$$(2.16) \quad \prod_{m=1}^{\infty} \frac{(1 - q^{2m})}{(1 - q^{30m})(1 - q^{5m-1})(1 - q^{5m-4})} \\ = \prod_{m=1}^{\infty} (1 - q^{30m-13})(1 - q^{30m-17}) + q \prod_{m=1}^{\infty} (1 - q^{30m-7})(1 - q^{30m-23}),$$

which was described by Rogers as a remarkable identity. And we know that

$$q^{-\frac{28}{24}} \prod_{m=1}^{\infty} \left(\frac{1 - q^{2m}}{1 - q^{30m}} \right), \quad q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-1})(1 - q^{5m-4}), \\ q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-3})(1 - q^{10m-7})$$

and

$$q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9})$$

are algebraic numbers ([17]).

Thus we claim by (2.16) that

$$(2.17) \quad D := q^{-\frac{28}{24}} \prod_{m=1}^{\infty} \left(\frac{1 - q^{2m}}{1 - q^{30m}} \right) \left(\frac{q^{\frac{1}{60}}}{(1 - q^{5m-1})(1 - q^{5m-4})} \right) \\ = q^{-\frac{71}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-13})(1 - q^{30m-17}) \\ + q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-7})(1 - q^{30m-23})$$

is an algebraic number. And,

$$E := \prod_{m=1}^{\infty} \frac{q^{-\frac{13}{60}} (1 - q^{10m-3})(1 - q^{10m-7})}{q^{\frac{69}{60}} (1 - q^{30m-3})(1 - q^{30m-27})}$$

$$(2.18) \quad \begin{aligned} &= \prod_{m=1}^{\infty} \left(q^{-\frac{71}{60}} (1 - q^{30m-13})(1 - q^{30m-17}) \right) \\ &\quad \cdot \left(q^{-\frac{11}{60}} (1 - q^{30m-7})(1 - q^{30m-23}) \right) \end{aligned}$$

is an algebraic number. Hence (2.17) and (2.18) enable us to get a polynomial $g(X) = X^2 - DX + E \in \overline{\mathbb{Q}}[X]$ satisfying $g(q^{-\frac{71}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-13})(1 - q^{30m-17})) = 0$. Thus, we find that

$$q^{-\frac{71}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-13})(1 - q^{30m-17}) \quad \text{and} \quad q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-7})(1 - q^{30m-23})$$

are algebraic numbers. And by Lemma 2.2 (a) we derive that

$$q^{-\frac{59}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-11})(1 - q^{30m-19}) - q^{\frac{121}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-1})(1 - q^{30m-29})$$

is an algebraic number. Moreover, it is not difficult to see that

$$\begin{aligned} &\left(q^{-\frac{59}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-11})(1 - q^{30m-19}) \right) \\ &\quad \cdot \left(-q^{\frac{121}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-1})(1 - q^{30m-29}) \right) \\ &= \frac{q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 - q^{10m-1})(1 - q^{10m-9})}{q^{-\frac{13}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-9})(1 - q^{30m-21})} \text{ is an algebraic number.} \end{aligned}$$

It then follows that

$$(2.19) \quad q^{-\frac{59}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-11})(1 - q^{30m-19}), \quad q^{\frac{121}{60}} \prod_{m=1}^{\infty} (1 - q^{30m-1})(1 - q^{30m-29})$$

are algebraic numbers.

If we use Son's identity ([28])

$$f^3(-q^{11}, -q^{19}) + q^9 f^3(-q, -q^{29}) = \left(\sum_{m=0}^{\infty} \frac{q^{3m(m+1)}}{(q^3; q^3)_m} \right) (q^6; q^6)_{\infty} \frac{(q^{10}; q^{10})_{\infty}^3}{(q^{30}; q^{30})_{\infty}}$$

we have by (2.19) that

$$\begin{aligned} &q^{\frac{11}{20}} \sum_{m=0}^{\infty} \frac{q^{3m(m+1)}}{(q^3; q^3)_m} \\ &= \prod_{m=1}^{\infty} (q^{-\frac{59}{60}} (1 - q^{30m-11})(1 - q^{30m-19}))^3 \left(q^{\frac{10}{24}} \frac{(1 - q^{30m})}{(1 - q^{10m})} \right)^3 \left(q \frac{(1 - q^{30m})}{(1 - q^{6m})} \right) \\ &\quad + \prod_{m=1}^{\infty} (q^{\frac{121}{60}} (1 - q^{30m-1})(1 - q^{30m-29}))^3 \left(q^{\frac{10}{24}} \frac{(1 - q^{30m})}{(1 - q^{10m})} \right)^3 \left(q \frac{(1 - q^{30m})}{(1 - q^{6m})} \right) \end{aligned}$$

is an algebraic number. Here we note that combining Slater identity A.14 ([17], [24], [27]) and Son's identity we can find a relation between sums and infinite products, i.e.,

$$\begin{aligned} & q^{\frac{11}{20}} \sum_{m=0}^{\infty} \frac{q^{3m(m+1)}}{(q^3; q^3)_m} \\ &= \prod_{m=1}^{\infty} (q^{-\frac{59}{60}}(1 - q^{30m-11})(1 - q^{30m-19}))^3 \left(q^{\frac{10}{24}} \frac{(1 - q^{30m})}{(1 - q^{10m})} \right)^3 \left(q \frac{(1 - q^{30m})}{(1 - q^{6m})} \right) \\ &+ \prod_{m=1}^{\infty} (q^{\frac{121}{60}}(1 - q^{30m-1})(1 - q^{30m-29}))^3 \left(q^{\frac{10}{24}} \frac{(1 - q^{30m})}{(1 - q^{10m})} \right)^3 \left(q \frac{(1 - q^{30m})}{(1 - q^{6m})} \right) \\ &= \prod_{m=1}^{\infty} (q^{\frac{1}{20}}(1 - q^{15m-3})(1 - q^{15m-12})) \left(q^{\frac{1}{2}} \frac{(1 - q^{15m})}{(1 - q^{3m})} \right). \end{aligned}$$

Example 2.8. Slater found in [26](or [3]) that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q)_{2m}} &= \prod_{m=1}^{\infty} \frac{(1 + q^{30m-14})(1 + q^{30m-16})(1 - q^{30m})}{(1 - q^m)} \\ &- \prod_{m=1}^{\infty} \frac{q^2(1 + q^{30m-4})(1 + q^{30m-26})(1 - q^{30m})}{(1 - q^m)}, \\ \sum_{m=1}^{\infty} \frac{q^{3m^2}}{(q^4)_{4,m}(q)_{2,m}} &= \prod_{m=1}^{\infty} \frac{(1 + q^{2m-1})}{(1 + (-1)^m q^{5m-1})(1 - (-1)^m q^{5m-4})} \end{aligned}$$

and

$$\sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}} = \prod_{m=1}^{\infty} \frac{(1 + q^m)(1 - q^{7m-2})(1 - q^{7m-5})(1 - q^{7m})}{(1 - q^m)(1 + q^{7m-1})(1 + q^{7m-6})}$$

where $(a)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$ and $(a)_{r,n} = (1 - a)(1 - aq^r) \cdots (1 - aq^{r(n-1)})$. By (2.3) and Example 2.7 we see that

$$\begin{aligned} & q^{-\frac{1}{40}} \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q)_{2m}} \\ &= \prod_{m=1}^{\infty} \left(q^{-\frac{74}{60}}(1 + q^{30m-14})(1 + q^{30m-16}) \right) \left(q^{\frac{29}{24}} \frac{(1 - q^{30m})}{(1 - q^m)} \right) \\ &- \prod_{m=1}^{\infty} \left(q^{\frac{46}{60}}(1 + q^{30m-4})(1 + q^{30m-26}) \right) \left(q^{\frac{29}{24}} \frac{(1 - q^{30m})}{(1 - q^m)} \right) \end{aligned}$$

is an algebraic number and hence $\sum_{m=1}^{\infty} \frac{q^{m^2}}{(q)_{2m}}$ is transcendental.

We showed in [17] that

$$q^{\frac{23}{60}} \prod_{m=1}^{\infty} (1 + q^{10m-1})(1 + q^{10m-9}) \quad \text{and} \quad q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-2})(1 - q^{5m-3})$$

are algebraic numbers. Thus,

$$q^{-\frac{7}{120}} \sum_{m=1}^{\infty} \frac{q^{3m^2}}{(q^4)_{4,m}(q)_{2,m}}$$

$$= \prod_{m=1}^{\infty} \frac{q^{-\frac{1}{24}}(1 + q^{2m-1})}{\left(q^{-\frac{22}{60}}(1 - q^{10m-4})(1 - q^{10m-6})\right) \left(q^{\frac{23}{60}}(1 + q^{10m-1})(1 + q^{10m-9})\right)}$$

is an algebraic number, and so $\sum_{m=1}^{\infty} \frac{q^{3m^2}}{(q^4)_{4,m}(q)_{2,m}}$ is a transcendental number.

And, by (2.3), Proposition 2.1 and Example 2.5 we derive that (2.20)

$$q^{\frac{1}{168}} \sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}} = \prod_{m=1}^{\infty} \left(q^{\frac{1}{24}}(1 + q^m) \right) \left(\frac{q^{-\frac{11}{84}}(1 - q^{7m-2})(1 - q^{7m-5})}{q^{\frac{13}{84}}(1 + q^{7m-1})(1 - q^{7m-6})} \right)$$

$$\cdot \left(q^{\frac{6}{24}} \frac{(1 - q^m)}{(1 - q^{7m})} \right)$$

is an algebraic number. On the other hand, we know from [3] and [22] that

$$\sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}} = \prod_{m=1}^{\infty} \frac{(1 + q^m)(1 + q^{21m-10})(1 + q^{21m-11})(1 - q^{21m})}{(1 - q^m)}$$

$$- q \prod_{m=1}^{\infty} \frac{(1 + q^m)(1 + q^{21m-4})(1 + q^{21m-17})(1 - q^{21m})}{(1 - q^m)}.$$

Hence, we claim by (2.20) and Example 2.5, that

$$H := q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-10})(1 + q^{21m-11}) - q^{\frac{1}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-4})(1 + q^{21m-17})$$

$$= \left(q^{\frac{1}{168}} \sum_{m=1}^{\infty} \frac{q^{\frac{1}{2}(m^2+m)}}{(q)_m(q)_{2,m}} \right) \left(\frac{1}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 + q^m)} \right) \left(q^{-\frac{20}{24}} \frac{(1 - q^m)}{(1 - q^{21m})} \right)$$

and

$$I := \left(q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-10})(1 + q^{21m-11}) \right) \left(-q^{\frac{1}{84}}(1 + q^{21m-4})(1 + q^{21m-17}) \right)$$

$$= - \prod_{m=1}^{\infty} \frac{q^{-\frac{23}{84}}(1 + q^{7m-3})(1 + q^{7m-4})}{q^{\frac{39}{84}}(1 + q^{21m-3})(1 + q^{21m-18})}$$

are algebraic numbers. Therefore we obtain an equation $g(X) = X^2 - HX + I$ satisfying $g(q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-10})(1 + q^{21m-11})) = 0$, from which we derive

that $q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-10})(1 + q^{21m-11})$ and $q^{\frac{1}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-4})(1 + q^{21m-17})$ are algebraic numbers.

Next, it follows from Slater identities A.80, 82 and routine calculations that

$$\begin{aligned} & \left(q^{\frac{25}{168}} \sum_{m=0}^{\infty} \frac{(-q; q)_m q^{\frac{1}{2}m(m+1)}}{(q; q)_{2m+1}} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}(1 - q^m)}{q^{\frac{21}{24}}(1 - q^{21m})} \right) \left(\frac{1}{q^{\frac{1}{24}}(1 + q^m)} \right) \\ &= \left(q^{-\frac{61}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-8})(1 + q^{21m-13}) \right) - \left(q^{\frac{107}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-1})(1 + q^{21m-20}) \right), \\ & \left(q^{\frac{121}{168}} \sum_{m=0}^{\infty} \frac{(-q; q)_m q^{\frac{1}{2}m(m+3)}}{(q; q)_{2m+1}} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}(1 - q^m)}{q^{\frac{21}{24}}(1 - q^{21m})} \right) \left(\frac{1}{q^{\frac{1}{24}}(1 + q^m)} \right) \\ &= \left(q^{-\frac{13}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-5})(1 + q^{21m-16}) \right) - \left(q^{\frac{71}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-2})(1 + q^{21m-19}) \right), \\ & - \left(q^{-\frac{61}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-8})(1 + q^{21m-13}) \right) \left(q^{\frac{107}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-1})(1 + q^{21m-20}) \right) \\ &= - \left(\prod_{m=1}^{\infty} \frac{q^{\frac{13}{84}}(1 + q^{7m-1})(1 + q^{7m-6})}{q^{-\frac{33}{84}}(1 + q^{21m-6})(1 + q^{21m-15})} \right) \end{aligned}$$

and

$$\begin{aligned} & - \left(q^{-\frac{13}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-5})(1 + q^{21m-16}) \right) \left(q^{\frac{71}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-2})(1 + q^{21m-19}) \right) \\ &= - \left(\prod_{m=1}^{\infty} \frac{q^{-\frac{11}{84}}(1 + q^{7m-2})(1 + q^{7m-5})}{q^{-\frac{23}{84}}(1 + q^{21m-9})(1 + q^{21m-12})} \right) \end{aligned}$$

are algebraic numbers. Therefore, we conclude that $q^{-\frac{13}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-5})(1 + q^{21m-16})$, $q^{\frac{71}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-2})(1 + q^{21m-19})$, $q^{-\frac{61}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-8})(1 + q^{21m-13})$ and $q^{\frac{107}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-1})(1 + q^{21m-20})$ are algebraic numbers.

Example 2.9. According to Slater ([27]), we have the following identity

$$\begin{aligned} & \prod_{m=1}^{\infty} (1 + q^{42m-19})(1 + q^{42m-23})(1 - q^{42m}) \\ (2.21) \quad & - q^3 \prod_{m=1}^{\infty} (1 + q^{42m-5})(1 + q^{42m-37})(1 - q^{42m}) \\ &= \prod_{m=1}^{\infty} \frac{(1 - q^{2m})}{(1 + q^{2m-1})} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n}}. \end{aligned}$$

From the infinite products derived from theta functions we know that

$$q^{-\frac{13}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n}}$$

is an algebraic number ([17]). Then by (2.21) we get that

$$(2.22) \quad \begin{aligned} & \left(q^{-\frac{13}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_{2n}} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{2}{24}} (1 - q^{2m})}{q^{\frac{42}{24}} (1 - q^{42m})} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}}{(1 + q^{2m-1})} \right) \\ &= \left(q^{-\frac{143}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-19})(1 + q^{42m-23}) \right) \\ & \quad - \left(q^{\frac{109}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-5})(1 + q^{42m-37}) \right) \end{aligned}$$

is an algebraic number and we can also check by Example 2.8 that

$$(2.23) \quad \begin{aligned} & - \prod_{m=1}^{\infty} \left(q^{-\frac{143}{84}} (1 + q^{42m-19})(1 + q^{42m-23}) \right) \left(q^{\frac{109}{84}} (1 + q^{42m-5})(1 + q^{42m-37}) \right) \\ &= - \frac{\left(q^{\frac{71}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-2})(1 + q^{21m-19}) \right) \left(q^{-\frac{13}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-5})(1 + q^{21m-16}) \right)}{\left(q^{\frac{107}{42}} \prod_{m=1}^{\infty} (1 + q^{42m-2})(1 + q^{42m-40}) \right) \left(q^{-\frac{61}{42}} \prod_{m=1}^{\infty} (1 + q^{42m-16})(1 + q^{42m-26}) \right)} \end{aligned}$$

is an algebraic number. Therefore, by (2.22) and (2.23) $\prod_{m=1}^{\infty} q^{-\frac{143}{84}} (1 + q^{42m-19})(1 + q^{42m-23})$ and $\prod_{m=1}^{\infty} q^{\frac{109}{84}} (1 + q^{42m-5})(1 + q^{42m-37})$ are algebraic numbers. In a similar way, due to Slater identities (A. 118), (A. 119) ([24], [27]) and [17] concerning algebraic numbers for infinite sums, we obtain that

$$(2.24) \quad \begin{aligned} & \left(q^{\frac{11}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n}} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{2}{24}} (1 - q^{2m})}{q^{\frac{42}{24}} (1 - q^{42m})} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}}{(1 + q^{2m-1})} \right) \\ &= \left(q^{-\frac{131}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-17})(1 + q^{42m-25}) \right) \\ & \quad - \left(q^{-\frac{47}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-11})(1 + q^{42m-31}) \right), \end{aligned}$$

$$(2.25) \quad \begin{aligned} & - \prod_{m=1}^{\infty} \left(q^{-\frac{131}{84}} (1 + q^{42m-17})(1 + q^{42m-25}) \right) \left(q^{-\frac{47}{84}} (1 + q^{42m-11})(1 + q^{42m-31}) \right) \\ &= - \frac{\left(q^{\frac{11}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-4})(1 + q^{21m-17}) \right) \left(q^{-\frac{73}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-10})(1 + q^{21m-11}) \right)}{\left(q^{\frac{71}{42}} \prod_{m=1}^{\infty} (1 + q^{42m-4})(1 + q^{42m-38}) \right) \left(q^{-\frac{13}{42}} \prod_{m=1}^{\infty} (1 + q^{42m-10})(1 + q^{42m-32}) \right)}, \end{aligned}$$

$$\begin{aligned}
 & \left(q^{\frac{107}{168}} \sum_{n=0}^{\infty} \frac{(-q; q^2)_n q^{n(n+2)}}{(q^2; q^2)_{2n+1}} \right) \left(\prod_{m=1}^{\infty} \frac{q^{\frac{2}{24}} (1 - q^{2m})}{q^{\frac{42}{24}} (1 - q^{42m})} \right) \\
 & \cdot \left(\prod_{m=1}^{\infty} \frac{q^{\frac{1}{24}}}{(1 + q^{2m-1})} \right) \\
 (2.26) \quad & = \left(q^{-\frac{83}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-13})(1 + q^{42m-29}) \right) \\
 & - \left(q^{\frac{253}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-1})(1 + q^{42m-41}) \right),
 \end{aligned}$$

and
(2.27)

$$\begin{aligned}
 & - \prod_{m=1}^{\infty} \left(q^{-\frac{83}{84}} (1 + q^{42m-13})(1 + q^{42m-29}) \right) \left(q^{\frac{253}{84}} (1 + q^{42m-1})(1 + q^{42m-41}) \right) \\
 & = - \frac{\left(q^{\frac{107}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-1})(1 + q^{21m-20}) \right) \left(q^{-\frac{61}{84}} \prod_{m=1}^{\infty} (1 + q^{21m-8})(1 + q^{21m-13}) \right)}{\left(q^{\frac{11}{42}} \prod_{m=1}^{\infty} (1 + q^{42m-8})(1 + q^{42m-34}) \right) \left(q^{-\frac{73}{42}} \prod_{m=1}^{\infty} (1 + q^{42m-20})(1 + q^{42m-22}) \right)}.
 \end{aligned}$$

Thus we claim by (2.24)~(2.27) that

$$\begin{aligned}
 & q^{-\frac{131}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-17})(1 + q^{42m-25}), \quad q^{-\frac{47}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-11})(1 + q^{42m-31}), \\
 & q^{-\frac{83}{84}} \prod_{m=1}^{\infty} (1 + q^{42m-13})(1 + q^{42m-29}), \quad q^{\frac{253}{84}} (1 + q^{42m-1})(1 + q^{42m-41})
 \end{aligned}$$

are algebraic numbers.

3. Rogers-Ramanujan identities

The Rogers-Ramanujan continue fraction, defined by

$$R(q) := \frac{q^{\frac{1}{5}}}{1} + \frac{q}{1 + \frac{q^2}{1} + \frac{q^3}{1 + \dots}},$$

first appeared in a paper of Rogers ([23]). Using such identities, Rogers proved that

$$R(q) = q^{\frac{1}{5}} \frac{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}.$$

By (2.9),

$$(3.1) \quad R(q) = \frac{q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-1})(1 - q^{5m-4})}{q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-2})(1 - q^{5m-3})}$$

is an algebraic number. Let

$$T(q) := \frac{q^{\frac{1}{5}}}{1} - \frac{q}{-1} + \frac{q^2}{1} - \frac{q^3}{-1} + \dots.$$

Berndt ([7, p.16]) wrote that

$$(3.2) \quad T(q)R(q)(T(q) - R(q))^4 - T(q)^2R(q)^2(T(q) - R(q))^2 + 2T(q)^3R(q)^3 = (T(q) - R(q))(1 + T(q)^5R(q)^5).$$

From (3.1) and (3.2), it follows that $T(q)$ is an algebraic number.

Set

$$f(-q) := f(-q; -q^2) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2} = (q; q)_{\infty}.$$

Note that the latter equality is known as Pentagonal number theorem.

Theorem 3.1. *Let $\alpha = \frac{1-\sqrt{5}}{2}$, $\beta = \frac{1+\sqrt{5}}{2}$ and $\zeta = e^{\frac{2\pi i}{5}}$.*

- (a) $q^{\frac{9}{40}} \sum_{m=-\infty}^{\infty} (-1)^m (10m + 3)q^{(5m+3)m/2}$,
 $q^{\frac{1}{40}} \sum_{m=-\infty}^{\infty} (-1)^m (10m + 1)q^{(5m+1)m/2}$ are transcendental numbers.
- (b) $\prod_{m=0}^{\infty} \frac{1}{q^{\frac{1}{12}}(1 + \alpha q^m + q^{2m})}$ and $\prod_{m=0}^{\infty} \frac{1}{q^{\frac{1}{12}}(1 + \beta q^m + q^{2m})}$ are algebraic numbers.
- (c) $q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 - \zeta^2 q^m)(1 - \zeta^3 q^m)$ and $q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 - \zeta q^m)(1 - \zeta^4 q^m)$ are algebraic numbers.

Proof. (a) We know from [8] that

$$\sum_{m=-\infty}^{\infty} (-1)^m (10m + 3)q^{(5m+3)m/2} = \left(\frac{3}{R^2(q)} + R^3(q) \right) q^{\frac{2}{5}} f^3(-q^5)$$

and

$$\sum_{m=-\infty}^{\infty} (-1)^m (10m + 1)q^{(5m+1)m/2} = \left(\frac{3}{R^2(q)} - 3R^3(q) \right) q^{\frac{3}{5}} f^3(-q^5).$$

Since $\left(q^{\frac{5}{24}} f^3(-q^5) = \prod_{m=1}^{\infty} (1 - q^{5m}) \right)^3$ is transcendental ([17]) and $R(q)$ is algebraic, we conclude that

$$q^{\frac{9}{40}} \sum_{m=-\infty}^{\infty} (-1)^m (10m+3)q^{(5m+3)m/2} \text{ and } q^{\frac{1}{40}} \sum_{m=-\infty}^{\infty} (-1)^m (10m+1)q^{(5m+1)m/2}$$

are transcendental numbers.

(b) It follows from [8] that

$$\frac{1}{\sqrt{R(q)}} - \alpha\sqrt{R(q)} = \frac{1}{q^{\frac{1}{10}}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{m=1}^{\infty} \frac{1}{1 + \alpha q^{\frac{m}{5}} + q^{\frac{2m}{5}}}$$

and

$$\frac{1}{\sqrt{R(q)}} - \beta\sqrt{R(q)} = \frac{1}{q^{\frac{1}{10}}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{m=1}^{\infty} \frac{1}{1 + \beta q^{\frac{m}{5}} + q^{\frac{2m}{5}}}.$$

Meanwhile, we know that $q^{-\frac{4}{24}} \prod_{m=1}^{\infty} \left(\frac{1 - q^m}{1 - q^{5m}}\right)$ and $R(q)$ are algebraic numbers. Since the values of the Dedekind $\eta(\tau)$ -function are nonzero ([25]),

$$\prod_{m=1}^{\infty} \frac{1}{q^{\frac{1}{60}}(1 + \alpha q^{\frac{m}{5}} + q^{\frac{2m}{5}})} \quad \text{and} \quad \prod_{m=1}^{\infty} \frac{1}{q^{\frac{1}{60}}(1 + \beta q^{\frac{m}{5}} + q^{\frac{2m}{5}})}$$

are algebraic numbers. Then the fact that $\tau, 5\tau \in \mathfrak{h} \cap k$ and $2 + \alpha, 2 + \beta$ are algebraic numbers implies the result.

(c) We find in [8] that

$$f(-q^2, -q^3) - \alpha q^{15} f(-q, -q^4) = f(-\zeta^2, -\zeta^3 q^{\frac{1}{5}})/(1 - \zeta^2)$$

and

$$f(-q^2, -q^3) - \beta q^{15} f(-q, -q^4) = f(-\zeta, -\zeta^4 q^{\frac{1}{5}})/(1 - \zeta).$$

If we multiply both sides by $q^{\frac{1}{60}} \prod_{m=1}^{\infty} \frac{1}{1 - q^m}$, then

$$\begin{aligned} & \left(q^{-\frac{11}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-2})(1 - q^{5m-3}) \right) \left(\frac{q^{\frac{5}{24}}(1 - q^{5m})}{q^{\frac{1}{24 \cdot 5}}(1 - q^{\frac{1}{5}m})} \right) \\ & + \alpha \left(q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - q^{5m-1})(1 - q^{5m-4}) \right) \left(\frac{q^{\frac{5}{24}}(1 - q^{5m})}{q^{\frac{1}{24 \cdot 5}}(1 - q^{\frac{1}{5}m})} \right) \\ & = q^{\frac{1}{60}} \prod_{m=1}^{\infty} (1 - \zeta^2 q^{\frac{1}{5}m})(1 - \zeta^3 q^{\frac{1}{5}m}) \end{aligned}$$

is an algebraic number. Replacing τ by 5τ , we conclude that $q^{\frac{1}{12}} \prod_{m=1}^{\infty} (1 - \zeta^2 q^m)(1 - \zeta^3 q^m)$ is an algebraic number. The other case can be done in a similar way. \square

Corollary 3.2. *If $\zeta = e^{\frac{2\pi i}{5}}$, then $\frac{\prod_{m=1}^{\infty} (1 - \zeta^2 q^m)(1 - \zeta^3 q^m)}{\prod_{m=1}^{\infty} (1 - \zeta q^m)(1 - \zeta^4 q^m)}$ is an algebraic number and $q^a \frac{\prod_{m=1}^{\infty} (1 - \zeta^2 q^m)(1 - \zeta^3 q^m)}{\prod_{m=1}^{\infty} (1 - \zeta q^m)(1 - \zeta^4 q^m)}$ is a transcendental number with $a \in \mathbb{Q} - \{0\}$.*

Proof. It is immediate by Theorem 3.1 (c). \square

Remark. Let m be a positive integer not divisible by 5, $\zeta = e^{\frac{2\pi i}{5}}$ and $\alpha = \frac{1-\sqrt{5}}{2}$. Then we know from [8] the following identity

$$\prod_{j=0}^4 (1 + \alpha \zeta^{mj} q^{\frac{jn}{5}} + \zeta^{2mj} q^{\frac{2jn}{5}}) = (1 - q^n)^2.$$

Hence, we get that

$$\prod_{\substack{m=1 \\ m \not\equiv 0 \pmod{5}}}^{\infty} \left(\prod_{j=0}^4 (1 + \alpha \zeta^{mj} q^{\frac{jn}{5}} + \zeta^{2mj} q^{\frac{2jn}{5}}) \right) = \prod_{m=1}^{\infty} \left(\frac{1 - q^m}{1 - q^{5m}} \right)^2$$

and

$$\prod_{m=1}^{\infty} \left(\prod_{j=0}^4 (1 + \alpha \zeta^{mj} q^{\frac{jn}{5}} + \zeta^{2mj} q^{\frac{2jn}{5}}) \right) = \prod_{m=1}^{\infty} \left(\frac{1 - q^m}{1 - q^{5m}} \right)^2 (1 + \alpha q^m + q^{2m})^5.$$

By Theorem 3.1 and (2.3) we conclude that

$$q^{\frac{1}{12}} \prod_{m=1}^{\infty} \left(\prod_{j=0}^4 (1 + \alpha \zeta^{mj} q^{\frac{jn}{5}} + \zeta^{2mj} q^{\frac{2jn}{5}}) \right)$$

is an algebraic number.

References

- [1] G. E. Andrews, *An introduction to Ramanujan's lost notebook*, Amer. Math. Monthly **86** (1979), 89–108.
- [2] ———, *Ramanujan's lost notebook. III. The Rogers-Ramanujan continued fraction*, Adv. Math. **41** (1981), 186–208.
- [3] W. N. Bailey, *On the simplification of some identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (3) (1951), 217–221.
- [4] B. C. Berndt, *Ramanujan's Notebooks II*, Springer-Verlag, 1989.
- [5] ———, *Ramanujan's Notebooks III*, Springer-Verlag, 1991.
- [6] ———, *Ramanujan's Notebooks IV*, Springer-Verlag, 1993.
- [7] ———, *Ramanujan's Notebooks V*, Springer-Verlag, 1997.
- [8] B. C. Berndt, S.-S. Huang, J. Sohn, and S. H. Son, *Some theorems on the Rogers - Ramanujan continued fraction in Ramanujan's lost notebook*, Trans. Amer. Math. Soc. **352** (2000), 2157–2177.
- [9] B. C. Berndt and H. H. Chan, *Some values for the Rogers-Ramanujan continued fraction*, Canad. J. Math. **47** (1995), 897–914.
- [10] B. C. Berndt, H. H. Chan, and L.-C. Zhang, *Explicit evaluations of the Rogers-Ramanujan continued fraction*, J. Reine Angew. Math. **480** (1996), 141–159.
- [11] P. B. Borwein and P. Zhou, *On the irrationality of a certain q series*, Proc. Amer. Math. Soc. **127** (1999), 1605–1613.
- [12] S.-S. Huang, *Ramanujan's evaluations of the Rogers-Ramanujan type continued fractions at primitive roots of unity*, Acta Arith. **80** (1997), 49–60.
- [13] N. Ishida, *Generators and equations for modular function fields of principal congruence subgroups*, Acta Arith. **85** (1998), 197–207.

- [14] S.-Y. Kang, *Some theorems on the Rogers-Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook*, The Ramanujan Journal **3** (1999), 91–111.
- [15] ———, *Ramanujan's formulas for the explicit evaluation of the Rogers-Ramanujan continued fraction and theta functions*, Acta Arith. **90** (1999), 49–68.
- [16] D. Kim and J. K. Koo, *Algebraic integer as values of elliptic functions*, Acta Arith. **100** (2001), 105–116.
- [17] ———, *On the infinite products derived from theta series I*, J. Korean Math. Soc. **44** (2007), 55–107.
- [18] D. Kubert and S. Lang, *Units in the modular function fields*, Math. Ann. **218** (1975), 175–189.
- [19] S. Lang, *Elliptic Functions*, Addison-Wesley, 1973.
- [20] S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
- [21] ———, *The Lost Notebook and Other Unpublished Papers*, Narosa, 1988.
- [22] L. J. Rogers, *On two theorems of combinatory analysis and some allied identities*, Proc. London Math. Soc. (1) **16** (1917), 315–336.
- [23] ———, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. (1) **25** (1894), 318–343.
- [24] A. V. Sills, *Finite Rogers-Ramanujan type identities*, Electron. J. Combin. **10** (2003), 1–122.
- [25] J. Silverman, *The Arithmetic of Elliptic Curves*, Springer -Verlag, New York, 1986.
- [26] L. J. Slater, *A new proof of Roger's transformations of series*, Proc. London Math. Soc. series 2 **53** (1951), 461–475.
- [27] ———, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. series 2 **54** (1952), 147–167.
- [28] S. Son, *Cubic identities of theta functions*, The Ramanujan Journal **2** (1998), 303–316.
- [29] ———, *Some theta function identities related to Rogers-Ramanujan continued fraction*, Proc. Amer. Math. Soc. **126** (1998), 2895–2902.
- [30] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Cambridge Press, 1978.

DAEYEOUL KIM
 NATIONAL INSTITUTE FOR MATHEMATICAL SCIENCES 385-16
 TAJEON 305-340, KOREA
 AND
 DEPARTMENT OF MATH. AND INSTITUTE OF PURE AND APPLIED MATHEMATICS
 CHONBUK NATIONAL UNIVERSITY
 CHONJU 561-756, KOREA
E-mail address: daeyeoul@chonbuk.ac.kr

JA KYUNG KOO
 KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY
 DEPARTMENT OF MATHEMATICS
 TAJEON 305-701, KOREA
E-mail address: jkkoo@math.kaist.ac.kr

YILMAZ SIMSEK
 UNIVERSITY OF AKDENIZ, FACULTY OF SCIENCE
 DEPARTMENT OF MATHEMATICS
 07058 ANTALYA, TURKEY
E-mail address: ysimsek@akdeniz.edu.tr