ON REES MATRIX REPRESENTATIONS OF ABUNDANT SEMIGROUPS WITH ADEQUATE TRANSVERSALS

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Abstract. The concepts of $*$-relation of a (Γ-)semigroup and $\bar{\Gamma}$-adequate transversal of a (Γ-)abundant semigroup are defined in this note. Then we develop a matrix type theory for abundant semigroups. We give some equivalent conditions of a Rees matrix semigroup being abundant and some equivalent conditions of an abundant Rees matrix semigroup having an adequate transversal. Then we obtain some Rees matrix representations for abundant semigroups with adequate transversals by the above theories.

Introduction

It is well-known that all abundant semigroups constitute an important class of generalized regular semigroups. An adequate transversal $S^0$ of an abundant semigroup $S$ is an adequate $*$-subsemigroup of $S$ which for any $x \in S$ there are unique element denoted by $x^0$ and two idempotents denoted by $e_x, f_x$ such that $x = e_x x^0 f_x$, where $e_x \mathcal{L}^* x^0 \mathcal{R}^* f_x$ ($\mathcal{L}^*, \mathcal{R}^*$ are Green’s $*$-relations). Here $e_x$ and $f_x$ are uniquely determined by $x$. Furthermore, $S^0$ is multiplicative if $f_x e_y \in E(S^0)$ for any $x, y \in S$.

By the Γ-semigroup $T$ (see [2, 10, 11]) means that for two non-empty sets $T$ and Γ in which an element denoted by $x, \nu$ respectively under multiplication

$x \circ y = xvy \in T$ satisfying $(x\alpha y)\beta z = x\alpha (y\beta z)$

for any $x, y, z \in T$ and $\nu, \alpha, \beta \in \Gamma$. A Γ-semigroup $T$ is Γ-commutative, if for any $x, y \in T, \alpha \in \Gamma, x\alpha y = y\alpha x$. Similar to the theory of semigroups, in the theory of Γ-semigroups we have also the well-known correlate concepts. Here we will apply them directly. Clearly, any semigroup $T$ is always a Γ-semigroup for any subset Γ of $T$ or $\Gamma = \{1\}$, where the member 1 is an outer identity. Conversely, a Γ-semigroup $T$ need not be a semigroup in general.

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We remark that Rees matrix semigroups have been defined in several slightly different ways. For example, in [9] the matrix semigroup is over an inverse semigroup. In [12], J. Fountain used some \((T_\alpha, T_\beta)\)-bisystem \(M_{\alpha\beta}\), where \(\alpha, \beta \in \Gamma\) and an outer zero and introduced blocked Rees matrix semigroup \(\mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma; P)\). Here, we say that a Rees matrix semigroup \(S_\mu\) is a Rees matrix representation of the semigroup \(S\) if \(S_\mu\) is isomorphic to \(S\).

Our purpose in this note is to develop a matrix type theory for abundant semigroups with adequate transversals, that is, to study the conditions of a Rees matrix semigroup being abundant and the conditions of an abundant Rees matrix semigroup having an adequate transversal. The above results such that we may obtain some Rees matrix representations of abundant semigroup with an adequate transversal. We proceed as follows:

We begin in Section 1 by defining \(\Gamma\)-relations on a \((\Gamma-)\)semigroup and discussing their properties. Particularly, the relations between Green’s \(\ast\)-relations and \(\Gamma\)-relations. By these relations, we show that the relations between a \(\Gamma\)-semigroup \(T\) and a Rees matrix semigroup over \(T\) and obtain some equivalent conditions of a Rees matrix semigroup being abundant. In Section 2, we define the concept of \(\overline{\Gamma}\)-adequate transversals of \(\Gamma\)-abundant semigroups. Then we show that the relation between \(\overline{\Gamma}\)-adequate transversals and adequate transversals of Rees matrix semigroups and obtain some equivalent conditions of an abundant Rees matrix semigroup having an adequate transversal. In Section 3, using the results in Sections 1 and 2, for an abundant semigroup \(S\) with an adequate transversal \(S^0\), we construct a Rees matrix semigroup \(S_\mu\) over some subset \(T\) of \(S\). Then we prove that \(S_\mu\) is a \((\Gamma-)\)Rees matrix representation of given semigroup \(S\).

For terminologies not given in this note the reader is referred to [3, 4, 5, 6, 8, 9, 11, 12].

1. The conditions of a Rees matrix semigroup being abundant

We present first some necessary notation and well-known results. For details consult [6, 9, 11, 12].

Given a semigroup \(T\), non-empty index sets \(I\) and \(\Lambda\) and defined a \(\Lambda \times I\) matrix \(P = (p_{i\lambda})_{\Lambda \times I}\) over \(T\). By [9], we may obtained a Rees matrix semigroup denoted by \(S_\mu = \mu(T; I, \Lambda, P)\). It’s elements consist of all triples \((x)_{i\lambda}\), where \(x \in T, (i, \lambda) \in I \times \Lambda\) with multiplication

\[
(1.1) \quad (\forall (x)_{i\lambda}, (y)_{j\mu} \in S_\mu) \quad (x)_{i\lambda}(y)_{j\mu} = (xp_{i\lambda}y)_{i\mu}.
\]

In general, \(S_\mu\) is only a semigroup. Put the set \(RS_\mu = \mathcal{R}\mu(T; I, \Lambda, P)\) of all regular elements of \(S_\mu\). It was proved in [9] that \(RS_\mu\) is a regular semigroup if \(T\) is regular. This result, we well generalized to abundant semigroups (see Theorem 1.9). Now we consider the converse problem.
Let $S$ be a semigroup with an outer zero. Like [6], we index the set of non-zero $\mathcal{R}^*$-classes of $S$ by $I$ and the set of non-zero $\mathcal{L}^*$-classes of $S$ by $\Lambda$, so that we write the $\mathcal{R}^*$-classes as $R_i^\ast$ ($i \in I$) and the $\mathcal{L}^*$-classes as $L_\lambda^\ast$ ($\lambda \in \Lambda$). Then we put $H_{i\lambda}^\ast = R_i^\ast \cap L_\lambda^\ast$ for $(i, \lambda) \in I \times \Lambda$, so that every non-zero $\mathcal{H}^*$-class of $S$ is some $H_{i\lambda}^\ast$ and each $H_{i\lambda}^\ast$ is either empty or a $\mathcal{H}^*$-class. Of course $S \setminus \{0\} = \bigcup\{H_{i\lambda}^\ast; (i, \lambda) \in I \times \Lambda\}$ we denote this structure express of $S$ by $(S, I, \Lambda)$.

Further, like [6] we may obtain another structure express denoted by $(S, I, \Lambda, \Gamma)$, where $\Gamma$ is the set of non-zero $\mathcal{D}$-classes of $S$ which contain idempotents. We write these $\mathcal{D}$-classes as $D_\alpha$ ($\alpha \in \Gamma$). Let $I_\alpha = \{i \in I; D_\alpha \cap R_i^\ast \neq \emptyset\}$, $\Lambda_\alpha = \{\lambda \in \Lambda; D_\alpha \cap L_\lambda^\ast \neq \emptyset\}$. In general, $I = \bigcup I_\alpha$ and $\Lambda = \bigcup \Lambda_\alpha$. When $S$ satisfies some conditions (for example, $S$ is abundant), they are disjoint union.

Now let the set

\[(1.2) \quad \Gamma = \{p_{\lambda i} \in T \mid (i, \lambda) \in I \times \Lambda\}\]

be any subset of $S$ determined by the index pair set $I \times \Lambda$. Since $S$ is a semigroup and $\Gamma \subseteq S$, so $S$ becomes a $\Gamma$-semigroup. Now let $T$ be a subset with zero of $S$. In this section we suppose always that $T$ is a $\Gamma$-semigroup where the set $\Gamma$ defined as (1.2). On $\Gamma$-semigroup $T$ (or $S$) we give the following concept.

**Definition 1.1.** For $\Gamma$-semigroup $T$ (resp. semigroup $T$) the $\Gamma$’s relations on $T$ denoted by $\forall \mathcal{L}_\lambda^\ast$ and $\forall \mathcal{R}_\lambda^\ast$ for $i \in I, \lambda \in \Lambda$ are defined by

\[
(a, b \in T) a \mathcal{L}_\lambda^\ast b : (x, y \in T, v, u \in I) \text{ for } \lambda \in \Lambda
\]

\[
ap_{\lambda i} x = ap_{\lambda i} y \Leftrightarrow b_{\lambda i} x = b_{\lambda i} y;
\]

\[
(a, b \in T) a \mathcal{R}_\lambda^\ast b : (x, y \in T, k, t \in I) \text{ for } i \in I
\]

\[
 xp_{\lambda i} a = yp_{\lambda i} a \Leftrightarrow xp_{\lambda i} b = yp_{\lambda i} b.
\]

The following results are clear.

**Lemma 1.2.** (1) For $\lambda \in \Lambda$ ($i \in I$), $\mathcal{L}^\ast_{\lambda i}(\mathcal{R}^\ast_{\lambda i})$ is an equivalence relation on $T$. We denote the $\mathcal{L}^\ast_{\lambda i}$-class ($\mathcal{R}^\ast_{\lambda i}$-class) by $L^\ast_{\lambda i}(\lambda \in \Lambda)$ ($R^\ast_{\lambda i}(i \in I)$).

(2) If $a \mathcal{L}^\ast b$ ($a \mathcal{R}^\ast b$) for $a, b \in T$, then $a \mathcal{L}^\ast_{\lambda i} b$ ($a \mathcal{R}^\ast_{\lambda i} b$) for any $\lambda \in \Lambda$ ($i \in I$).

By the structure express $(S, I, \Lambda)$ of $S$, we know that for $a, b \in S$, if $a, b \in L_\lambda^\ast$ ($R_i^\ast$), then $a \mathcal{L}^\ast b$ ($a \mathcal{R}^\ast b$). Conversely, if $a \mathcal{L}^\ast b$ ($a \mathcal{R}^\ast b$), then there is a member $\lambda \in \Lambda$ ($i \in I$) such that $a, b \in L_\lambda^\ast$ ($R_i^\ast$). Thus we may think that the Green’s $\ast$-relation $\mathcal{L}^\ast$ ($\mathcal{R}^\ast$) can be written in the form $\mathcal{L}^\ast_{\lambda i}$ ($\mathcal{R}^\ast_{\lambda i}$) for some $\lambda \in \Lambda$ ($i \in I$).

The following we suppose always that the matrix $P = (p_{\lambda i})_{\lambda \times I}$ over $\Gamma$ and $S_\mu = \mu(T; I, \Lambda, P)$ is a Rees matrix semigroup over $T$ under the multiplication (1.1).

**Lemma 1.3.** (1) For any non-zero elements $(a)_{\lambda i}$, $(b)_{j\mu} \in S_\mu$, if $\lambda = \mu$ and $a \mathcal{L}^\ast_{\lambda i} b$, then $(a)_{\lambda i} \mathcal{L}^\ast_{\lambda i} (b)_{j\mu}$, if $i = j$ and $a \mathcal{R}^\ast_{\lambda i} b$, then $(a)_{\lambda i} \mathcal{R}^\ast_{\lambda i} (b)_{j\mu}$.

Further, if $S_\mu$ is abundant, then the converse case is also true.

(2) We denote the $\mathcal{L}^\ast$-classes ($\mathcal{R}^\ast$-classes) of $S_\mu$ by $L_\lambda^\ast (R_i^\ast)$. If $S_\mu$ is abundant, then
(i ∈ I) \( R^*_i \) = \( \{ (x)_\lambda \mid x \in R^*(i), \lambda \in \Lambda \} \); 
(\lambda \in \Lambda) \( L^*_\lambda \) = \( \{ (x)_\lambda \mid x \in L^*(\lambda), i \in I \} \).

Proof. (1) Let \( 0 \neq (a)_{i\lambda}, 0 \neq (b)_{j\mu} \), if \( \lambda = \mu \) and \( a.L^*_\lambda b \), then for \((x)_{vt}, (y)_{uk} \) in \( S_\mu \)

\[ (a)_{i\lambda}(x)_{vt} = (a)_{i\lambda}(y)_{uk} \iff (ap_{\lambda x})_{vt} = (ap_{\lambda y})_{uk} \]
\[ \iff t = k \text{ and } ap_{\lambda x} = ap_{\lambda y}. \]

Similarly, for the element \((b)_{j\mu} = (b)_{j\lambda} \)

\[ (b)_{j\lambda}(x)_{vt} = (b)_{j\lambda}(y)_{uk} \iff t = k \text{ and } bp_{\lambda x} = bp_{\lambda y}. \]

Thus by \( \lambda = \mu \) and \( a.L^*_\lambda b \) may imply that

\[ (a)_{i\lambda}(x)_{vt} = (a)_{i\lambda}(y)_{uk} \iff (t = k, ap_{\lambda x} = ap_{\lambda y}) \]
\[ \iff t = k \text{ and } bp_{\lambda x} = bp_{\lambda y}. \]
\[ \iff (b)_{j\lambda}(x)_{vt} = (b)_{j\lambda}(y)_{uk}. \]

That is \( (a)_{i\lambda}L^*_\lambda (b)_{j\lambda} \). The another result is dual.

If \( S_\mu \) is abundant, let \( e = (e)_{wt} \in L^*_\lambda \cap E(S) \), by \( e^2 = e \) implies \( (a)_{i\lambda}e = (a)_{i\lambda} \) so \( t = \lambda \) and if \( (b)_{j\mu} \in L^*_\mu \cap L^*_\lambda \), similarly \( \mu = \lambda \). Thus we obtain \( (a)_{i\lambda}L^*_\lambda (b)_{j\lambda} \), that is, for \((x)_{vt}, (y)_{uk} \) in \( S_\mu \)

\[ (a)_{i\lambda}(x)_{vt} = (a)_{i\lambda}(y)_{uk} \iff (b)_{j\lambda}(x)_{vt} = (b)_{j\lambda}(y)_{uk}. \]

Computing we may imply that \( t = k \) and \( a.L^*_\lambda b \) for \( \lambda \in \Lambda \).

(2) Let \( S_\mu \) be abundant, by part (1) we know that for \( 0 \neq (a)_{i\lambda}, 0 \neq (b)_{j\mu} \in S \)

\[ (a)_{i\lambda}L^*_\lambda (b)_{j\mu} \iff \lambda = \mu \text{ and } a.L^*_\lambda b. \]

So that \( a, b \in L^*(\lambda) \). Conversely, if \( a, b \in L^*(\lambda) \), then \( (a)_{i\lambda}, (b)_{j\lambda} \in L^*_\lambda \) for any \( i, j \in I \). Thus we have that

\[ L^*_\lambda = \{ (x)_{\lambda} \mid x \in L^*(\lambda), i \in I \} (\lambda \in \Lambda) \].

The other result is dual. \( \square \)

Corollary 1.4. For \((i, \lambda) \in I \times \Lambda, a, b \in T \)

(1) if \( a.L^*_\lambda b \) \((a.R^*_\lambda b) \), then \( a.L^*_\lambda b \) \((a.R^*_\lambda b) \).

(2) if \( a.L^*_\lambda b \) \((a.R^*_\lambda b) \), then \( ap_{\lambda x}L^*_\lambda bp_{\lambda y} \((p_{\lambda x}, a.R^*_\lambda p_{\lambda y}) \).

Lemma 1.5. For any \( p_{\lambda x} \in \Gamma \) if there are two elements \( q_x \) and \( r_i \) in \( S \) such that \( p_{\lambda x} = q_x r_i \), then

(1) the mapping \( \phi \) defined by

\[ (\forall (a)_{i\lambda} \in S_\mu) \; (a)_{i\lambda} \phi = r_i a q_x \]

is a homomorphism from \( S_\mu \) to \( S \).
(2) If \( S_\mu \) is abundant and \( \phi \) is an isomorphism, then
\[
(a)_{i, \lambda} \mathcal{L}_\lambda^* (b)_{j, \lambda} \text{ in } S_\mu \Leftrightarrow r_\lambda a p_\lambda \mathcal{L}_\lambda^* r_\lambda b \in S
\]
\[
(a)_{i, \lambda} \mathcal{T}_\lambda^* (b)_{i, \lambda} \text{ in } S_\mu \Leftrightarrow r_\lambda a p_\lambda \mathcal{T}_\lambda^* r_\lambda b \mu \in S
\]

Proof. (1) Since \( S \) is a semigroup, so \( \phi \) is a mapping from \( S_\mu \) to \( S \). For \((a)_{i, \lambda}, (b)_{j, \mu} \in S_\mu\),
\[
[(a)_{i, \lambda} (b)_{j, \mu}] \phi = (a p_\lambda b)_{i, \mu} = r_\lambda a p_\lambda b \mu
\]
\[
= (r_\lambda a q_\lambda) (r_\lambda b \mu) = (a)_{i, \lambda} \phi \cdot (b)_{j, \mu} \phi.
\]

(2) By Lemma 1.3, we obtain directly this result. \( \Box \)

The following concepts are different from that in the theory of \( \Gamma \)-semigroups.

Definition 1.6. An element \( a \in T \) is a \( p_\lambda \)-regular element means that \( a p_\lambda a = a \). The set of all \( p_\lambda \)-regular elements of \( T \) is denoted by \( \text{V}(p_\lambda) \). Let \( a, b \in \text{V}(p_\lambda) \) if \( a \) and \( b \) are \( p_\lambda \)-commutative, then we say that \( a \) and \( b \) are two \( p_\lambda \)-commutative regular elements. Let the set
\[
\text{CV}(p_\lambda) = \{ a \in \text{V}(p_\lambda) ; \forall x \in \text{V}(p_\lambda), a p_\lambda x = x p_\lambda a \},
\]
then \( \text{CV}(p_\lambda) \) is called the center of \( \text{V}(p_\lambda) \). \( T \) is called \( \Gamma \)-abundant if for any \((i, \lambda) \in I \times \Lambda, L^*(\lambda) \cap V(p_\lambda) \neq \phi \) and \( R^*(i) \cap V(p_\lambda) \neq \phi \). The \( \Gamma \)-abundant semigroup \( T \) is called \( \Gamma \)-adequate, if \( \text{V}(p_\lambda) = \text{CV}(p_\lambda) \) for any \((i, \lambda) \in I \times \Lambda \).

Particularly, for only one \( p_\lambda, \in \Gamma \) we have the concepts of \( p_\lambda \)-abundant and \( p_\lambda \)-adequate.

Clearly, since \( S \) is a semigroup, so the set \( \text{V}(p_\lambda) \) is the set of inverse elements of the non-zero element \( p_\lambda \) for \((i, \lambda) \in I \times \Lambda \). \( T \) is \( \Gamma \)-abundant (adequate) if and only if for any \( p_\lambda, \in \Gamma, T \) is \( p_\lambda \)-abundant (adequate). If \( T \) is \( \Gamma \)-abundant (adequate), then by Lemma 1.3 and Corollary 1.4, \( T \) is necessarily \( \Gamma \)-abundant (adequate), but the converse case is not always true.

Lemma 1.7. If \( T \) is \( \Gamma \)-adequate, then \( |L^*(\lambda) \cap \text{V}(p_\lambda)| = |R^*(i) \cap \text{V}(p_\lambda)| = 1 \) for any \( p_\lambda, \in \Gamma \).

Proof. Let \( a_1, a_2 \in L^*(\lambda) \cap \text{V}(p_\lambda) \) then \( 0 \neq a_1 \mathcal{L}_\lambda^* a_2 \neq 0 \). By Corollary 1.4,
\[
a_1 p_\lambda \mathcal{L}_\lambda a_2 p_\lambda \implies a_1 = a_1 p_\lambda a_1 \mathcal{L}_\lambda a_2 p_\lambda = a_1 p_\lambda a_1 \mathcal{L}_\lambda a_2 p_\lambda = a_2 \text{ by } V(p_\lambda) = \text{CV}(p_\lambda).
\]
Since \( p_\lambda \in \text{V}(a_k) \) \((k = 1, 2) \), so by \( a_1 \mathcal{L}_\lambda a_2 \) implies \( a_1 p_\lambda = a_2 p_\lambda \). Thus we have that \( a_1 = a_1 p_\lambda a_1 = a_2 p_\lambda a_1 = a_1 p_\lambda a_2 = a_2 p_\lambda a_2 = a_2 \). Dually, we can show that the other result for any \( p_\lambda, \in \Gamma \). \( \Box \)

Lemma 1.8. (1) A non-zero element \((a)_{i, \lambda} \) in \( S_\mu \) is regular if and only if \( a \in \text{Reg}(S) \) (the regular element set of \( S \)) and \((\exists \ (j, \mu) \in I \times \Lambda)\)
\[
p_\lambda T \phi, \cap \text{V}(a) \neq \phi.
\]
(2) A non-zero element \((a)_{i\lambda}\) in \(S_p\) is an idempotent if and only if \(a \in V(p_{i\lambda})\).
(3) Two idempotents \((a)_{i\lambda}, (b)_{j\mu}\) are commutative if and only if \((i, \lambda) = (j, \mu)\) and \(a, b \in p_{i\lambda}\)-commutative regular elements.

Proof. Here we omit the checking process of part (1) to part (3). \(\square\)

**Theorem 1.9.** The following are equivalent:
(1) \(S_\mu\) is abundant.
(2) \(T\) is \(\Gamma\)-abundant.
(3) \(P\) is an abundant matrix (i.e., each row and each column of \(P\) contain a regular element of \(S\)).

Proof. By Definition 1.6, Lemma 1.8, it is easy to show that they are equivalent. We here omit this proof. \(\square\)

**Theorem 1.10.** If \(T\) is \(\Gamma\)-abundant, then the following equivalent:
(1) \(S_\mu\) is adequate.
(2) \(T\) is \(\Gamma\)-adequate where the Gamma set \(\Gamma\) denoted by (1.2) and satisfies that \(I = \Lambda, |V(p_{i\lambda})| = 1\) for any \(i \in I\) and \(p_{i\lambda} = 0\) for \(\lambda \neq i, i, \lambda \in I\).

Proof. (1) \(\Rightarrow\) (2) Let \(S_\mu\) be adequate, then \(E(S_\mu)\) is a semilattice. For \(0 \neq (a)_{i\lambda}, 0 \neq (b)_{j\mu} \in E(S_\mu)\) computing we know that \(i = j\) and \(\lambda = \mu\) and \(a, b \in V(p_{i\lambda})\). Since the \(L_{\ast}^1(\mathfrak{A}^\ast)\)-class \(L_{\ast}^1(R_{\ast})\) of \(S_\mu\) has only an idempotent, so \(|V(p_{i\lambda})| = 1\) and \(CV(p_{i\lambda}) = V(p_{i\lambda})\). Suppose that \(\lambda \neq i\), and \(i \in \Lambda\) (or \(\lambda \in I\)), then \(L_{\ast}^1\) has an idempotent \((b)_{ji}\) for some \(j \in I\). Since for \((a)_{i\lambda} \in L_{\ast}^1 \bigcap E(S_\mu)\) and \((b)_{ji} \in L_{\ast}^1 \bigcap E(S_\mu)\), \((a)_{i\lambda}(b)_{ji} = (b)_{ji}(a)_{i\lambda}\). So we imply that \(i = j = \lambda\). It is a contradiction. Thus we obtain that \(I = \Lambda\) and \(|CV(p_{i\lambda})| = 1\) for any \(I\). Since \(H_{\ast}^\ast = L_{\ast}^1 \bigcap R_{\ast}\) has only an idempotent so by [6, Lemma 1.12] \(H_{\ast}^\ast\) is a cancellative monoid with the identity denote by \((e)_{ii}\). So we may write
\[
E(S_\mu) = \{(e_i)_{ii} | i \in I = \Lambda, e_i \in V(p_{i\lambda})\} \cup \{0\}.
\]
Let \((e_i)_{ii}, (f_j)_{jj} \in E(S_\mu), i \neq j\) by \(E(S_\mu)\) is a semilattice we obtain \((e_i)p_{i\lambda} f_j)_{ij} = (f_j)p_{i\lambda} e_i\) and \(i \neq j, i, j \in I\). Suppose that \(p_{ij} \neq 0\). Since \(T\) is \(\Gamma\)-abundant so \(L_{\ast}^1(j) \bigcap V(p_{i\lambda}) \neq \phi\). Let \(a \in L_{\ast}^1(j) \bigcap V(p_{i\lambda})\), by Theorem 1.9 and Lemma 1.8, \((a)_{ij} \in L_{\ast}^1 \bigcap E(S_\mu)\) and \(i \neq j, i, j \in I\). Since \(E(S_\mu)\) has above express, it is also a contradiction. So we know that \(p_{ij} = 0\). Similarly, we can prove \(p_{ij} = 0\) for \(i \neq j, i, j \in I\). Concluding we know that \(I\) satisfies the following conditions: \(I = \Lambda, |V(p_{i\lambda})| = 1\) for \(i \in I, p_{i\lambda} = 0\) for \(\lambda \neq i, i, \lambda \in I\). Finally, since \(T\) is \(\Gamma\)-abundant, by the above results we know \(V(p_{i\lambda}) = CV(p_{i\lambda})\) for \(i \in I\), and \(V(p_{i\lambda}) = V(0) = CV(0) = CV(p_{i\lambda})\) for \(\lambda \neq i, i, \lambda \in I\). So \(T\) is \(\Gamma\)-adequate.
(2) \(\Rightarrow\) (1) Suppose that \(\Gamma\) is as required and \(T\) is \(\Gamma\)-adequate, then \(E(S_\mu)\) can be written in the form
\[
E(S_\mu) = \{(e_i)_{ii} | i \in I, e_i \in V(p_{i\lambda}), |V(p_{i\lambda})| = 1\} \cup \{0\}.
\]
It is easy to check that $E(S_\mu)$ is a semilattice with zero. Thus $S_\mu$ is adequate.

The above results (see Theorems 1.9 and 1.10) generalized the corresponding results in [6] and [9].

**Corollary 1.11.** $S_\mu$ contains an adequate subsemigroup $S^0_\mu$ if and only if the matrix $P = (p_{x_\lambda})_{x \times \lambda}$ satisfies conditions

1. There is a subset $\bar{I} \times \bar{I} \subseteq I \times \Lambda$ (or $\bar{A} \times \bar{A} \subseteq I \times \Lambda$).
   
2. On $\bar{I} \times \bar{I}, |V(p_{x_\lambda})| = 1$ (i.e., $i \in \bar{I}$) and $p_{x_\lambda} = 0$ (i.e., $i, j \in \bar{I}$).

**2. The conditions of an abundant Rees matrix semigroup having an adequate transversal**

In this section the semigroup $S$ and $\Gamma$-semigroup $T$, the sets $I, \Lambda, \Gamma = \{p_{x_\lambda}, (i, \lambda) \in I \times \Lambda\}$, the matrix $P = (p_{x_\lambda})_{x \times \lambda}$, and the Rees matrix semigroup $S_\mu$ are as required in Section 1. We continue to discuss the relations between a $\bar{\Gamma}$-adequate transversal of $T$ and adequate transversals of $S_\mu$ and $S$. We begin by defining the following concept.

**Definition 2.1.** Let $T^0$ be a $\bar{\Gamma}$-adequate subsemigroup of $T$, where $\bar{\Gamma} \subseteq \Gamma$. By Theorem 1.10, we may denote the subset $\bar{\Gamma}$ of $\Gamma$ by

$$\bar{\Gamma} = \{p_{x_\lambda} \in \Gamma \mid (i, \lambda) \in \bar{I} \times \bar{I}, |V(p_{x_\lambda})| = 1 \text{ for } i \in \bar{I}, p_{x_\lambda} = 0 \text{ if } \lambda \neq i, i, \lambda \in \bar{I}\}.$$ 

We denote an element of $T^0$ by $x^0$. If for any $x \in T$ and $(i, \lambda) \in I \times \Lambda$ there are a unique element $x^0 \in T^0$ and two elements $a \in CV(p_{x_\lambda})$ and $b \in CV(p_{x_\lambda})$ for a unique $(i, \lambda) \in I \times \bar{I}$, such that $x$ can be uniquely written in the form

$$x = b_{x_\lambda} x^0 p_{x_\lambda} a,$$

where $b_{x_\lambda} \in x^0 \bar{\mathcal{R}}_{\bar{\Gamma}} x^0 a$. Then $T^0$ is called a $\bar{\Gamma}$-adequate transversal of $T$.

For this concept we have:

**Lemma 2.2.** (1) $b_{x_\lambda} x^0 \bar{\mathcal{R}}_{\bar{\Gamma}} x^0 a, 0^0 = 0$, otherwise the element $x^0$ is uniquely determined by $x$ and $(i, \lambda)$.

(2) The elements $a$ and $b$ are uniquely determined by $x$ and $(i, \lambda)$. We denote them by $a_x$ and $b_x$.

**Proof.** (1) By Lemma 1.3 and Definition 2.1 the results may be directly obtained.

(2) Let $0 \neq x \in T$, suppose that there are $b_1, b_2 \in CV(p_{x_\lambda})$ such that $b_k \bar{\mathcal{L}}_{\bar{\Gamma}} x^0$ $(k = 1, 2)$ and satisfy all conditions in Definition 2.1. Then $p_{x_\lambda} b_1 = p_{x_\lambda} b_2 p_{x_\lambda} b_1$ and $p_{x_\lambda} b_2 = p_{x_\lambda} b_1 p_{x_\lambda} b_2$ imply that $p_{x_\lambda} b_1 p_{x_\lambda} b_1$ and $p_{x_\lambda} b_2 p_{x_\lambda} b_2$. Since for any $(p_{x_\lambda} b_1) \in V(p_{x_\lambda} b_1)$ we have $p_{x_\lambda} b_1 \mathcal{L}(p_{x_\lambda} b_1)$. So we obtain that $p_{x_\lambda} b_1 \mathcal{L} p_{x_\lambda} b_2$. This implies $b_1 b_1 = b_2 b_2$ and so $b_1 = b_1 p_{x_\lambda} b_1 = b_1 p_{x_\lambda} b_2 = b_2 p_{x_\lambda} b_1 = b_2 b_1 b_2$. The other case is dual.

By Lemma 2.2, we may denote the sets by
Let (1) By Theorem 1.9, we know that $a$ where of course $(T, T)$ is multiplicative.

Therefore (2), if $T^0$ is a $\Gamma$-multiplicative, then $S^0_\mu$ is multiplicative.

Proof. (1) By Theorem 1.9, we know that $S_\mu$ is abundant. Let $(x^0)i_\mu, (y^0)\lambda_\mu \in S^0_\mu$ then

$$(x^0)i_\mu(y^0)\lambda_\mu = \begin{cases} (x^0)p_{i_\mu}y^0 \mid \bar{i} = \bar{\lambda} \\ 0 \mid \bar{i} \neq \bar{\lambda} \end{cases}.$$ 

Since $T^0$ is a $\bar{\Gamma}$-semigroup with zero, so $(x^0)i_\mu(y^0)\lambda_\mu \in S^0_\mu$. Since $L_i^* \cap S^0_\mu \neq \phi$ and $R_i^* \cap S^0_\mu \neq \phi$. Thus $S^0_\mu$ is a $\ast$-subsemigroup. Clearly, Theorem 1.10 demonstrates that $S^0_\mu$ is adequate. Since $T^0$ is a $\bar{\Gamma}$-adequate transversal of $T$, so for any $x \in T$ and $(i, \lambda) \in I \times \Lambda$, $x$ can be uniquely written in the form $x = b_xp_{i_\mu}x^0p_{i_\mu}a_x$, where $a_x \in A_\Lambda, b_x \in B_I, (i, \bar{i}) \in \bar{I} \times \bar{I}$ and $b_xL_i^*x^0R_i^*a_x$. Thus for any $(x)\lambda_\mu \in S_\mu, (x)\lambda_\mu$ can be also uniquely written in the form

$$(x)\lambda_\mu = (bx)p_{i_\mu}x^0p_{i_\mu}a_x \lambda_\mu = (bx)i_\mu(x^0)i_\mu(a_x)\lambda_\mu,$$

where of course $(b_x)i_\mu L_i^*(x^0)i_\mu R_i^*(a_x)i_\mu \lambda_\mu$. Since $a_x \in A_\Lambda$ and $b_x \in B_I$, so $(a_x)\lambda_\mu$ and $(b_x)i_\mu$ are uniquely determined by $(x)\lambda_\mu$ and $(a_x)\lambda_\mu, (b_x)i_\mu \in E(S)$. Thus $S^0_\mu$ is indeed an adequate transversal of $S_\mu$.

(2) Since $T^0$ is $\Gamma$-multiplicative, for any $a_x \in A_\Lambda, b_y \in B_I$ and $p_{i_\mu} \in \Gamma, a_xp_{i_\mu}b_y \in V(p_{i_\mu})$ for a unique $(i, \bar{i}) \in \bar{I} \times \bar{I}$ (or $a_xp_{i_\mu}b_y = 0$, so for any $f(x)\lambda_\mu = (a_x)\lambda_\mu$ and $e(y)\mu_\mu = (b_y)\lambda_\mu$

$$f(x)\lambda_\mu e(y)\mu_\mu = (a_x)\lambda_\mu(b_y)\lambda_\mu = (a_xp_{i_\mu}b_y)\lambda_\mu = \begin{cases} (a_xp_{i_\mu}b_y)\mu_\mu \mid \bar{i} = \bar{\lambda} \\ 0 \mid \bar{i} \neq \bar{\lambda} \end{cases}$$

where $a_xp_{i_\mu}b_y \in V(p_{i_\mu}) \subseteq T^0$ for $\bar{i} = \bar{\lambda}$, and so $f(x)\lambda_\mu e(y)\mu_\mu \in E(S_\mu^0)$. Thus $S^0_\mu$ is multiplicative.
Lemma 2.5. If $S^0_\mu$ is an adequate transversal of $S_\mu$, then

(1) $T$ contains a $\bar{\Gamma}$-adequate transversal $T^0$ for some $\bar{\Gamma} \subseteq \Gamma$.

(2) If $S^0_\mu$ is multiplicative, then $T^0$ is $\Gamma$-multiplicative.

Proof. (1) By Theorem 1.9 we know that $T$ is $\Gamma$-abundant. Let $(x)_{i\lambda} \in S^0_\mu$ then the element $x$ is belongs to some subset $T^0$ of $T$. By Theorem 1.10 we can always denote the set by

$$S^0_{\mu} = \{(x^0)_{i\lambda} | x^0 \in T^0, (\bar{i}, \bar{\lambda}) \in \bar{I} \times \bar{I})\},$$

where $(x^0)_{i\lambda}$ means that $(x^0)_{i\lambda} \in S^0_\mu$ is uniquely determined by $(x)_{i\lambda}$ and $\bar{I} \times \bar{I}$ is some subset of $I \times \Lambda$. Since $S^0_\mu$ is an adequate subsemigroup of $S_\mu$, by Theorem 1.10 $T^0$ is $\bar{\Gamma}$-adequate subsemigroup of $T$, so we may suppose that $\bar{\Gamma} = \{p_{i\lambda} | (\bar{i}, \bar{\lambda}) \in \bar{I} \times \bar{I})\} \subseteq \Gamma$ satisfies $p_{i\lambda} = \begin{cases} p_i & \lambda = \bar{i} \\ 0 & \lambda \neq \bar{i} \end{cases}$ and $|V(p_{i\lambda})| = 1$ for any $\lambda \in \bar{I}$. This time we may write the set

$$E(S^0_{\mu}) = \{(a^0)_{i\lambda} | a^0 \in V(p_{i\lambda}), \lambda \in \bar{I} \text{ or } a^0 = 0\}.$$

Since $S^0_{\mu}$ is an adequate transversal of $S_\mu$, for any $(x)_{i\lambda} \in S_\mu$, $(x)_{i\lambda}$ can be uniquely written in the form

$$(x)_{i\lambda} = (b)_{i\bar{\lambda}}(x^0)_{i\lambda}(a)_{i\lambda} = (bp_{i\lambda}x^0p_{i\lambda}a)_{i\lambda},$$

where $(b)_{i\bar{\lambda}}$ denote $e((x)_{i\lambda})$ and $(a)_{i\lambda}$ denote $f((x)_{i\lambda})$. This implies that $x = bp_{i\lambda}x^0p_{i\lambda}a$ by the unique property of $e((x)_{i\lambda})$ and $f((x)_{i\lambda})$, we know that $b = b_x \in B_I$ and $a = a_x \in A_\Lambda$, and so that for any $x \in T$, $x$ can be uniquely written in the form $x = b_xp_{i\lambda}x^0p_{i\lambda}a_x$, where $x^0 \in T^0$, $b_x \in B_I$ and $a_x \in A_\Lambda$ are uniquely determined by $x$ and $(i, \lambda)$. Thus $T^0$ is indeed a $\bar{\Gamma}$-adequate transversal of $T$.

(2) If $S^0_{\mu}$ is multiplicative in part (1), then for any $f((x)_{i\lambda}) = (a_x)_{i\lambda}$ and $e((y)_{i\mu}) = (b_y)_{i\lambda}$, where $a_x \in A_\Lambda, b_y \in B_I$,

$$f((x)_{i\lambda})e((y)_{i\mu}) = (a_x)_{i\lambda}(b_y)_{i\lambda} = (a_xp_{i\lambda}b_y)_{i\lambda} = \begin{cases} (a_xp_{i\lambda}b_y)_{i\lambda} & \text{if } \bar{i} = \bar{\lambda} \\ 0 & \text{if } \bar{i} \neq \bar{\lambda} \end{cases} \in E(S^0_{\mu}).$$

So $a_xp_{i\lambda}b_y \in V(p_{i\lambda}) \subseteq T^0$ (or $a_xp_{i\lambda}b_y = 0$) for any $a_x \in A_\Lambda, b_y \in B_I$ and $p_{i\lambda} \in \Gamma$, that is, $T^0$ is a $\Gamma$-multiplicative $\bar{\Gamma}$-adequate transversal of $T$. \qed

According as the results of Lemmas 2.4 and 2.5 we obtain:

Theorem 2.6. (1) The $\Gamma$-semigroup $T$ contains a $\bar{\Gamma}$-adequate transversal $T^0$ as in Lemma 2.4 if and only if $S_\mu$ contains an adequate transversal $S^0_{\mu} = \{(x^0)_{i\lambda} | x^0 \in T^0, (\bar{i}, \bar{\lambda}) \in \bar{I} \times \bar{I})\}$ with the idempotents semilattice $E(S^0_{\mu}) = \{(a^0)_{i\lambda} | a^0 \in CV(p_{i\lambda}), (\bar{i}, \bar{\lambda}) \in \bar{I} \times \bar{I} \text{ or } a^0 = 0\}$ for some subset $\bar{I} \times \bar{I} \subseteq I \times \Lambda$.

(2) $T^0$ is $\Gamma$-multiplicative if and only if $S^0_{\mu}$ is multiplicative where $T^0$ and $S^0_{\mu}$ as that in part (1).

For the relations between the semigroup $S$ and the Rees matrix semigroup $S_\mu = \mu(T; I, \Lambda, P)$, we have:
Theorem 2.7. If the mapping \( \phi \) as in Lemma 1.5 is a semigroup isomorphism from \( S_\mu \) to \( T \), then the following arguments hold.

1. \( S \) is abundant if and only if and only if \( T \) is \( \Gamma \)-abundant if and only if \( S_\mu \) is abundant.

2. \( S \) contains an adequate transversal \( S^0 \) if and only if \( T \) contains a \( \bar{\Gamma} \)-adequate transversal \( T^0 \) where \( \bar{\Gamma} \) as above required if and only if \( S_\mu \) contains an adequate transversal \( S^0_\mu = \{(x^0)_{\bar{i}} \mid x^0 \in T^0, i \in \bar{1}\} \).

3. \( S^0 \) is multiplicative if and only if \( T^0 \) is \( \Gamma \)-multiplicative if and only if \( S^0_\mu \) is multiplicative where \( S^0, T^0, S^0_\mu \) are as that in part (2).

**Proof.** (1) Since \( \phi \) is an isomorphism from \( S_\mu \) to \( T \), by Lemma 1.5 and Theorem 1.9 we directly obtain part (1).

(2) Let \( S^0 \) be an adequate transversal of \( S \). Since \( \phi \) is an isomorphism from \( S_\mu \) to \( S \) and for any \( (t)_\lambda \in S_\mu \), where \( t \in T \), \( (t)_\lambda \phi = r_t q_\lambda = x \). Let \( x = e_x x^0 f_x \) so for \( x^0 \in S^0 \), \( e_x, f_x \in E(S) \). There are a unique element \( t^0 \in T^0 \) which is a subset of \( T \) and two elements \( b_t, a_t \in T \). Such that for some \( (i, v), (j, \mu), (k, \lambda) \in I \times \Lambda \)

\[
(b_t)_{iv} \phi = e_x, \quad (t^0)_{j\mu} \phi = x^0, \quad (a_t)_{k\lambda} \phi = f_x.
\]

Since \( x = e_x x^0 f_x \) where \( e_x L_e x^0 \bar{\Phi} f_x \) so that

\[
x = e_x x^0 f_x = r_t b_t q_v \cdot r_j t^0 q_\mu \cdot r_k a_t q_\lambda = r_t b_t p_{ij} t^0 p_{k\lambda} a_t q_\lambda.
\]

Since \( T \) is a \( \Gamma \) semigroup, so \( t = b_t p_{ij} t^0 p_{k\lambda} a_t \in T \) and such that \( (t)_\lambda \phi = \lambda = e_x x^0 f_x = (b_t)_{iv} \phi \cdot (t^0)_{j\mu} \phi \cdot (a_t)_{k\lambda} \phi = [(b_t)_{iv} (t^0)_{j\mu} (a_t)_{k\lambda}] \phi \). This implies that

\[
(t)_\lambda = (b_t)_{iv} (t^0)_{j\mu} (a_t)_{k\lambda}.
\]

Further, by the fact that \( S^0 \) is an adequate transversal of \( S \), like the proof of Lemma 2.5. We may prove that \( S_\mu \) has an adequate transversal \( S^0_\mu = \{(t^0)_{\bar{i}} \mid t^0 \in T^0, (i, \bar{i}) \in \bar{1} \times \bar{1} \} \), where \( T^0 \) is an \( \bar{\Gamma} \)-adequate transversal of \( T \) and \( \bar{\Gamma} = \{p_{ij} \mid (i, \lambda) \in \bar{1} \times \bar{1} \} \) as required in (2.1). Here we omit this proof.

Conversely, if \( T^0 \) is a \( \bar{\Gamma} \)-adequate transversal of \( T \) by Theorem 2.6, then \( S^0_\mu \) as above described is an adequate transversal of \( S_\mu \). Since \( \phi \) is a semigroup isomorphism from \( S_\mu \) to \( S \), so the set

\[
S^0 = \{r_t t^0 q_i = x^0 \in S \mid \forall t^0 \in T^0, (i, \bar{i}) \in \bar{1} \times \bar{1} \}
\]

is necessarily an adequate transversal of \( S \). In fact, let \( (t)_\lambda \phi = r_t q_\lambda = x \in S \), by Theorem 2.6

\[
(t)_\lambda = (b_t)_{iv} (t^0)_{j\mu} (a_t)_{k\lambda} = (b_t p_{ij} t^0 p_{k\lambda} a_t)_{i\lambda}
\]

implies that \( t = b_t p_{ij} t^0 p_{k\lambda} a_t \) and

\[
x = r_t q_\lambda = r_t b_t p_{ij} t^0 p_{k\lambda} a_t = r_t b_t (r_t t^0 q_i) (r_t a_t q_\lambda) = e_x x^0 f_x.
\]

Since \( b_t \in B_I, a_t \in A_\Lambda \), so
Consider the monoid \( S \) of Definition 2.1 according to Definition 1.4, here \( P \) is a matrix.

Example 2.8. Consider the monoid \( T \) with zero \( a \) which is not abundant under multiplication

\[
\begin{array}{cccc}
T & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & b & b \\
c & a & b & c & d \\
d & a & b & d & c \\
\end{array}
\]

According to the structure express \( (T, I, \Lambda) \), here let \( I = \{1, 2\} = \Lambda \), where \( R_1^3 = \{c, d\}, R_2^3 = \{a\}, R_1^2 = \{b\} \) and \( L_1^3 = \{c, d\}, L_1^2 = \{a\}, L_2^2 = \{b\} \). Therefore we have \( H_{11} = \{c, d\} \) is a non-zero group with identity \( c \). \( H_{22}^3 \) contains no idempotent, \( H_{22}^2 = \phi (i \neq \lambda, 1, \lambda \in I = \{1, 2\}) \). According \( (i, \lambda) \) positions, let \( \Gamma = \{p_{\lambda}\} \mid (i, \lambda) \in I \times \Lambda \) where \( p_{\lambda} \) is as in the above matrix \( P \). Clearly, the matrix \( P \) is abundant. By Theorem 1.9 we know that \( T \) is \( \Gamma \)-abundant and so that \( T_n = \mu(T; I, \Lambda, P) \) is also abundant. Computing by Lemma 1.3 and Corollary 1.4, here \( L^1 = L_1^2 = L_2^2 = \phi \). Similarly, \( R^1 = R_1^3 = R_2^3 = R_2^2, \) and \( V(p_{11}) = \{c\}, V(p_{22}) = V(p_{33}) = V(p_{21}) = \{d\} \). We will see that \( \Gamma \)-abundant semigroup \( T \) has a \( \Gamma \)-adequate transversal. Put \( \Gamma = \{p_{11}\} \subseteq \Gamma \) and \( T^0 = T \), then \( \Gamma \) satisfies the conditions in Theorem 2.6, so by Theorem 2.6, \( T^0 \) is a \( \Gamma \)-adequate transversal of \( T \). In fact, for any \( x \in T \) and \( (i, \lambda) \in I \times \Lambda \), under the sense of Definition 2.1 according \( (i, \lambda) \) position \( x \) can be uniquely written in the form

\[
\begin{aligned}
a = a_{p_{11}} & a_{p_{11}} a \\
b = c_{p_{11}} & b_{p_{11}} c \\
c = d_{p_{11}} & d_{p_{11}} b \\
d = c_{p_{11}} & d_{p_{11}} c \\
\end{aligned}
\]

where \( a \in A \cap B_1, a^0 = a \)

\[
\begin{aligned}
c = c_{p_{11}} & c_{p_{11}} c \\
d = d_{p_{11}} & d_{p_{11}} d \\
\end{aligned}
\]

where \( c \in V(p_{11}), d \in V(p_{12}) \).

Similarly, \( b \) and \( a \) follow similarly.

\[
\begin{aligned}
a = r_i b_i q_i r_i b_i q_i = r_i b_i p_i b_i q_i = r_i b_i q_i = e_x.
\end{aligned}
\]
By Theorem 2.6 we obtain an adequate transversal $T^0_\mu$ of $T_\mu$ as follows
\[ T^0_\mu = \{(x\alpha)_{11} \mid x\alpha \in T^0\} \]
For any $(x)_{\alpha} \in T_\mu$, $(x)_{\alpha}$ can be uniquely written in the form $(x)_{\alpha} = (b_\beta)_{11}$ $(x^0)_{11} (a_\gamma)_{11}$. Here we omit these expressions. But since there are $a_\alpha = b_\gamma = d$ such that $a_\alpha b_\gamma b_\gamma = c \cdot c \cdot d = d \notin V(p_{i_1})$, so $T^0$ is not $\Gamma$-multiplicative by Theorem 2.6 and we know that $T^0_\mu$ is also not multiplicative.

Clearly, since $|T| = 4, |T_\mu| = 13$, so if there is a mapping $\phi$ as that in Lemma 1.5 then $\phi$ is impossible an isomorphism from $T_\mu$ to $T$. From this example we may see that for a semigroup $T$ being not abundant, there possible is a set $\Gamma$ such that $T$ becomes a $\Gamma$-adequate semigroup and such that the Rees matrix semigroup $T_\mu = \mu(T, I, \Lambda, \Gamma)$ over $T$ may be an abundant semigroup and it may contain an adequate transversal $T^0_\mu$.

3. Rees matrix representations of an abundant semigroup with an adequate transversal

In this section $S$ is always an abundant semigroup with an adequate transversal $S^0$. Our aim in this section is to give some Rees matrix representations of $S$. We begin by blocked Rees matrix semigroups to give a representation of $S$.

**Lemma 3.1.** Let $0 \neq e \in E(S)$ and $a_\alpha^* e \ (a_\alpha^* e)$. Then $a \in \alpha e$. 

**Proof.** By $a_\alpha^* e$ implies $a_\alpha^* e$ and so $a = a_\alpha^* e$. Dually, if $a_\alpha^* e$, then $a \in \alpha e$.

**Lemma 3.2.** Let $0 \neq x, 0 \neq y \in S$. If $xy \neq 0$, then $xy \in R^*_x \cap L^*_y$.

**Proof.** Let $e, f \in E(S)$ with $e_\alpha^* x, f_\beta^* y$, then $x \in e S, y \in S f$ so that $xy \in e S \cap S f$ and by Lemma 3.1 $xy \in R^*_x \cap R^*_y = R^*_x \cap R^*_y$.

**Lemma 3.3.** Let $0 \neq x, 0 \neq y \in S$ and $e, f \in E(S)$ with $x_\alpha^* e, y_\beta^* f$. Then $xy = 0$ if and only if $fe = 0$.

**Proof.** If $yx = 0$, then $yx = 0$, so that $fx = f0$ and then $fx = 0 x$, which gives $fe = 0 e = 0$. If $fe = 0$, then $yx = yf ex = 0$ by $y = yf$ and $x = ex$.

By the process of shaping a semigroup into the form of blocked Rees matrix semigroup in [6], we may obtain a blocked Rees matrix representation of $S$ when the condition (M) in [6] holds. By Lemma 3.1 to 3.3 we may show this point. Now suppose that we have shaped $S$ into a blocked set denoted by $S_\alpha = \mu(M_\alpha, \beta; I, \Lambda, \Gamma')$ where $\Gamma'$ is the set of non-zero $\mathcal{D}$-classes of $S$ which contain idempotents and each $M_\alpha \beta$ is a torsion-free $(T_\alpha, T_\beta)$-biset system where $T_\alpha, T_\beta$ are two cancellative monoid with an outer zero denoted by 0.

Let $\alpha, \beta, \gamma \in \Gamma'$. Suppose that $M_\alpha \beta, M_\beta \gamma$ are both non-empty. If $a \in M_\alpha \beta, b \in M_\beta \gamma$, then $a_\alpha^* e_\beta^* b$ and it follows from Lemma 3.3 that $ab \neq 0$. 

\[ = dp_{i_1} c, \quad d \in V(p_{i_1}), \quad c \in V(P_{i_1}), \quad d \in B_1, \quad c \in A_\Lambda, \quad d^0 = c \]
\[ = dp_{i_1} c, \quad d \in V(p_{i_1}), \quad c \in V(P_{i_1}), \quad d \in B_1, \quad c \in A_\Lambda, \quad d^0 = d. \]
Since $ab \in aS \cap Sb \subseteq e_\alpha S \cap Se_\gamma$, we have $e_\alpha R^* ab \subseteq \mathbb{L}^* e_\gamma$ by Lemma 3.1. Thus $ab \in R^*_{\alpha(\alpha)} \cap L^*_\gamma = M_{\alpha\gamma}$ and $M_{\alpha\gamma} \neq \phi$. We now define $\varphi_{\alpha\beta\gamma} : M_{\alpha\beta} \otimes M_{\beta\gamma} \rightarrow M_{\alpha\gamma}$ by $(a \otimes b)\varphi_{\alpha\beta\gamma} = ab$ (see [5, ch. 7]). It is easy to see that this is a well-defined $(T_\alpha, T_\beta)$-homomorphism (see [6]) and that the condition (M) in [6] is satisfied.

For each $\alpha \in \Gamma'$ we define the sets as

$$I_\alpha = \{ i \in I : D_\alpha \cap R^*_\alpha \neq \phi \}; \quad \Lambda_\alpha = \{ \mu \in \Lambda : D_\alpha \cap L^*_\mu \neq \phi \}.$$ 

Since $S$ is an abundant semigroup, so by [6] we know that $I = \bigcup I_\alpha$ and $\Lambda = \bigcup \Lambda_\alpha$ are disjoint union. Like [6], we define $P$ as the $\Lambda \times I$ matrix $(p_{\lambda i})$ where for $(\lambda, j) \in \Lambda_\alpha \times I_\beta$, $p_{\lambda i} = q^\alpha_{ij} r^\beta_j$, $q^\alpha_{ij} \in H^*_{\iota(\alpha), \lambda}$ and $r^\beta_j \in H^*_{\iota(\beta)}$ where $q^\alpha_{ij}$, $r^\beta_j$ are regular elements and $q^\alpha_{ij} r^\beta_j = r^\beta_j q^\alpha_{ij} = e_\alpha$ (e. $e_\alpha$ is the identity of $M_{\alpha\alpha}$). So that $q^\alpha_{ij} \in R^*_{\iota(\alpha)} \cap L^*_\lambda = \mathbb{L}^*_{\lambda(\beta)}$ and hence either $q^\alpha_{ij} r^\beta_j = 0$ or $q^\alpha_{ij} r^\beta_j \in R^*_{\iota(\alpha)} \cap L^*_\lambda = \mathbb{L}^*_{\lambda(\beta)}$. Thus any non-zero entry in the $(\alpha, \beta)$-block of $P$ is a member of $M_{\alpha\beta}$. By [6] we know that $S_\mu = \mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$ (see [6]) is a blocked Rees matrix semigroup.

Note, here $S_\mu = \mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$ is not necessarily a PA blocked Rees matrix semigroup, that is, the abundant semigroup $S_\mu$ need not to satisfy the conditions (U) and (R) in [6], so $S_\mu$ is not necessarily a primitive abundant semigroup.

We next show that the bijection $\phi : S_\mu \rightarrow S$ given by

$$(3.1) \quad 0\phi = 0 \text{ and } (a)_{i\lambda} \phi = r^\alpha_i a q^\alpha_{ij} ((i, \lambda) \in I_\alpha \times \Lambda_\beta, a \in M_{\alpha\beta})$$

is an isomorphism from $S_\mu$ to $S$. Clearly, $S_\mu \setminus \{0\} = \bigcup \{H^*_{\lambda(\beta)} \mid (i, \lambda) \in I \times \Lambda\}$ and $S$ is disjoint unions, so it is straightforward to show that $\phi$ is a bijection and it is also an isomorphism. Thus we have already proved that $S_\mu$ is a blocked Rees matrix representation of $S$. It is such that we may obtain the following representation theorem. It is a generalization of [6, Theorem 3.8].

**Theorem 3.4.** Let $S$ be an abundant semigroup with an adequate transversal $S^0$ then $S$ has a blocked Rees matrix representation $S_\mu = \mu^0(M_{\alpha\beta}; I, \Lambda, \Gamma'; P)$ with an adequate transversal $S^0_\mu$ isomorphic to $S^0$. Furthermore, $S^0$ is multiplicative if and only if $S^0_\mu$ is multiplicative.

**Proof.** We first show that $S_\mu$ has adequate transversal $S^0_\mu$ isomorphic to $S^0$. For any $x^0 \in S^0$, since $\phi$ is an isomorphism from $S_\mu$ to $S$, there is a unique element denoted by $(t_{x^0})_{i\lambda} \in S_\mu$ such that $(t_{x^0})_{i\lambda} \phi = r^\alpha_i t_{x^0} q^\alpha_{ij} = x^0$ for $(i, \lambda) \in I_\alpha \times \Lambda_\beta$ and $t_{x^0} \in M_{\alpha\beta}$ for some $\alpha, \beta \in \Gamma'$. Similar to the proof of Theorem 2.7, there is a subset $I \times \bar{I}$ of $I \times \Lambda$ such that

$$S^0_\mu = \{(t_{x^0})_{i\lambda} \mid \forall x^0 \in S^0 \quad (t_{x^0})_{i\lambda} \phi = x^0, (i, \lambda) \in I \times \bar{I}\}.$$ 

Since $\phi$ is an isomorphism, $S^0_\mu$ is an adequate *-subsemigroup of $S_\mu$, and for any $x \in S$ there are a unique element $x^0 \in S^0$ and two idempotents $e_x, f_x$ in
\[ E(S) \] such that \( x = e_x x^0 f_x \), where \( e_x L^* x^0 R^* f_x \). We denote \( x \varphi^{-1}, e_x \varphi^{-1} \) and \( f_x \varphi^{-1} \) by \((t_x)_{1, \lambda}, e(t_x)_{1, \lambda} \) and \( f(t_x)_{1, \lambda} \) respectively, then

\[
e_{(t_x)_{1, \lambda}} f_{(t_x)_{1, \lambda}} \in E(S_\mu) \quad \text{and} \quad e_{(t_x)_{1, \lambda}} R^* (t_x)_{1, \lambda} L^* f_{(t_x)_{1, \lambda}}.
\]

The element \((t_x)_{1, \lambda} \) of \( S_\mu \) can be uniquely written in the form

\[
(t_x)_{1, \lambda} = e_{(t_x)_{1, \lambda}} (t_x)_{0, \lambda}^0 f_{(t_x)_{1, \lambda}} = e_{(t_x)_{1, \lambda}} (t_x)_{0, \lambda}^0 f_{(t_x)_{1, \lambda}}.
\]

where \( e_{(t_x)_{1, \lambda}} L^* (t_x)_{0, \lambda}^0 R^* f_{(t_x)_{1, \lambda}} \). Like Theorem 2.7 we may write \( e_{(t_x)_{1, \lambda}} = (b_{t_x})_{1, \lambda}, f_{(t_x)_{1, \lambda}} = (a_{t_x})_{1, \lambda} \), then \((t_x)_{1, \lambda} = (b_{t_x})_{1, \lambda} (a_{t_x})_{1, \lambda} \). Since \((t_x)_{1, \lambda} \) is any element of \( S_\mu \), we know that \( S_\mu^0 \) is an adequate transversal of \( S_\mu \). We next show that \( S^0 \) is multiplicative if and only if \( S_\mu^0 \) is also. We denote \( y \varphi^{-1} = (b_y)_{1, \mu} \) for \( y \in S \) and \( e_y \varphi^{-1} = e_{(b_y)_{1, \mu}} \), then by \( \varphi \) being an isomorphism, \( f_x e_y \in E(S^0 \), if and only if \( f_{(a_x)_{1, \lambda}} e_{(b_y)_{1, \mu}} \in E(S^0_\mu) \). It is as required.

The blocked Rees matrix semigroup is over \((T_\alpha, T_\beta)\)-bisystem \( M_{\alpha \beta} (\alpha, \beta \in \Gamma) \) where \( T_\alpha = M_{\alpha \alpha}, T_\beta = M_{\beta \beta} \) are two cancellative monoid with an outer zero. Using \( M_{\alpha \beta} = H^*_{\{(\alpha), \lambda(\beta) \}} (\alpha, \beta \in \Gamma') \), we define the set \( T = \bigcup \{M_{\alpha \beta} \mid \alpha, \beta \in \Gamma' \} \). Thus we may think that the blocked Rees matrix semigroup over \( T \). Using here expression that is \( S_\mu = \mu(T, I, \Lambda, P) \), where \( I \times \Lambda = \bigcup \{I_\alpha \times I_\beta \mid \alpha, \beta \in \Gamma' \} \). But \( T \) is not necessarily a semigroup. When \( p_{\lambda j} = q^\alpha j^\beta, \) where \( q^\alpha \in H^*_{\{(\alpha), \lambda \}}, \) and \( r^\beta \in H^*_{\{\beta \}}, \) \( S_\mu \) is isomorphic to \( S \).

It is clear that \( S \) and \( T \) are \( \Gamma \)-semigroups, where

\[
\Gamma = \{p_{\lambda j}; \ p_{\lambda j} = q^\alpha j^\beta \in M_{\alpha \beta}, \ q^\alpha \in H^*_{\{(\alpha), \lambda \}}, r^\beta \in H^*_{\{\beta \}}, \alpha, \beta \in \Gamma', \ (j, \lambda) \in I_\beta \times \Lambda_\alpha \}.
\]

under \( \Gamma \)-operation “ \( \circ \) ” as that for any \( x, y \in S, \ p_{\lambda i} \in \Gamma \)

\[
x \circ y = x p_{\lambda i} y = \begin{cases} x p_{\lambda i} y & \text{if} \ x \in M_{\alpha \beta}, \ y \in M_{\gamma \delta}, \ p_{\lambda i} \in M_{\beta \gamma}, \\ 0 & \text{otherwise.} \end{cases}
\]

So we say that \( S_\mu = \mu(T, I, \Lambda, P) \) is a Rees matrix semigroup over \( \Gamma \)-semigroup \( T \). Like Theorem 2.7 we can prove the following representation theorem.

**Theorem 3.5.** Let \( S \) be an abundant semigroup with an adequate transversal \( S^0 \). Then \( S \) has a Rees matrix representation \( S_\mu = \mu(T; I, \Lambda, P) \) over \( \Gamma \)-semigroup \( T \) and the following argument hold.

1. \( S_\mu \) contains an adequate transversal \( S^0_\mu \) may be expressed by

\[
S^0_\mu = \{(x^0)_{1, \lambda} | x^0 \in T^0, (\bar{i}, \bar{j}) \in \bar{I} \times \bar{I} \}
\]

and \( S^0_\mu \) is isomorphic to \( S^0 \) where \( T^0 \) is a \( \bar{\Gamma} \)-adequate transversal of \( T \) and \( \bar{\Gamma} \) as \( (2.1) \).

2. \( S^0 \) is multiplicative if and only if \( S^0_\mu \) is multiplicative.
Proof. We have proved that $S_\mu = \mu(T; I, \Lambda, P)$ is a Rees matrix representation of $S$. By the proof of Lemma 2.4, we can similarly obtain that $S_\mu$ contains an adequate transversal denoted by $S_0^\ast$, and $S_0^\ast$ is isomorphic to $S^0$. The following we show that $S_0^\ast$ may be expressed as required form. In fact, let $T^0 = \{ t^0 \in T \mid \forall x^0 \in S^0, \exists t^0 \in T, x^0 = r_{t^0}^1 t^0 q_{t^0}^\beta \}$. Since $S^0$ is a subsemigroup of $S$, let $x^0 = r_{t^0}^1 t^0 q_{t^0}^\beta$, $y^0 = r_{t^0}^0 t^0 q_{t^0}^\mu \in S^0$, where $(i, \bar{\lambda}), (j, \bar{\mu}) \in \bar{I} \times \bar{A} \subseteq I \times \Lambda$, then $x^0 y^0 = r_{t^0}^1 t^0 p_{3j}^\gamma q_{3j}^\nu$ implies that $t^0_0 p_{3j}^\gamma s_0^0 \in T^0$. Let $\bar{\Phi} = \{ p_{i, \bar{\lambda}} \in \Gamma \mid (i, \bar{\lambda}) \in \bar{I} \times \bar{A} \}$, since $S^0$ is a subsemigroup of $S$, so $T^0$ is necessarily a $\bar{\Phi}$-subsemigroup of $T$. Since $S^0$ is adequate by Theorem 1.10, $T^0$ is $\bar{\Phi}$-adequate and $I = \bar{A}$. By Theorem 2.7, $T^0$ is a $\bar{\Phi}$-adequate transversal of $T$ and $S_0^\ast$ may be denoted by

$$S_0^\ast = \{ (t^0)_{i, \bar{\lambda}} \mid t^0 \in T^0, (i, \bar{\lambda}) \in \bar{I} \times \bar{I} \}$$

which is an adequate transversal of $S_\mu$. Since $\phi|_{S_0^\ast}$ as

$$\forall (t^0)_{i, \bar{\lambda}} \in S_0^\ast \quad (t^0)_{i, \bar{\lambda}} \phi = r_{i, \bar{\lambda}}^0 t^0 q_{i, \bar{\lambda}}^{\mu} = x^0 \in S^0$$

is an isomorphism from $S_0^\ast$ to $S^0$. We complete the proof of part (1). By Theorem 2.7 we know that part (2) holds.

Our final aim in this section is that given a $\Gamma$-Rees matrix representation for semigroup $S$. This means that taking a some set $\Gamma_1$ need not belong to $S$ and a $\Lambda \times I$ matrix $\rho = (\rho_{\lambda, i})_{\lambda \times I}$ over $\Gamma_1$ which is called a $\Gamma$-Rees matrix, we can obtain a $\Gamma$-semigroup $T$ for some subset $T$ of $S$ and the $\Gamma_1$-semigroup $S$. Then taking some set $I_2$ such that we can obtain a $\Gamma_2$-Rees matrix semigroup $T_\mu$ denoted by $T_\mu = \mu(T; I, \Lambda, \rho)$ over $T$. Then we will prove that $\Gamma_2$-semigroup $T_\mu$ is $\Gamma$-isomorphic to $\Gamma_1$-semigroup $S$. Since the set $T \subseteq S$ and $\rho$ is over $\Gamma_1$, so we call that $T_\mu$ is a $\Gamma$-Rees matrix representation of $S$.

Firstly, we recall the concept of $\Gamma$-semigroup isomorphism.

**Definition 3.6** ([11]). Let $T_1$ be a $\Gamma_1$-semigroup and $T_2$ be a $\Gamma_2$-semigroup, a mapping pair denoted by $\phi = (\phi_1, \phi_2)$ from $(T_1, \Gamma_1)$ to $(T_2, \Gamma_2)$ as follows

$$\phi_1 : T_1 \rightarrow T_2, \quad \phi_2 : \Gamma_1 \rightarrow \Gamma_2$$

$$x_1 \mapsto x_2, \quad \gamma_1 \mapsto \gamma_2$$

If $\phi$ satisfies that for any $x_1, y_1 \in T_1, \gamma_1 \in \Gamma_1$

$$(x_1 \gamma_1 y_1) \phi = x_1 \phi_1 \cdot \gamma_1 \phi_2 \cdot y_1 \phi_1 \phi_2 = x_2 \gamma_2 y_2,$$

then $\phi$ is called a $((\Gamma_1, \Gamma_2)$ homomorphism from $T_1$ to $T_2$. If $\phi$ is a surjection (resp. injection), then $\phi$ is called a surjection (resp. injection) homomorphism. If $\phi$ is a bijection, then $\phi$ is called a $((\Gamma_1, \Gamma_2)$-isomorphism from $\Gamma_1$-semigroup $T_1$ to $\Gamma_2$-semigroup $T_2$.

Note here $\phi$ is bijective (surjective, injective) means that $\phi_1$ and $\phi_2$ are bijective (surjective, injective).

Now, we suppose that $S$ is an abundant semigroup with an adequate transversal $S^0$, then $S$ has the structure express $(S; I, \Lambda, \Gamma')$ as Section 1. We denote
Clearly, \( \phi \) is bijective. Since \( r_i^\alpha xq_\alpha^\beta \) on semigroup \( S \), by Theorem 3.5 we know that \( \phi_1 \) is a semigroup isomorphism from \( \mu(T; I, \Lambda, P) \) to \( S \). Thus we know that \( \phi \) is a bijection from \( \Gamma \)-semigroup \( T_\mu \) to \( \Gamma \)-semigroup \( S \). Let \( (x)_{i(\alpha, \lambda), (y)}_{j(\gamma)(\mu)} \in T_\mu \), \( \eta_{\alpha i} \in \Gamma_2 \) then

\[
[(x)_{i(\alpha, \lambda), (y)}_{j(\gamma)(\mu)} \phi] = \begin{cases} 
(xq_\alpha^\beta \rho_{j(\gamma)(\mu)} r_j^\gamma y)_\mu \phi & \text{if } (u, t) = (\lambda, j), \\
0 & \text{otherwise}
\end{cases}
\]

Finally, we define a mapping \( \phi = (\phi_1, \phi_2) \) from \( \Gamma \)-semigroup \( T_\mu \) to \( \Gamma \)-semigroup \( S \) as follows

\[
\phi_1 : T_\mu \rightarrow S, \quad \phi_2 : \Gamma_2 \rightarrow \Gamma_1
\]

\[
(x)_{i(\alpha, \lambda)} \mapsto r_i^\alpha xq_\alpha^\beta, \quad \eta_{\alpha i} \mapsto \rho_{i(\alpha)}
\]

Clearly, \( \phi_2 \) is bijective. Since \( r_i^\alpha xq_\alpha^\beta \) on semigroup \( S \), by Theorem 3.5 we know that \( \phi_1 \) is a semigroup isomorphism from \( \mu(T; I, \Lambda, P) \) to \( S \). Thus we know that \( \phi \) is a bijection from \( \Gamma \)-semigroup \( T_\mu \) to \( \Gamma \)-semigroup \( S \).
Let $\bar{\Gamma}$ such that $\phi$ is $\bar{\Gamma}$-isomorphic to a $\Gamma$-isomorphic to a $\Gamma$-Rees matrix semigroup $S_\bar{\Gamma}$ 

Further, we put the subset of

$$\{r_\alpha^\nu y_\nu^\alpha x_q^\gamma \rho_{\lambda(\beta)}, j(\gamma)(r_q^j y_q^\mu) \} \text{ if } (u, t) = (\lambda, j)$$

otherwise.

On the other hand,

$$(x)_{i\lambda} \phi_{\nu\mu}(y)_{j\nu} = \{ (r_\alpha^\nu x_q^\gamma \rho_{\lambda(\beta)}, j(\gamma)(r_q^j y_q^\mu) \} \text{ if } (u, t) = (\lambda, j),$$

otherwise.

Thus we obtain that for any $(u, t) \in \Lambda \times I$,

$$[(x)_{i\lambda} \eta_{\nu\mu}(y)_{j\nu}] = (x)_{i\lambda} \phi_{\nu\mu}(y)_{j\nu} \phi.$$

By Definition 3.6, we know that $\phi = (\phi_1, \phi_2)$ is a $\Gamma_1(\Gamma_2)$-isomorphism from $\Gamma_2$-semigroup $T_\mu$ to $\Gamma_1$-semigroup $S$, that is, we have:

**Theorem 3.7.** Let $S$ be an abundant semigroup with zero, then $S$ is $\Gamma_2$-isomorphic to a $\Gamma_1$-Rees matrix representation $T_\mu = \mu(T; I, \Lambda, \rho)$ where $T = \bigcup \{ M_{\alpha, \beta} \mid \alpha, \beta \in \bar{\Gamma} \}$ is a $\Gamma$-semigroup and $\rho$ is a $\Gamma$-Rees matrix over $\Gamma_1$, that is, any abundant semigroup $S$ has a $\Gamma$-Rees matrix representation.

Furthermore, we can prove the following result.

**Theorem 3.8.** Let $S$ be an abundant semigroup with zero, if $S^0$ is an adequate transversal of $S$, then $S$ is $\Gamma$-isomorphic to a $\Gamma_1$-Rees matrix semigroup $T_\mu = \mu(T; I, \Lambda, \rho)$ with a $\Gamma$-adequate transversal $T_\mu^0$ $\Gamma$-isomorphic to $S^0$. Further, $T_\mu^0$ is $\Gamma$-multiplicative if and only if $S^0$ is $\Gamma$-multiplicative. That is, any abundant semigroup $S$ with an adequate transversal $S^0$ has a $\Gamma$-Rees matrix representation $T_\mu$ such that $T_\mu$ has a $\Gamma$-adequate transversal $T_\mu^0$ $\Gamma$-isomorphic to $S^0$.

**Proof.** By Theorem 3.7, we know that there is a $\Gamma_2$-Rees matrix semigroup $T_\mu$ such that $T_\mu$ is $\Gamma$-isomorphic to $S$ by $\phi$. Let $T^0 = \{ t_0^\alpha \in T \mid \forall x_0 \in S^0 \exists t_0 \in T, r_\alpha^\nu x_q^\gamma \in S, \exists x_0 = r_\alpha^\nu x_q^\gamma \}$. Similar to the proof of Theorem 3.5, $T^0$ is a $\bar{\Gamma}$-subsemigroup of $T$, under the multiplication (3.4) for some subset

$$(3.7) \quad \bar{\Gamma}_1 = \left\{ \rho_{\lambda i} = \left\{ \begin{array}{ll} \rho_{\lambda i} & \bar{i} = \lambda \\ 0 & \bar{i} \neq \lambda \end{array} \right\} \forall \bar{i} \in \bar{I} \exists a \in T, a \rho_{\bar{i} a} = a, \bar{I} = \bar{A} \right\}.$$

Further, we put the subset of $T_\mu$ as follows

$$T_\mu^0 = \{ (t_0^\alpha)_{\bar{i} \lambda} \mid t_0^\alpha \in T^0, (\bar{i}, \lambda) \in \bar{I} \times \bar{I} \}.$$

Since $\phi = (\phi_1, \phi_2)$ given by (3.6) is a $\Gamma$-isomorphism from $\Gamma_2$-semigroup $T_\mu$ to $\Gamma_1$-semigroup $S$, consider $\phi|_{t_0^\alpha}$ as belows for any $(t_0^\alpha)_{\bar{i} \lambda} \in T_\mu^0$

$$(t_0^\alpha)_{\bar{i} \lambda} \phi_1 = r_\alpha^\mu \rho_{\lambda(\alpha)}, j(\alpha) t_0^\alpha \rho_{\lambda(\beta)}, j(\beta) y_q^\mu = (r_\alpha^\mu t_0^\alpha y_q^\mu) \in S^0,$$

$$(t_0^\alpha)_{\bar{i} \lambda} \phi_2 = \rho_{\lambda i}.$$
Thus $\phi|_{\mu}$ is a $\Gamma$-isomorphism from $\tilde{\Gamma}_2 = \{\eta_{(i, \lambda)} \mid (i, \lambda) \in \tilde{I} \times \tilde{I}\}$-semigroup $T^0_\mu$ to $\tilde{\Gamma}_1$-semigroup $S^0$. Now for any $(t, \lambda) \in T^0_\mu$, where $t \in T$, $(i, \lambda) \in I \times \Lambda$, let $t = c t^0 f_t$ by $S^0$ is an adequate transversal of $S$. Then

\begin{align*}
(t)_{(i, \lambda)} \phi &= r^0_i \rho_{(i)(\alpha)\iota(\alpha)} \iota \rho_{(\beta)\iota(\beta)} q^\beta_{\lambda} \\
 &= r^0_i t q^\beta_{\lambda} \\
 &= r^0_i e t^0 f_t q^\beta_{\lambda} \\
 &= r^0_i e t^0 q_i t^0 q_i r_i f_t q^\beta_{\lambda} (\because q_i r_i \text{ is the identity of } T_\alpha) \\
 &= (r^0_i e t^0 q_i)(r^0_i t^0 q_i) p_i t^0 f_t q^\beta_{\lambda} \\
 &= (e)_{i, \lambda} \phi_{\iota}(t^0)_{i, \lambda} \phi_{\iota}(f_t)_{i, \lambda} \phi \ (\text{by } (3.4)) \\
 &= [(e)_{i, \lambda} \eta_{i}(t^0)_{i, \lambda} \eta_{i}(f_t)_{i, \lambda}] \phi \ (\text{by } \Gamma \text{ isomorphism } \phi)
\end{align*}

for some $(\tilde{i}, \tilde{\lambda}) \in \tilde{I} \times \tilde{I}$. So we have that

\begin{align*}
(t)_{(i, \lambda)} &= (e)_{i, \lambda} \eta_{i}(t^0)_{i, \lambda} \eta_{i}(f_t)_{i, \lambda},
\end{align*}

where $(t^0)_{i, \lambda} \in T^0_\mu$ and $(e)_{i, \lambda} \cdot \mathcal{L}(t^0)_{i, \lambda} \mathcal{R}(f_t)_{i, \lambda} \text{ by Lemma 1.3. Thus by Definition 2.1 and Definition 3.6 we know that } T^0_\mu \text{ is an } \tilde{\Gamma} \text{adequate transversal of } T_\mu^0 \text{ and } T^0_\mu \text{ is } \Gamma \text{-isomorphic to } S^0. \text{ The remanent proofs are omitted. We complete the proof of this theorem.} \quad \Box

Finally, we use an example to conclude this note, at the same time to illustrate the application of Theorem 3.8.

**Example 3.9.** Let $M$ be a regular idempotent generated semigroup with zero and having a multiplicative semilattice transversal $M^0 = \{a, e\}$. $M$ is not orthodox with Cayley table as below.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a$</td>
<td>$a$</td>
<td>$b$</td>
<td>$b$</td>
<td>$e$</td>
</tr>
<tr>
<td>$b$</td>
<td>$a$</td>
<td>$b$</td>
<td>$e$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c$</td>
<td>$d$</td>
<td>$c$</td>
<td>$d$</td>
<td>$e$</td>
</tr>
<tr>
<td>$d$</td>
<td>$c$</td>
<td>$d$</td>
<td>$e$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$e$</td>
<td>$e$</td>
<td>$e$</td>
<td>$e$</td>
</tr>
</tbody>
</table>

Under structure express $(M, I, \Lambda, \cdot)$ of $M$, computing we let the mappings

$\rho_{\lambda} : H_{11} \rightarrow H_{1\lambda}$, $x \rho_{\lambda} = x q_{\lambda}$, where $q_{\lambda} = \{a, \lambda = 1, b, \lambda = 2\}$

$\rho_i : H_{1\lambda} \rightarrow H_i \lambda$, $\rho_i y = r_i y$, where $r_{\lambda} = \{a, i = 1, b, i = 2\}$.

Further, let $\rho_{\lambda_i} = \rho_i \rho_i$ for $(i, \lambda) \in I \times \Lambda$ and the $\Gamma_1$-Rees matrix $\rho = (\rho_{\lambda_i})_{\Lambda \times I}$, where $\Gamma_1 = \{\eta_{(i, \lambda)} \mid (i, \lambda) \in I \times \Lambda\}$. Let $\Gamma_2 = \{\eta_{(i, \lambda)} \mid (i, \lambda) \in I \times \Lambda\}$ where $\eta_{i, \lambda}$ as required in (3.5). Then $M$ becomes a $\Gamma_1$-semigroup under the multiplication (3.4), $M^0 = m(H_{11}, I, \Lambda, \cdot)$ becomes a $\Gamma_2$-semigroup under the multiplication
we define the following bijection
\[ a \Gamma \]
we know that \( S \) (3.5). By Theorem 3.8, \( M \) is \( \Gamma \)-isomorphic to the \( \Gamma_2 \)-Rees matrix semigroup \( M_\mu \) over \( \Gamma_1 \).

Nextly, Let \( N \cup \{0\} \) be cancellative monoid of natural number with an outer zero under multiplication. Using the above set \( I \times \Lambda \) take the matrix \( \theta = (1)_{\Lambda \times I} \). It is easy to show that the Rees semigroup \( K = \mu(N, I, \Lambda, \theta) \) is an abundant semigroup with a multiplicative transversal \( K^0 = \{(n)_{11} \mid \forall n \in N \cup \{0\}\} \). We denote the element of \( K \) by \( x_{i\lambda} \) for \( (i, \lambda) \in I \times \Lambda \). Computing we have that \( E(K) = \{1_{11}, 1_{12}, 1_{21}, 1_{22}\} \cup \{0\} \) and the mappings
\[ \rho_\lambda' : H_{11}^* \rightarrow H_{11}^*, \quad x_{11}y_{11} = x_{11}1_{11} = x_{11}, \quad \text{where} \quad y_{11} \lambda = \begin{cases} 1_{11} & \lambda = 1 \\ 1_{12} & \lambda = 2 \end{cases}; \]
\[ \rho_{i\lambda}' : H_{11}^* \rightarrow H_{11}^*, \quad \rho_{y1}\lambda = 1_{11}y_{11} = y_{11}, \quad \text{where} \quad y_{11} \lambda = \begin{cases} 1_{11} & i = 1 \\ 1_{21} & i = 2 \end{cases}. \]
Let \( \rho_{\lambda}' \) be \( \rho_{\lambda}' \rho_{\lambda} \) for \((i, \lambda) \in I \times \Lambda \) and the \( \Gamma'_1 \)-Rees matrix \( \rho' = (\rho_{\lambda}')_{\Lambda \times I} \), where \( \Gamma'_1 = \{(\rho_{\lambda})' \mid (i, \lambda) \in I \times \Lambda \} \). Let \( \Gamma'_2 = \{(\eta_{\lambda})' \mid (i, \lambda) \in I \times \Lambda \} \), where \( \eta_{\lambda}' \) as required in (3.5). Then \( K \) becomes a \( \Gamma_1 \)-semigroup under the multiplication (3.4), \( K_\mu = \mu(H_{11}'1, I, \Lambda, \rho') \) becomes a \( \Gamma_2 \)-semigroup under the multiplication (3.5). Similar to \( M \), by Theorem 3.8, \( K \) is \( \Gamma \)-isomorphic to the \( \Gamma_2 \)-Rees matrix semigroup \( K_\mu \) over \( \Gamma_1 \).

Let \( S = M \times K \), \( S^0 = M^0 \times K^0 \) be two direct product sets. Let \( \Gamma_1' = \{\Gamma_1, \Gamma_2\} = \{(\rho_{\lambda}), (\rho_{\lambda}') \mid (i, \lambda) \in I \times \Lambda \} \) and \( \Gamma_2' = \{(\eta_{\lambda}), (\eta_{\lambda}') \mid (i, \lambda) \in I \times \Lambda \} \). Under the multiplications of (3.4) and (3.5), we define the following multiplications of \( S \) and \( M_\mu \times K_\mu \).

\[
(\forall (x, n_{i\lambda}), (y, m_{j\mu}) \in S) (x, n_{i\lambda}) \circ (y, m_{j\mu}) = (x, n_{i\lambda})(\rho_{ut}, \rho_{ut}')y_{j\mu} = (x, n_{i\lambda})(\rho_{ut}, \rho_{ut}')y_{j\mu};
\]
\[
(\forall ((x, n_{i\lambda}), (n_{ut}), (y, m_{j\mu})), (m_{vk}), (m_{j\mu}))) \in M_\mu \times K_\mu
\]
\[
((x, n_{i\lambda}), (n_{ut})) \circ ((y, m_{j\mu}), (m_{vk})) = ((x, n_{i\lambda}), (n_{ut}))((y, m_{j\mu}), (m_{vk}));
\]
where \( (\eta, \eta') \) denote some \( (\eta_{\lambda}), (\eta_{\lambda}') \) for \((i, \lambda) \in I \times \Lambda \). Thus \( S = M \times K \) becomes a \( \Gamma_1 \)-semigroup and \( M_\mu \times K_\mu \) becomes a \( \Gamma_2 \)-semigroup. Further, using \( \rho_{\lambda}, \rho_{\lambda}' \) we define the following bijection \( \phi = (\phi_1, \phi_2) \)
\[ \phi_1 : M_\mu \times K_\mu \rightarrow S, \quad ((x, n_{i\lambda}), (n_{ut})) \mapsto ((x, n_{i\lambda}), (n_{ut})); \]
\[ \phi_2 : \Gamma_2' \rightarrow \Gamma_1', \quad (\eta_{\lambda}), (\eta_{\lambda}') \mapsto (\rho_{\lambda}, \rho_{\lambda}'). \]

It is easy to check that \( \phi \) is a \( \Gamma \)-isomorphism from \( \Gamma_2 \)-semigroup \( M_\mu \times K_\mu \) to \( \Gamma_1 \)-semigroup \( S \). By Theorem 3.8 we know that \( M_\mu \times K_\mu \) has a \( \Gamma_2 \)-multiplicative adequate transversal \( M_\mu^0 \times K_\mu^0 \) where \( M_\mu^0 (K_\mu^0) \) is the \( \Gamma_2 \)-\( \Gamma_2' \) multiplicative adequate transversal of \( M_\mu (K_\mu) \) and \( M_\mu^0 \times K_\mu^0 \) is \( \Gamma \)-isomorphic to \( S^0 \). Computing we know that \( M_\mu^0 \) and \( K_\mu^0 \) may be described by
\[ M_\mu^0 = \{(a)_{11}, (e)\}, \quad K_\mu^0 = \{(n_{11})_{11} \mid n_{11} \in K^0\}. \]
Note that under the multiplication (1.1) and the multiplication of direct product $S$ is indeed an abundant semigroup with a multiplicative adequate transversal $S^0$. Therefore concluding above results we may say that the abundant semigroup $S$ has a $\Gamma$-Rees matrix representation $M_\mu \times K_\mu$ with a $\Gamma$-multiplicative $\bar{\Gamma}$-adequate transversal $M_\mu^0 \times K_\mu^0$ $\Gamma$-isomorphic to $S^0$.

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