Swapping Hats: A Generalization of Montmort's Problem

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1 Montmort's Matching Problem

The following problem was first proposed by the mathematician Pierre Rémond de Montmort [7] in *Essay d'Analyse sur les Jeux de Hazard*, his 1708 treatise on the analysis of games of chance: Suppose you have a deck of N cards, numbered $1, 2, 3, \ldots, N$. After shuffling, you draw one card at a time, without replacement, counting out loud as each card is drawn: "1, 2, 3, … ". What is the probability that there will be no coincidence, i.e., no drawing of a card bearing the number just called out? In Montmort's version of the problem, the deck had 13 cards, so the game was called *Treize*, French for thirteen. The game has also been called *Rencontres* (Coincidences), or Montmort's Matching Problem.

Montmort discusses a generalized version of this problem in his correspondence with Nicholas Bernoulli (1687–1759) from 1710 to 1712; these letters are included in the second edition of Montmort's work on gaming [8]. In the generalization, N cards are drawn from a deck of Ns cards; there are s cards bearing each number from 1 to N. Again, one seeks the probability of at least one coincidence, for which Montmort and Bernoulli find a formula.

Other mathematicians who have generalized and discussed this problem include de Moivre [6], Euler [1], Lambert [4], Laplace [5], and Waring [11]. For a more extensive account of the history of this problem, see [2, pages 326–345] and [10].

2 Calculation of $P_m(N)$

Montmort's Matching Problem is often posed in the following more amusing form: N men, attending a banquet, check their hats. When each man leaves he takes a hat at random. What is the probability that at least one man gets his own hat?

If there are no such coincidences, the next best thing might be a two-way swap. So one might ask for the likelihood of no matches but at least one swap, or the likelihood of no matches and no swaps but at least one three-way swap. More generally, one is interested in the probability that for any m from 1 to N, m is the size of the smallest subset of N men who exchange hats among themselves.

We let $P_m(N)$ denote the probability that among N men, m is the size of the smallest subset of men that swap hats. $P_1(N)$, then, is the probability of at least one match, which is Montmort's original problem. The usual way of calculating $P_1(N)$ is to let E_i be the event that the ith man gets his own hat back. Then we use the inclusion-exclusion principle to calculate the probability of at least one match, as follows:

$$\begin{aligned}
P_1(N) = P(\cup_i E_i) &= \sum_i P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) - \cdots + (-1)^{N+1} P(E_1E_2\ldots E_N) \\
&= \sum_i \frac{(N-1)!}{N!} - \sum_{i<j} \frac{(N-2)!}{N!} + \sum_{i<j<k} \frac{(N-3)!}{N!} - \cdots + (-1)^{N+1}\frac{1}{N!} \\
&= N\frac{(N-1)!}{N!} - \binom{N}{2}\frac{(N-2)!}{N!} + \binom{N}{3}\frac{(N-3)!}{N!} - \cdots + (-1)^{N+1}\frac{1}{N!} \\
&= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{N+1}}{N!}.
\end{aligned}$$

The series converges to $1 - 1/e \approx 0.63$ as N tends to infinity.

Let us now calculate $P_m(N)$ for some small values of N. If $N = 3$, there are $3! = 6$ ways of distributing the hats. In fact, the sample space is just S_3, the group of permutations of 3 elements. Let $(i\,j\,k)$ indicate that the first man gets hat i, the second hat j, and the third hat k. Then our sample space becomes $S_3 = \{(1\,2\,3), (1\,3\,2), (2\,1\,3), (2\,3\,1), (3\,1\,2), (3\,2\,1)\}$. Then

$$\begin{aligned} P_1(3) &= P(\text{at least one match}) = P(\{(1\,2\,3), (1\,3\,2), (2\,1\,3), (3\,2\,1)\}) = 2/3 \\ P_2(3) &= P(\text{no matches but at least one swap}) = P(\emptyset) = 0 \\ P_3(3) &= P(\text{no matches, no swaps, but at least one 3-way swap}) = P(\{(2\,3\,1), (3\,1\,2)\}) = 1/3 \end{aligned}$$

With four men and four hats, there are $4! = 24$ sample points. Let $(i_1\, i_2\, i_3\, i_4)$ represent the outcome where the jth man gets hat i_j. We know that the probability of at least one match is $P_1(4) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} = \frac{15}{24} = \frac{5}{8}$. In how many ways can we distribute the four hats so that nobody gets his own, but at least one pair of men swaps? Notice that if two men swap, the other two must also swap (since no matches are allowed). Hence we want to count the number of ways of dividing four men into two pairs. This is $\frac{1}{2}\binom{4}{2} = 3$, so $P_2(4) = \frac{3}{24} = \frac{1}{8}$.

In any three-way swap, the fourth man gets his own hat back, so $P_3(4) = 0$. The last possibility is of a four-way swap; it has probability $P_4(4) = 1 - \frac{5}{8} - \frac{1}{8} = \frac{1}{4}$.

We now give a general formula for $P_m(N)$:

Theorem 1 $P_m(N) = \displaystyle\sum_{k=1}^{\lfloor N/m \rfloor} \frac{(-1)^{k+1}}{m^k k!} \left(1 - \sum_{i=1}^{m-1} P_i(N - mk)\right)$.

Proof. Let $E_{i_1\, i_2\, \ldots\, i_m}$ be the event where men $i_1 \ldots, i_m$ exchange hats among themselves and no smaller subset of men exchange hats. Then by the inclusion-exclusion principle,

$$P_m(N) = \sum_{k=1}^{\lfloor N/m \rfloor} (-1)^{k+1} \sum P(E_{i_{11} \ldots i_{m1}} E_{i_{12} \ldots i_{m2}} \ldots E_{i_{1k} \ldots i_{mk}}),$$

where the second summation is taken over all possible choices of disjoint subsets $\{i_{11}, \ldots i_{m1}\}$, $\ldots$, $\{i_{1k}, \ldots, i_{mk}\}$ of $\{1, 2, \ldots, N\}$.

For a specific choice of these subsets, let us calculate the probability that each subset of men exchange hats among themselves and no smaller subset of these men exchange hats among themselves. This means that the men in $\{i_{1j}, \ldots, i_{mj}\}$ exchange hats in some cyclical manner. There are $(m-1)!$

such cyclical permutations of each subset $\{i_{1j},\ldots,i_{mj}\}$. For a specific choice, the probability of the chosen cyclical permutations occurring among the members of the subsets, with no j-way swaps ($j < m$) occurring among the remaining $N-mk$ men, is $\frac{1}{N}\frac{1}{N-1}\cdots\frac{1}{N-mk+1}\left(1-\sum_{i=1}^{m-1}P_i(N-mk)\right)$.

Therefore

$$\begin{aligned}
P_m(N) &= \sum_{k=1}^{\lfloor N/m\rfloor}(-1)^{k+1}\sum((m-1)!)^k\frac{1}{N}\frac{1}{N-1}\cdots\frac{1}{N-mk+1}\left(1-\sum_{i=1}^{m-1}P_i(N-mk)\right)\\
&= \sum_{k=1}^{\lfloor N/m\rfloor}(-1)^{k+1}\sum((m-1)!)^k\frac{(N-mk)!}{N!}\left(1-\sum_{i=1}^{m-1}P_i(N-mk)\right),
\end{aligned}$$

where the second summation is over all possible choices of disjoint subsets $\{i_{11},\ldots i_{m1}\},\ldots,\{i_{1k},\ldots,i_{mk}\}$ of $\{1,2,\ldots,N\}$. The number of choices of such disjoint subsets is $\binom{N}{m}\binom{N-m}{m}\cdots\binom{N-mk+m}{m}\frac{1}{k!} = \frac{N!}{k!(m!)^k(N-mk)!}$. Therefore, finally,

$$\begin{aligned}
P_m(N) &= \sum_{k=1}^{\lfloor N/m\rfloor}(-1)^{k+1}\frac{N!}{k!(m!)^k(N-mk)!}((m-1)!)^k\frac{(N-mk)!}{N!}\left(1-\sum_{i=1}^{m-1}P_i(N-mk)\right)\\
&= \sum_{k=1}^{\lfloor N/m\rfloor}\frac{(-1)^{k+1}}{m^k k!}\left(1-\sum_{i=1}^{m-1}P_i(N-mk)\right).
\end{aligned}$$

This completes the proof. □

3 The Limiting Value of $P_m(N)$

In this section, we find a general formula for the limit of $P_m(N)$ as N tends to infinity. Recall that for $m=1$, this reduces to Montmort's problem, so that $\lim_{N\to\infty}P_1(N) = 1-1/e$. Theorem 2, which gives a formula for evaluating $P_m \equiv \lim_{N\to\infty}P_m(N)$, makes use of the following lemma (see e.g. [9, pages 73–74]):

Lemma *Let $\{a_k\}$ and $\{b_k\}$ be sequences such that a_k converges to a and $\sum b_k$ converges absolutely and the sum is b. Let $c_l = \sum_{k=0}^{l} b_k a_{l-k}$. Then $\lim_{l\to\infty} c_l = ab$.*

Theorem 2 *Let* $P_m = \lim_{N\to\infty} P_m(N)$*,* $m = 1, 2, 3, \ldots N$*. Then* P_m *exists and*

$$P_m = e^{-\sum_{k=1}^{m-1}\frac{1}{k}} - e^{-\sum_{k=1}^{m}\frac{1}{k}}. \tag{1}$$

Proof. The proof is by complete induction on m. We know (1) is true for $m = 1$. Now assume $m > 1$ and whenever $i = 1, ..., m-1$ then P_i exists and $P_i = e^{-\sum_{k=1}^{i-1}\frac{1}{k}} - e^{-\sum_{k=1}^{i}\frac{1}{k}}$. To show (1), we will first show that

$$P_m = (1 - e^{-1/m}) \left(1 - \sum_{i=1}^{m-1} P_i\right). \tag{2}$$

It is enough to show that for each value of $r = 0, 1, 2, \ldots, m-1$, $P_m(mq + r)$ approaches $(1 - e^{-1/m}) \left(1 - \sum_{i=1}^{m-1} P_i\right)$ as $q \to \infty$. To this end, fix r and apply the lemma with $a_k = 1 - \sum_{i=1}^{m-1} P_i(mk + r)$ and $b_k = \frac{(-1)^k}{m^k k!}$. Then $a = 1 - \sum_{i=1}^{m-1} P_i$ and $b = e^{-1/m}$. Therefore as $q \to \infty$,

$$\sum_{k=0}^{q} \frac{(-1)^k}{m^k k!} \left(1 - \sum_{i=1}^{m-1} P_i(mq + r - mk)\right) \to e^{-1/m} \left(1 - \sum_{i=1}^{m-1} P_i\right).$$

Now we have, as $q \to \infty$,

$$\begin{aligned}
P_m(mq + r) &= \sum_{k=1}^{q} \frac{(-1)^{k+1}}{m^k k!} \left(1 - \sum_{i=1}^{m-1} P_i(mq + r - mk)\right) \\
&= 1 - \sum_{i=1}^{m-1} P_i(mq + r) - \sum_{k=0}^{q} \frac{(-1)^k}{m^k k!} \left(1 - \sum_{i=1}^{m-1} P_i(mq + r - mk)\right), \\
&\to 1 - \sum_{i=1}^{m-1} P_i - e^{-1/m} \left(1 - \sum_{i=1}^{m-1} P_i\right) \\
&= (1 - e^{-1/m}) \left(1 - \sum_{i=1}^{m-1} P_i\right),
\end{aligned}$$

which proves (2).

By the induction hypothesis, we know that $\sum_{i=1}^{m-1} P_i = \sum_{i=1}^{m-1} \left(e^{-\sum_{k=1}^{i-1}\frac{1}{k}} - e^{-\sum_{k=1}^{i}\frac{1}{k}}\right) = 1 -$

$e^{-\sum_{k=1}^{m-1}\frac{1}{k}}$, from which we obtain

$$P_m = \left(1-e^{-1/m}\right)\left(1-\sum_{i=1}^{m-1} P_i\right) = \left(1-e^{-1/m}\right)\left(e^{-\sum_{k=1}^{m-1}\frac{1}{k}}\right) = e^{-\sum_{k=1}^{m-1}\frac{1}{k}} - e^{-\sum_{k=1}^{m}\frac{1}{k}}.$$

This completes the proof. □

We can now state our main result:

Theorem 3 *The probability that the size of the smallest subset of N men that exchange hats among themselves exceeds m approaches $e^{-\sum_{k=1}^{m}\frac{1}{k}}$ as $N \to \infty$.*

Proof. Theorem 3 follows immediately from Theorem 2, since the probability in Theorem 3 is simply $1-\sum_{i=1}^{m} P_i$. □

This result is not new. In [3], for instance, it is shown that if α_r is the number of cycles of length r in a random permutation, then

$$P(\alpha_{r_1}=k_1, \alpha_{r_2}=k_2, \ldots, \alpha_{r_s}=k_s) \quad = \quad \frac{1}{k_1!k_2\ldots k_s!} e^{-\frac{1}{r_1}-\cdots-\frac{1}{r_s}} + o(1)$$

as $n \to \infty$. In particular, $P(\alpha_1=0, \alpha_2=0, \ldots, \alpha_m=0) = e^{-1-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{m}} + o(1)$, from which Theorem 3 follows. The proof given by Kolchin uses sophisticated tools, including local limit theorems in probability and integrals of complex-valued functions. The proof given above is relatively elementary and illustrates the method of inclusion-exclusion.

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