FEYNMAN’S OPERATIONAL CALCULI FOR 
TIME DEPENDENT NONCOMMUTING OPERATORS

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Abstract. We study Feynman’s Operational Calculus for operator-valued functions of time and for measures which are not necessarily probability measures; we also permit the presence of certain unbounded operators. Further, we relate the disentangling map defined within to the solutions of evolution equations and, finally, remark on the application of stability results to the present paper.

1. Introduction

The problem of forming functions of noncommuting operators, say $A$ and $B$, is a difficult one. Even if the function is as simple as multiplication $M$, it is not clear what $M(A, B)$ should mean: $AB$, $BA$, $\frac{1}{2}(AB + BA)$, ..., (Note that this ambiguity is present even if $A$ and $B$ are self-adjoint and bounded.) This problem was of concern to Richard Feynman because of his interest in quantum theory where the observables are self-adjoint operators that frequently do not commute with one another.

Feynman invented some ‘rules’ for forming functions of noncommuting operators in conjunction with his famous contributions to quantum electrodynamics:

(R1) Attach time indices to operators to indicate the order in which they act in products. (Note: Operators sometimes come with time indices naturally attached.) Feynman’s time ordering convention was that an operator with a smaller (or earlier) time index should act before one with a larger (or later) index no matter how they are ordered on the page.

Received July 14, 1999.
2000 Mathematics Subject Classification: Primary 47A56, 47A60, 46J15; Secondary 47N50, 28A33, 28A25.
Key words and phrases: Feynman’s operational calculus for time dependent noncommuting operators, disentangling, evolution equations, stability.
(R2) With time indices attached, form functions of these operators as if they were commuting. (If one stops naively at this point, the ‘equality’ involved is usually false. For example, it might say that $e^{A+B} = e^A e^B$ even though $A$ and $B$ do not commute.)

(R3) After (R2) is completed, ‘disentangle’ the resulting expression; that is, restore the conventional ordering of the operators. In practice, this means to manipulate the expression (if possible) until the ordering on the page coincides with the time ordering.

A much more extensive introduction to Feynman’s operational calculus including a discussion of its connection with the Feynman path integral can be found in the book of Johnson and Lapidus [5], see especially Chapter 14. Further work that will soon be referenced below can also be consulted.

The recent paper of Jefferies and Johnson [4] is the one most closely related to the present paper. The authors began by constructing a commutative Banach algebra, called the disentangling algebra, in which (R2) above can be rigorously applied. They then go on to define an elementary disentangling map from the commutative setting into the noncommutative Banach algebra of operators. The commutative setting, the disentangling map, and various theorems that are established in Part I of [4] permit the ‘rule’ (R3) above to be carried out in a variety of circumstances.

The operators involved in [4] are required to be time independent and bounded. Following [5,6], [7,8] and then [2,3], the directions for disentangling are specified by a set of measures on an appropriate time interval where there is one measure for each operator. Different sets of measures may well define different disentangling maps and so different operational calculi. The variety of operational calculi is associated with the ‘problem’ of ambiguity mentioned at the beginning of this introduction. We will see below in connection with evolution equations that this ‘problem’ is actually an asset. In most of [4] the measures are taken to be probability measures. This is a natural choice for most problems in which a functional calculus is useful. However, the main application in this paper will be to solving evolution equations (in integral form) and probability measures are not the natural choice for this purpose.

The authors of [4] believed after finishing Part I of that paper that many of the results would carry through to the case of time dependent operators with appropriate adjustments in the disentangling algebra and only modest changes in the proof. Nielsen showed that this is true for a portion of that material in Chapter 2 of his thesis [9]. One of the key results from [4, Part I] on, in Feynman’s words, “disentangling of an experimental factor”, was nontrivial to extend to the time dependent
case. This was done in Nielsen’s thesis [9, Theorem 2.3.1] and we report on it here with a detailed proof. We will also finish extending the rest of Part I of [4] to the time dependent case.

We remark that, in Part I, the functions in the disentangling algebras are required to be analytic in a polydisk centered at the origin. The norms of the bounded operators determine the radii of the polydisk. Spectral information about the operators plays no role. This suggests that a richer functional calculus might be available. Indeed, Part II of [4] treats this topic. It is likely that such matters will be much more difficult in the present time dependent setting and, in any case these issues will not be dealt with in this paper. Indeed a good deal of our focus will be on evolution equations and hence on exponential functions where spectral considerations are unnecessary.

Motivated by Corollary 3.1 below on disentangling an experimental (or exponential) factor, we give, in Definition 4.1 a disentangling formula which involves the ‘exponential’ of an unbounded operator, specifically the generator $-A$ of a ($C_0$) contraction semigroup $T(s) = e^{-sA}$. This allows us to make a connection with the disentangling formula in [2] which is given below in equation (37). The expression from [2] was ‘derived’ there by a series of steps some of which depended on Feynman’s heuristic ideas. However, the final expression was rigorously shown in [2] to make sense mathematically and to satisfy the evolution equation in (41). Combining the work here with the rigorous part of [2] we derive the disentangled expression and see that it is the same as that arrived at in [2] and so satisfies the evolution equation. The advantage here is that no heuristic steps are used.

The main objective of Nielsen’s thesis [9] was to study stability properties in the setting of Feynman’s Operational Calculus. His results included theorems related to the disentangling map in the time independent case as in [4] and for exponential functions as in [2]. Stability results were obtained both with respect to the time ordering measures and the operators involved. We state one of these below as Theorem 5.1 and briefly discuss some of the other results. No proofs are given here as the material will be submitted elsewhere. Theorem 5.1 concerns the stability of the exponential function disentangled in [2] and developed more fully in this paper.

In summary: We study Feynman’s Operational Calculus for operator-valued functions of time and for measures which are not necessarily probability measures; we also permit the presence of certain unbounded operators. Further, we relate the disentangling map defined within to the solutions of evolution equations and, finally, remark on the application of stability results in [9] to the present paper.
2. The time dependent disentangling map

In this section we extend the definition of the disentangling map from the time independent setting of [4] to the time dependent situation that was investigated in Chapter 2 of [9].

**Definition 2.1.** Let $X$ be a separable Banach space over the complex numbers and let $\mathcal{L}(X)$ denote the space of bounded linear operators on $X$. Fix $T > 0$. For $i = 1, \ldots, k$ let $A_i : [0, T] \to \mathcal{L}(X)$ be maps that are measurable in the sense that $A_i^{-1}(E)$ is a Borel set in $[0, T]$ for any strong operator open set $E \subset \mathcal{L}(X)$. To each $A_i(\cdot)$ we associate a finite continuous Borel measure $\mu_i$ on $[0, T]$ and we require that, for each $i$,

$$r_i = \int_{[0,T]} \|A_i(s)\|_{\mathcal{L}(X)}|\mu_i|(ds) < \infty.$$  

We obtain, exactly as in Section 1 of [4], a Banach algebra $A_T(r_1, \ldots, r_k)$ of functions analytic in the open polydisk with radii $r_1, \ldots, r_k$ centered at the origin and continuous on $|z_1| = r_1, \ldots, |z_k| = r_k$. (It is perhaps worth emphasizing that the weights we are using here depend on the operator-valued functions as well as on $T$ and on the measures.) The norm for this Banach algebra is defined to be

$$\|f\|_{A_T} = \sum_{n_1, \ldots, n_k=0}^{\infty} |a_{n_1, \ldots, n_k}|r_1^{n_1} \cdots r_k^{n_k}. \tag{2}$$

**Definition 2.2.** To the algebra $A_T$ we associate as in [4] a disentangling algebra by replacing the $z_i$'s with formal commuting objects $(A_i(\cdot), \mu_i)^\sim$, $i = 1, \ldots, k$. We define the disentangling algebra $D_T((A_1(\cdot), \mu_1)^\sim, \ldots, (A_k(\cdot), \mu_k)^\sim)$ as in [9] and much as in [4]. However, rather than using the notation $(A_i(\cdot), \mu_i)^\sim$ below, we will often abbreviate to $A_i(\cdot)^\sim$, especially when carrying out calculations. The norm for $D_T$ is the same as defined in (2) for the Banach algebra $A_T$ though we will refer to it as $\|\cdot\|_{D_T}$ if a distinction needs to be made.

It is immediate that $A_T$ and $D_T$ are commutative Banach algebras which are isomorphic to one another (see Propositions 1.1–1.3 in [4]).

**Remark 2.1.** We will often write $D_T$ in place of $D_T((A_1(\cdot), \mu_1)^\sim, \ldots, (A_k(\cdot), \mu_k)^\sim)$ or $\mathcal{D}_T ((A_1(\cdot), \mu_1)^\sim, \ldots, (A_k(\cdot), \mu_k)^\sim)$.

For each $t \in [0, T]$ we will define below the disentangling map

$$T_{\mu_1, \ldots, \mu_k}^t : D_T((A_1(\cdot), \mu_1)^\sim, \ldots, (A_k(\cdot), \mu_k)^\sim) \to \mathcal{L}(X) \tag{3}$$
much as in [4] and [9]. In order to state the following proposition, which will suggest the essential action of the disentangling map, we must first introduce some notation. (This notation is essentially the same as used in [4] and [9].) For a nonnegative integer $n$ and a permutation $\pi \in S_n$, the set of all permutations of $\{1, \ldots, n\}$, we define subsets $\Delta_n^t(\pi)$ of $[0, t]^n$ by

$$\Delta_n^t(\pi) = \{(s_1, \ldots, s_n) \in [0, t]^n : 0 < s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(n)} < t\}.$$  

We next define, for nonnegative integers $n_1, \ldots, n_k$ and a permutation $\pi \in S_n$ with $n := n_1 + \cdots + n_k$,

$$\tilde{C}_{\pi(i)}(s_{\pi(i)}) = \begin{cases} A_1(s_{\pi(i)})^\sim, & \text{if } \pi(i) \in \{1, \ldots, n_1\} \\ A_2(s_{\pi(i)})^\sim, & \text{if } \pi(i) \in \{n_1, \ldots, n_1 + n_2\} \\ \vdots \\ A_k(s_{\pi(i)})^\sim, & \text{if } \pi(i) \in \{n_1 + \cdots + n_{k-1} + 1, \ldots, n\}. \end{cases}$$

With this notation recorded, we can state the following proposition.

**Proposition 2.1.** We have, for any nonnegative integers $n_1, \ldots, n_k$,

$$P_t^{n_1, \ldots, n_k} (A_1(\cdot)^\sim, \ldots, A_k(\cdot)^\sim) = \sum_{\pi \in S_n} \int_{\Delta_n^t(\pi)} \tilde{C}_{\pi(n)}(s_{\pi(n)}) \cdots \tilde{C}_{\pi(1)}(s_{\pi(1)}) \left( \mu_1^{n_1} \times \cdots \times \mu_k^{n_k} \right) (ds_1, \ldots, ds_n)$$

where, for any $t$ in $[0, T]$, $P_t^{n_1, \ldots, n_k}$ denotes the monomial $z_1^{n_1} \cdots z_k^{n_k}$ in the $k$ complex variables $z_1, \ldots, z_k$.

**Proof.** The proof is the same as that for Proposition 2.2 of [4] when we normalize the measures appropriately in order to obtain probability measures. (See Section 6 of [4].) The essential idea is to work in the commutative disentangling algebra and carry out the time-ordering in a manner consistent with Feynman’s heuristic ideas.

Now, for every $t \in [0, T]$, we define the action of the disentangling map on monomials.
Definition 2.3. Let $P_{t}^{n_1,\ldots,n_k}$ be as above. We define the action of the disentangling map on this monomial by

\[
\mathcal{T}_{\mu_1,\ldots,\mu_k}^{t} P_{t}^{n_1,\ldots,n_k} (A_1(\cdot)^\sim, \ldots, A_k(\cdot)^\sim) = \sum_{\pi \in S_n} \int_{\Delta_t^{\pi}(\pi)} C_{\pi(n)}(s_{\pi(n)}) \cdot \cdots \cdot C_{\pi(1)}(s_{\pi(1)}) \left( \mu_1^{n_1} \times \cdots \times \mu_k^{n_k} \right) (ds_1, \ldots, ds_n)
\]

where the notation is as defined in (4)–(6) except that here we omit the tildes and so obtain the appropriate operator-valued functions in place of the formal commuting objects.

Finally, for $f \in D_T ((A_1(\cdot), \mu_1)^\sim, \ldots, (A_k(\cdot), \mu_k)^\sim)$ written as

\[
f (A_1(\cdot)^\sim, \ldots, A_k(\cdot)^\sim) = \sum_{n_1,\ldots,n_k=0}^{\infty} c_{n_1,\ldots,n_k} (A_1(\cdot)^\sim)^{n_1} \cdots (A_k(\cdot)^\sim)^{n_k}
\]

we define the action of the disentangling map on $f$ by

\[
\mathcal{T}_{\mu_1,\ldots,\mu_k}^{t} f (A_1(\cdot)^\sim, \ldots, A_k(\cdot)^\sim) = f_{\mu_1,\ldots,\mu_k} (A_1(\cdot)^\sim, \ldots, A_k(\cdot)^\sim)
\]

\[
= \sum_{n_1,\ldots,n_k=0}^{\infty} c_{n_1,\ldots,n_k} \mathcal{T}_{\mu_1,\ldots,\mu_k}^{t} P_{t}^{n_1,\ldots,n_k} (A_1(\cdot)^\sim, \ldots, A_k(\cdot)^\sim).
\]

Before we move on to the next proposition we need to state some definitions. First we define the subset $\Delta_t^l$ of $[0,t]^l$ by

\[
\Delta_t^l = \left\{ (s_1, \ldots, s_l) \in [0,t]^l : 0 < s_1 < \cdots < s_l < t \right\}.
\]

Next we need to define a subset $\mathcal{P}_{n_1,\ldots,n_k}$ of the set of all permutations of $n = n_1 + \cdots + n_k$ objects. It consists of all those permutations $\pi$ of $\{1, \ldots, n\}$ which leave the components of each $S_{p,n_p}$ contained in $(S_{1,n_1}, \ldots, S_{k,n_k})$ in the same relative order where

\[
S_{p,n_p} := (s_{p,1}, \ldots, s_{p,n_p}).
\]

We now turn to the next proposition.

Proposition 2.2. The disentangling map $\mathcal{T}_{\mu_1,\ldots,\mu_k}^{t}$ is a bounded linear operator from $D_T ((A_1(\cdot), \mu_1)^\sim, \ldots, (A_k(\cdot), \mu_k)^\sim)$ to $\mathcal{L}(X)$. In fact $\| \mathcal{T}_{\mu_1,\ldots,\mu_k}^{t} \| \leq 1$. 
Proof. The proof that $T_{t\mu_1,\ldots,\mu_k}$ is linear is immediate. The rest of the proof proceeds as follows. Using Proposition 2.5 of [4] and ideas similar to those from Proposition 2.4 of that paper we have

$$\|T_{t\mu_1,\ldots,\mu_k}f\|_{L(X)} \leq \sum_{n_1,\ldots,n_k=0}^{\infty} |c_{n_1,\ldots,n_k}| n_1! \cdots n_k! \left( \sum_{\pi \in P_{n_1,\ldots,n_k}} \left\{ \prod_{i=1}^{k} \|A_i(s_{\pi(i)})\| \right\} \right) \sum_{\mu_1,\ldots,\mu_k} \left( \int_{\Delta_{n_1}} \cdots \int_{\Delta_{n_k}} (ds_{n_1}) \cdots (ds_{n_k}) \right) \leq \sum_{n_1,\ldots,n_k=0}^{\infty} |c_{n_1,\ldots,n_k}| n_1! \cdots n_k! \left( \int_{[0,\tau]} \|A_i(s)\| |\mu_i|(ds) \right)^{n_1} \cdots \left( \int_{[0,\tau]} \|A_k(s)\| |\mu_k|(ds) \right)^{n_k}.$$

Therefore we see that

$$\|T_{t\mu_1,\ldots,\mu_k}f\|_{L(X)} \leq \sum_{n_1,\ldots,n_k=0}^{\infty} |c_{n_1,\ldots,n_k}| \left( \int_{[0,\tau]} \|A_i(s)\| |\mu_i|(ds) \right)^{n_1} \cdots \left( \int_{[0,\tau]} \|A_k(s)\| |\mu_k|(ds) \right)^{n_k} = \|f\|_{D^{T(\infty)\tau}} < \infty.$$

The boundedness of the map follows. In fact we see that the operator norm of the disentangling map is less than or equal to 1. This finishes the proof.
Remark 2.2. Note that the proof above only shows, even when \( t = T \), that the time-dependent disentangling map is a contraction and not a norm-one contraction as in the time-independent case (Proposition 2.4 of [4]).

3. Disentangling an experimental or exponential factor

This section parallels section 8 of [4] but the time-dependence makes the proofs more complicated. We introduce the notation used in [4] in detail to make the exposition as clear as possible. Let \( A \in \mathcal{L}(X) \) and suppose that \( B_i : [0, T] \to \mathcal{L}(X) \), \( i = 1, \ldots, k \) are measurable in the usual sense. We want to disentangle expressions of the form \((A^\sim)^m(B_1(\cdot)^\sim)^{n_1}\cdots(B_k(\cdot)^\sim)^{n_k}\) and also of the form \(e^{A^\sim} - (B_1(\cdot)^\sim)^{n_1}\cdots(B_k(\cdot)^\sim)^{n_k}\) where we will write \( tA = \int_0^t A(s) \, ds \) with \( A_i(s) := A \) for \( 0 < s < t \). The idea is to keep the special role of terms corresponding to \( A^\sim \) separate from the other terms.

Set \( n = n_1 + \cdots + n_k \). Suppose that \( \mu, \nu_1, \ldots, \nu_k \) are finite continuous measures on the Borel \( \sigma \)-algebra of \([0, T] \). Assume as well that

\[
\int_{[0,T]} \|B_i(s)\| |\nu_i|(ds) < \infty
\]

for all \( i = 1, \ldots, k \).

The disentangling \( P_{t_1,\nu_1;\ldots;\nu_k}^{\mu,n_1;\ldots;n_k}(A, B_1(\cdot), \ldots, B_k(\cdot))\) is given, according to equation (7), by the expression

\[
P_{t_1,\nu_1;\ldots;\nu_k}^{\mu,n_1;\ldots;n_k}(A, B_1(\cdot), \ldots, B_k(\cdot)) = \sum_{\pi \in S_{m+n}} \int_{\Delta_{m+n}(\pi)} C_{\pi(m+n)}(s_{\pi(m+n)}) \cdots C_{\pi(1)}(s_{\pi(1)})
\]

\[
(\mu^m \times \nu_1^{n_1} \times \cdots \times \nu_k^{n_k})(ds_1, \ldots, ds_{m+n}).
\]

We wish to write this in terms of \( A, B_1, \ldots, B_k \) in a way that keeps track explicitly of the \( m \) powers of \( A \). Given \( \pi \in S_{m+n} \), let \( J_\pi = \pi^{-1}\{ \{ m + 1, \ldots, m + n \} \} \). Then \( C_{\pi(j)}(s_{\pi(j)}) \in \{ B_1(s_{\pi(j)}), \ldots, B_k(s_{\pi(j)}) \} \) for all \( j \in J_\pi \) and \( C_{\pi(j)}(s) = A \) for all \( j \notin J_\pi \) and all \( s \in [0, t] \).

Suppose that \( J_\pi = \{ u_1, \ldots, u_n \} \) for \( n \) integers \( u_1 < u_2 < \cdots < u_n \) between 1 and \( m + n \). Suppose that for each \( k = 0, \ldots, n \) there are \( j_k \) integers greater than \( u_k \) and less than \( u_{k+1} \). Take \( u_0 = 0, u_{n+1} = m + n + 1 \), that is,

\[
\begin{align*}
j_0 &= u_1 - 1, & j_1 &= u_2 - u_1 - 1, \ldots, \\
j_{n-1} &= u_n - u_{n-1} - 1, & j_n &= m + n - u_n.
\end{align*}
\]
Then for every \((m + n)\)-tuple \((s_1, \ldots, s_{m+n})\) \(\in \Delta^t_{m+n}(\pi)\), we have

\[
C_{\pi(m+n)}(s_{\pi(m+n)}) \cdots C_{\pi(1)}(s_{\pi(1)}) = A^{j_n}B^{\pi(u_n)}(s_{\pi(u_n)})A^{j_{n-1}} \cdots A^{j_1}B^{\pi(u_1)}(s_{\pi(u_1)})A^{j_0},
\]

and this tells us how to rewrite the integrand in equation (13). We will take advantage of this in equation (18). By the construction above, each permutation \(\pi \in S_{m+n}\) determines nonnegative integers \((j_0, \ldots, j_n)\) such that \(j_0 + \cdots + j_n = m\). On the other hand, given nonnegative integers \((j_0, \ldots, j_n)\) such that \(j_0 + \cdots + j_n = m\) a unique \(n\)-element subset \(J\) of \(\{1, \ldots, m+n\}\) is determined which we will continue to refer to as \(J = \{u_1, \ldots, u_n\}\). Using this \(J\), there are \(m!\) permutations \(\pi \in S_{m+n}\) such that \(J = \pi^{-1}\left(\{m + 1, \ldots, m + n\}\right)\) corresponding to the set \(B_{j_0, \ldots, j_n}\) of \(m\) bijections from \(J\) onto \(\{m + 1, \ldots, m + n\}\) and the set \(A_{j_0, \ldots, j_n}\) of \(m\) bijections from \(\{1, \ldots, m + n\}\) \(\setminus J\) to \(\{1, \ldots, m\}\).

Let \((j_0, \ldots, j_n)\) be nonnegative integers such that \(m = j_0 + \cdots + j_n\). Let \(\beta \in B_{j_0, \ldots, j_n}\), \(\alpha \in A_{j_0, \ldots, j_n}\). For each \(0 \leq u < v \leq t\) and each \(k = 0, \ldots, n\), set

\[
\Delta^\alpha_{j_0, \ldots, j_n}[u, v; k] = \left\{(s_1, \ldots, s_m) \in \mathbb{R}^m : u \leq s_{\alpha(u_k+1)} < \cdots < s_{\alpha(u_{k+1})} \leq v\right\}
\]

\[
\Gamma^\beta_{j_0, \ldots, j_n} = \left\{(s_{m+1}, \ldots, s_{m+n}) \in \mathbb{R}^n : 0 \leq s_{\beta(u_1)} < \cdots < s_{\beta(u_n)} \leq t\right\}.
\]

Note that the sets defined in (16) depend on \(t\) although that dependence is suppressed in the notation. The integers \((j_0, \ldots, j_n)\) and the maps \(\alpha \in A_{j_0, \ldots, j_n}\), \(\beta \in B_{j_0, \ldots, j_n}\) uniquely determine \(\pi \in S_{m+n}\). By our construction and definitions it is apparent that

\[
\bigcup_{(s_1, \ldots, s_n) \in \Gamma^\beta_{j_0, \ldots, j_n}} \left\{\Delta^\alpha_{j_0, \ldots, j_n}[0, s_{\beta(u_1)}; 0] \cap \cdots \cap \Delta^\alpha_{j_0, \ldots, j_n}[s_{\beta(u_n)}; t] \right\}(s_1, \ldots, s_n)
\]

\[
= \bigcup_{(s_1, \ldots, s_n) \in \Gamma^\beta_{j_0, \ldots, j_n}} \left\{g^\alpha_{j_0, \ldots, j_n}(s_1, \ldots, s_n)\right\}
\]

\[
= \Delta^t_{m+n}(\pi).
\]
Because of our integrability assumption on the $B_i$'s, our integrand is Bochner integrable. Fubini's theorem can be applied to obtain

$$\int_{\Delta_{m+n}(\pi)} C_{\pi(m+n)}(s_{\pi(m+n)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \left( \mu^m \times \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right) (ds_1, \ldots, ds_{m+n})$$

$$= \int_{1\beta} \left( \int_{[0, \infty]} A^{jn} B_{\pi(u_n)}(s_{\pi(u_n)}) A^{jn-1} \cdots \right.$$

$$\left. \cdots A^{j_1} B_{\pi(u_1)}(s_{\pi(u_1)}) A^{j_0} d\mu^m \right) \left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right)$$

(18)

$$= \frac{1}{j_0! \cdots j_n!} \int_{1\beta} \epsilon_{j_0, \ldots, j_n} A^{jn} B_{\pi(u_n)}(s_{\pi(u_n)}) A^{jn-1} \cdots$$

$$A^{j_1} B_{\pi(u_1)}(s_{\pi(u_1)}) A^{j_0} d\left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right)$$

where

(19) \( \epsilon_{j_0, \ldots, j_n} = \left( \mu \left( \Delta_{j_0}[0, s_{\beta(u_1)}] \right) \cdots \mu \left( \Delta_{j_n}[s_{\beta(u_n)}, t] \right) \right)^{j_0} \cdots \right.$$
and so the disentangling $P^{m,n_1,...,n_k}_{\mu,\nu_1,...,\nu_k}(A,B_1(\cdot),\ldots,B_k(\cdot))$ is given by

$$
P^{m,n_1,...,n_k}_{\mu,\nu_1,...,\nu_k}(A,B_1(\cdot),\ldots,B_k(\cdot)) = \sum_{j_0+\cdots+j_n=m} \sum_{\sigma \in S_n} \frac{m!}{j_0! \cdots j_n!} \int_{\Delta_n(\sigma)} e^{\sigma} \cdot A^{j_n} B_{\sigma(n)}(s_{\sigma(n)}) A^{j_{n-1}} \cdots A^{j_1} B_{\sigma(1)}(s_{\sigma(1)}) A^{j_0} \cdot \left(\nu_1^{n_1} \times \cdots \times \nu_k^{n_k}\right) (ds_1,\ldots,ds_n).
$$

Hence we have proved the following theorem.

**Theorem 3.1.** Using the notation introduced above and assuming the hypotheses on $A,B_1(s),\ldots,B_k(s)$ and on the measures $\mu,\nu_1,\ldots,\nu_k$ stated at the beginning of this section, it follows that $P^{m,n_1,...,n_k}_{\mu,\nu_1,...,\nu_k}(A,B_1(\cdot),\ldots,B_k(\cdot))$ is given by the expression (21) and further, if

$$
f(z_0,z_1,\ldots,z_k) \in A_T(\|\mu\|_A,\int_{[0,T]} \|B_1(s)\| \|\nu_1\|(ds),\ldots,\int_{[0,T]} \|B_k(s)\| \|\nu_k\|(ds))
$$

has the usual power series expansion, we can write, for all $t \in [0,T]$

$$
f_{t;\mu,\nu_1,...,\nu_k}(A,B_1(\cdot),\ldots,B_k(\cdot)) = \sum_{m,n_1,...,n_k=0}^{\infty} c_{m,n_1,...,n_k} \sum_{j_0+\cdots+j_n=m} \sum_{\sigma \in S_n} \frac{m!}{j_0! \cdots j_n!} \int_{\Delta_n(\sigma)} e^{\sigma} \cdot A^{j_n} B_{\sigma(n)}(s_{\sigma(n)}) A^{j_{n-1}} \cdots A^{j_1} B_{\sigma(1)}(s_{\sigma(1)}) A^{j_0} \left(\nu_1^{n_1} \times \cdots \times \nu_k^{n_k}\right) (ds_1,\ldots,ds_n).
$$

We give a corollary to Theorem 3.1 which will point in the direction of the definition of the disentangling map when the infinitesimal generator of a $(C_0)$ contraction semigroup is involved. Keep in mind that, for now, the operator $A$ below is still a bounded operator. We are interested in the case where the function to be disentangled is

$$
f(z_0,z_1,\ldots,z_k) = e^{z_0} z_1^{n_1} \cdots z_k^{n_k} = e^{z_0} P^{n_1,...,n_k}(z_1,\ldots,z_k)
$$

with $n_1,\ldots,n_k$ fixed, and we want a formula for

$$
f_{t;\mu,\nu_1,...,\nu_k}(A^\sim,B_1(\cdot)^\sim,\ldots,B_k(\cdot)^\sim).
$$
Since \( f \) in (24) is entire, we need no restriction on the disentangling algebra; that is \( f \in \mathcal{A}(r_1, \ldots, r_k) \) for all choices of the radii \( r_1, \ldots, r_k \).

Using (21) and letting \( n = n_1 + \cdots + n_k \) we have

\[
(26) \quad f_{\nu_1,\nu_2,\ldots,\nu_k}(A^n, B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim)
= \sum_{m=0}^\infty \frac{1}{m!} P_{\nu_1,\nu_2,\ldots,\nu_k}(m, m_1, \ldots, m_k)(A^n, B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim)
= \sum_{n_0+\cdots+j_n=m} \sum_{s_\sigma \in S_n} \frac{1}{j_0! \cdots j_n!} \int_{\Delta^m_{\sigma}(\sigma)} A^{j_n} B_{\sigma(n)}(s_{\sigma(n)}) A^{j_n-1} \cdots
\]

\[
A^{j_n} B_{\sigma(1)}(s_{\sigma(1)}) A^{j_n} (\mu([0, \sigma(1)]))\frac{j_0!}{j_n!}
\]

\[
(\nu_1^{n_1} \times \cdots \times \nu_k^{n_k})(ds_1, \ldots, ds_n)
\]

\[
= \sum_{n_0+\cdots+j_n=m} \sum_{s_\sigma \in S_n} \int_{\Delta^m_{\sigma}(\sigma)} A^{j_n} B_{\sigma(n)}(s_{\sigma(n)}) A^{j_n-1} \cdots
\]

\[
A^{j_n} B_{\sigma(1)}(s_{\sigma(1)}) A^{j_n} (\mu([0, \sigma(1)]))\frac{j_0!}{j_n!}
\]

\[
(\nu_1^{n_1} \times \cdots \times \nu_k^{n_k})(ds_1, \ldots, ds_n)
\]

\[
= \sum_{s_\sigma \in S_n} \int_{\Delta^m_{\sigma}(\sigma)} \left\{ \sum_{j_n=0}^{\infty} \left( \frac{\mu([s_{\sigma(n)}, t])] A^{j_n}}{j_n!} \right) \right\} B_{\sigma(n)}(s_{\sigma(n)}) \cdot
\]

\[
\left\{ \sum_{j_n=0}^{\infty} \left( \frac{\mu([s_{\sigma(n)}, t))] A^{j_n}}{j_n!} \right) \right\} \left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right)(ds_1, \ldots, ds_n)
\]

\[
= \sum_{s_\sigma \in S_n} \int_{\Delta^m_{\sigma}(\sigma)} e^{\mu([s_{\sigma(n)}, d]]) A B_{\sigma(n)}(s_{\sigma(n)}) e^{\mu([s_{\sigma(n)}, t])] A} \cdots e^{\mu([s_{\sigma(2)}, t])] A} \cdot
\]

\[
B_{\sigma(1)}(s_{\sigma(1)}) \cdot e^{\mu([0, s_{\sigma(1)]}) A (\nu_1^{n_1} \times \cdots \times \nu_k^{n_k}) (ds_1, \ldots, ds_n)
\]

The corollary to Theorem 3.1 that we wish to record is the end result of the preceding calculation.

**Corollary 3.1.** **Under the assumptions of this section and with the function**
\[ f(z_0, z_1, \ldots, z_k) \text{ given by } (24), \text{ we obtain} \]

\[ f_{t; \mu, \nu_1, \ldots, \nu_k}(A^\sim, B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim) \]
\[ = \sum_{\sigma \in S_n} \int_{\Delta^\sigma_k} e^{\mu([s_{\sigma(n)}, t]_n)} A B_{\sigma(n)}(s_{\sigma(n)}) e^{\mu([s_{\sigma(n-1)}, s_{\sigma(n)}]_n)} A \ldots e^{\mu([s_{\sigma(1)}, s_{\sigma(2)}]_n)} A \cdot B_{\sigma(1)}(s_{\sigma(1)}) e^{\mu([0, 0, s_{\sigma(1)}]_n)} (\nu_1^{n_1} \times \cdots \times \nu_k^{n_k}) (ds_1, \ldots, ds_n). \]

The results above will motivate our definition of disentangling in the presence of a semigroup. We will make this definition in the next section and go on to consider evolution equations. To supply the needed motivation we carry out the following calculations where the operator \( A \), which will play the role of the semigroup generator below, remains a bounded operator. This allows the following calculations to be done rigorously.

With \( g(z_1, \ldots, z_k) \in \mathcal{A}(r_1, \ldots, r_k) \) where for \( i \in \{1, \ldots, k\} \)

\[ r_i = \int_{[0, T]} \| B_i(s) \| |\nu_i|(ds) < \infty, \]

we want a formula for

\[ h_{t; \mu, \nu_1, \ldots, \nu_k}(A^\sim, B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim), \quad 0 \leq t \leq T \]

where

\[ h(z_0, z_1, \ldots, z_k) = e^{z_0} g(z_1, \ldots, z_k). \]

Write \( g \) as

\[ g(z_1, \ldots, z_k) = \sum_{n_1, \ldots, n_k = 0}^{\infty} c_{n_1, \ldots, n_k} z_1^{n_1} \cdots z_k^{n_k}. \]

It is clear from (26) what that formula should be:

\[ h_{t; \mu, \nu_1, \ldots, \nu_k}(A^\sim, B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim) \]
\[ = \sum_{n_1, \ldots, n_k = 0}^{\infty} c_{n_1, \ldots, n_k} \sum_{\sigma \in S_{n_1 + \cdots + n_k}} \int_{\Delta^\sigma_{n_1 + \cdots + n_k}} e^{\mu([s_{\sigma(n)}, t]_{n_1 + \cdots + n_k})} A \cdot B_{\sigma(n)}(s_{\sigma(n)}) e^{\mu([s_{\sigma(n-1)}, s_{\sigma(n)}]_{n_1 + \cdots + n_k})} A \ldots e^{\mu([s_{\sigma(1)}, s_{\sigma(2)}]_{n_1 + \cdots + n_k})} A \cdot B_{\sigma(1)}(s_{\sigma(1)}) e^{\mu([0, 0, s_{\sigma(1)}]_{n_1 + \cdots + n_k})} (\nu_1^{n_1} \times \cdots \times \nu_k^{n_k}) (ds_1, \ldots, ds_{n_1 + \cdots + n_k}). \]
4. Evolution equations

Our next goal is to relate the work here to the solution of evolution equations. We will make use below of results from the paper of DeFacio, Johnson, and Lapidus [2]. Accordingly, we will make the same assumptions here as in that paper. (We will indicate in Remark 4.1 below ways in which the hypotheses of [2] can be reduced. These reductions are due to Reyes and can be found in his unpublished thesis along with some treatment of the case where the measures involved are allowed to have nonzero discrete parts.)

4.1. Hypotheses

(H.1) The operator \(-A\) will be the (not necessarily bounded) infinitesimal generator of a \((C_0)\) contraction semigroup of operators on a separable Hilbert space \(H\).

We will use the suggestive notation \(T(s) = e^{-sA}, 0 \leq s < \infty\), for the operators in the semigroup. These operators all belong to \(L(H)\) and the contraction assumption assures us that \(\|T(s)\| \leq 1\) for all \(s \in [0, \infty)\). (Actually the contraction assumption is not necessary but it simplifies the estimates and is satisfied anyway in the applications that we have primarily in mind.)

(H.2a) For each \(i = 1, \ldots, k\), let \(\mu_i\) be a \(\mathbb{C}\)-valued measure on \(B([0, T])\) such that \(|\mu_i|([0, T]) < \infty\).

The reader should think mainly in terms of nonnegative measures but signed or \(\mathbb{C}\)-valued measures are allowable.

(H.2b) Each \(\mu_i\) is a continuous measure; that is, \(\mu_i(\{s\}) = 0\) for every \(s \in [0, T]\).

(H.3a) Each \(B_i : [0, T] \to \mathcal{L}(H)\) has the property that \(B_i^{-1}(E) \in B([0, T])\) for every strong operator open subset \(E\) of \(\mathcal{L}(H)\).

(H.3b) For each \(i = 1, \ldots, k\),

\[
\int_{[0,T]} \|B_i(s)\| |\mu_i|(ds) < \infty.
\]

(H.3c) The family of operators \(\{B_i(s) : 0 \leq s \leq T\}\) is a commuting family for each \(i = 1, \ldots, k\).

We should emphasize that there is no assumption that \(B_i(s_1)\) commutes with \(B_j(s_2)\) when \(i \neq j\) nor that any \(B_i(s)\) commutes with the operators in the semigroup.
Remark 4.1. (a) Reyes has shown [10] that the main results of [2] carry over to the case where the separable Hilbert space \( \mathcal{H} \) is replaced by a separable Banach space \( X \) and the assumption (H.3c) of commutativity within the \( i \)-th family of operators for each fixed \( i \) is eliminated.

(b) The results to follow carry over without any essential change in the proofs to the interval \([0, \infty)\) provided that the following modest changes in the hypotheses are made: (i) The interval \([0, T]\) is replaced by \([0, \infty)\) throughout except that \( \mu_i([0, T]) \) is required to be finite only for every \( T > 0 \) with a similar requirement in formula (33). We further remark that if the semigroup from (H.1) is a unitary group, the hypotheses can be adjusted to fit the interval \((-\infty, \infty)\).

We are now ready to define the disentangling map in the case where the bounded operator \( A \) in formulas (26) and (32) is replaced by the infinitesimal generator \(-A\) of a \( (C_0) \) contraction semigroup \( \{T(t)\} \).

Definition 4.1. Let hypotheses (H.1)-(H.3) be satisfied, let \( \mu \) be a finite positive measure on \([0, T]\) and let \( f \) be given by (24). Then, motivated by (26) and its consistency with the ideas of Feynman’s operational calculus we define

\[
\begin{aligned}
    &f_{\nu_1, \nu_2, \ldots, \nu_n} (A^\sim, B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim) \\
    &= \exp_{\nu_1} (-A^\sim) P_{\nu_2, \ldots, \nu_k} (B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim) \\
    &= \sum_{\sigma \in S_n} \int_{\Delta^\sigma_n} e^{-\mu([s_1, t])A} B_{s_2} (s_1) e^{-\mu([s_1, s_2])A} B_{s_3} (s_2) e^{-\mu([s_2, s_3])A} B_{s_4} (s_3) \cdots \\
    &\quad \times (\nu_{s_1}^{n_1} \times \cdots \times \nu_{s_k}^{n_k}) (ds_1, \ldots, ds_n)
\end{aligned}
\]

where, as before, \( n = n_1 + \cdots + n_k \).

Next, motivated by (32), it is natural to define

\[
\exp_{\nu_1} (-A^\sim) g_{\nu_1, \nu_2, \ldots, \nu_n} (B_1(\cdot)^\sim, \ldots, B_k(\cdot)^\sim)
\]

\[
\begin{aligned}
    &:= \sum_{n_1, \ldots, n_k = 0} \infty c_{n_1, \ldots, n_k} \sum_{\sigma \in S_n} \int_{\Delta^\sigma_n} e^{-\mu([s_1, t])A} B_{s_2} (s_1) e^{-\mu([s_1, s_2])A} B_{s_3} (s_2) e^{-\mu([s_2, s_3])A} B_{s_4} (s_3) \cdots \\
    &\quad \times (\nu_{s_1}^{n_1} \times \cdots \times \nu_{s_k}^{n_k}) (ds_1, \ldots, ds_n)
\end{aligned}
\]

where \( g \) is given by (31) and where (28) is satisfied. As before, \( n = n_1 + \cdots + n_k \).

Formula (37) below is ‘derived’ in [2] by a series of steps some of which freely use Feynman’s heuristic ideas. Hence, the derivation there
is not mathematically rigorous. However, once formula (37) is arrived at in [2], it is shown that the expression makes sense mathematically and that it provides the unique solution to the evolution equation (41) below. Hence Feynman’s intuition that disentangling the exponential of a sum of integrals of operators gives the evolution of a physical system is justified mathematically under the present assumptions. (The operators just referred to come from the forces associated with the physical problem.) The difference in this paper is that (36) is derived from equation (9), Theorem 3.1 (which verifies consistency with earlier results for bounded operators), and Definition 4.1 without use of heuristic steps.

Before the statement of theorems giving solutions to evolution equations, we will first relate the disentangling given in (34) to the disentangling of the exponential function

\[
\exp \left( -tA + \int_{[0,t]} B_1(s)\nu_1(ds) + \cdots + \int_{[0,t]} B_k(s)\nu_k(ds) \right).
\]

as it appears in [2]. The result, in the notation being used here, is

\[
\exp_{\nu}(\cdot) \exp_{\nu}(\cdot + \cdots + \nu_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{\pi \in \mathcal{P}_{n_1,\ldots,n_k}} \int_{\Delta_{t_1} \times \cdots \times \Delta_{t_k}(\pi)} \exp(-t - t_{\pi(n)}A) \times \\
\times C_{\pi(n)}(t_{\pi(1)}) \cdots C_{\pi(n)}(t_{\pi(k)}) \exp(-(t_{\pi(2)} - t_{\pi(1)})A) \times \\
\times C_{\pi(1)}(t_{\pi(1)}) \exp(-t_{\pi(1)}A) \nu_{t_{\pi(1)}}^{m_1}(dt_{\pi(1)}, \ldots, dt_{n_1}) \times \cdots \times \\
\times \nu_{t_k}^{m_k}(dt_{n_1} + \cdots + n_{k-1} + 1, \ldots, dt_{n_k}).
\]

The second equation above is equation (3.14) in [2]. (It is convenient at this time to introduce, in the first equation above the notation used in [2] for the particular exponential function we are working with.) If we look at equation (34) we see that it is very similar to equation (37). In fact, the integrands are the same when we use \( \mu = \text{Lebesgue measure} \) in equation (34). If we use equation (34) to obtain a disentangling of the
exponential function (36) we obtain
\[
\exp \left( -tA + \int_{[0,t]} B_1(s)\nu_1(ds) + \ldots + \int_{[0,t]} B_k(s)\nu_k(ds) \right)
\]

(38) \quad = \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{n_1!\cdots n_k!} \sum_{\sigma \in S_n} \int_{\Delta_{n_1}(\sigma)} e^{-\mu([s_{\sigma(n)},t])A} B_{\sigma(n)}(s_{\sigma(n)}) \n\]
\[
e^{-\mu([s_{\sigma(n-1)},s_{\sigma(n)}])A} \ldots e^{-\mu([s_{\sigma(1)},s_{\sigma(2)}])A} B_{\sigma(1)}(s_{\sigma(1)}) e^{-\mu([0,s_{\sigma(1)})A} \n\]
\[
\left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right) (ds_1,\ldots,ds_n).
\]

On inspection of this result and on comparing it with (37), it is easy
to see that the main difference is in the sums over the different sets of
permutations. (The factorials are present in (38) and not in (34) since
the sums of the integrals of the \( B_i(\cdot) \)'s are exponentiated in (38) and
not in (34).) However, this difference is straight forward to eliminate.
The key is Proposition 2.5 of [4] and in particular in the discussion that
precedes this proposition. The immediate consequence of Proposition
2.5 in [4] is that

(39) \quad \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{n_1!\cdots n_k!} \sum_{\sigma \in S_n} \int_{\Delta_{n_1}(\sigma)} e^{-\mu([s_{\sigma(n)},t])A} B_{\sigma(n)}(s_{\sigma(n)}) \n\]
\[
e^{-\mu([s_{\sigma(n-1)},s_{\sigma(n)}])A} \ldots e^{-\mu([s_{\sigma(1)},s_{\sigma(2)}])A} B_{\sigma(1)}(s_{\sigma(1)}) e^{-\mu([0,s_{\sigma(1)})A} \n\]
\[
\left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right) (ds_1,\ldots,ds_n)
\]
\[
= \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{n_1!\cdots n_k!} \sum_{\sigma \in P_{n_1},\ldots,n_k} \int_{\Delta_{n_1} \times \cdots \times \Delta_{n_k}(\sigma)} e^{-\mu([s_{\sigma(n)},t])A}
\]
\[
B_{\sigma(n)}(s_{\sigma(n)}) e^{-\mu([s_{\sigma(n-1)},s_{\sigma(n)}])A} \ldots e^{-\mu([s_{\sigma(1)},s_{\sigma(2)}])A} B_{\sigma(1)}(s_{\sigma(1)}) \n\]
\[
e^{-\mu([0,s_{\sigma(1)})A} \left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right) (ds_1,\ldots,ds_n)
\]
\[
= \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{\sigma \in P_{n_1},\ldots,n_k} \int_{\Delta_{n_1} \times \cdots \times \Delta_{n_k}(\sigma)} e^{-\mu([s_{\sigma(n)},t])A} B_{\sigma(n)}(s_{\sigma(n)}) \n\]
\[
e^{-\mu([s_{\sigma(n-1)},s_{\sigma(n)}])A} \ldots e^{-\mu([s_{\sigma(1)},s_{\sigma(2)}])A} B_{\sigma(1)}(s_{\sigma(1)}) e^{-\mu([0,s_{\sigma(1)})A} \n\]
\[
\left( \nu_1^{n_1} \times \cdots \times \nu_k^{n_k} \right) (ds_1,\ldots,ds_n).
\]

This last expression is indeed identical to the result in [2] when \( \mu =
Lebesgue measure.

In view of this last calculation the following theorem is now conse-
quence of Theorem 5.1 in [2].
Theorem 4.1. Suppose that the hypotheses (H.1)-(H.3) above are satisfied. Then $L_t$, defined by

$$L_t := T_{t}^{\nu_1, \ldots, \nu_k} \exp \left( -t\tilde{A} + \int_0^t \tilde{B}_1(s)\nu_1(ds) + \cdots + \int_0^t \tilde{B}_k(s)\nu_k(ds) \right)$$

which is equal to the final expression in equation (39) above in the case where $\mu$ is Lebesgue measure, satisfies the integral equation

$$L_t = e^{-tA} + \sum_{l=1}^{k} \int_0^t e^{-(t-s)A}B_l(s)L_s\nu_l(ds)$$

where as usual the integrals are Bochner integrals in the strong operator sense.

Since (44) above has given us the equality of the disentangled expression from this paper and from (3.14) of [2], the following theorems which deal with continuity in time and uniqueness, respectively, are immediate consequences of Theorems 5.1 and 6.2 from [2].

Theorem 4.2. Suppose that hypotheses (H.1)-(H.3) above are satisfied. Then the function $L_t$ is strong operator continuous as a function of $t$ on $[0, \infty)$.

Theorem 4.3. Suppose that hypotheses (H.1)-(H.3) are satisfied. Let $M : [0, \infty) \to \mathcal{L}({\mathcal{H}})$ be any function which is: (i) measurable as a function of $t$ in the sense described in (H.3a), (ii) bounded in operator norm on $[0, T]$ for every $T > 0$, and (iii) satisfies the integral equation (41) on $[0, T]$ for every $T > 0$. Then $M_t = L_t$ for every $t \in [0, \infty)$.

We finish this section with some brief remarks about applications. The integrated forms of many familiar partial differential equations are of the type (41) for some choice of a semigroup generator-$A$, possibly time-dependent operators $B_1(\cdot), \ldots, B_k(\cdot)$ and measures $\nu_1, \ldots, \nu_k$. For example, if $A = (i/2)\Delta, k = 1, B_1(\cdot) = -iV$ where $V$ represents the operators of multiplication by a potential $V$, and $\nu_1 = l$, then (41) is the integrated form of the Schrödinger equation. The $B_i(\cdot)$’s need not be restricted to those coming from models which tend to be familiar to mathematicians. For example, nonlocal potentials which are represented mathematically by certain integral operators are used often in connection with ‘few body problems’ as effective phenomenological models in nuclear theory and in other areas of science. A somewhat more detailed discussion of such matters can be found in [5] and [2].
5. Stability results for the exponential disentangling

In this section we state a stability result (Theorem 3.2.1 of [9]) for the disentangling series for the exponential function

\[
\exp \left( -tA + \int_{[0,t]} B_1(s) \mu_1(ds) + \cdots + \int_{[0,t]} B_k(s) \mu_n(ds) \right).
\]

where \(-A\) is the generator of a \((C_0)\) contraction semigroup and where the notation in (42) was introduced in equation (37). Using the notation in (42) and the hypothesis (H.1) - (H.3) above, DeFacio, Johnson, and Lapidus used Feynman’s heuristic rules to write down the disentangled exponential. The following expression is arrived at in [2]. (The reader should compare this expression to that in Definition 4.1.)

\[
\exp \left( -tA + \int_{[0,t]} B_1(s) \mu_1(ds) + \cdots + \int_{[0,t]} B_k(s) \mu_k(ds) \right)
= \sum_{n_1, \ldots, n_k = 0}^{\infty} \sum_{\pi \in P_{n_1, \ldots, n_k}} \int (\Delta_{n_1} \times \cdots \times \Delta_{n_k}) \exp([-t - t_{\pi(n)}] A) \times \\
\times C_{\pi(n)}(t_{\pi(n)}) \cdots C_{\pi(2)}(t_{\pi(2)}) \exp([-t_{\pi(2)} - t_{\pi(1)}] A) \times \\
\times C_{\pi(1)}(t_{\pi(1)}) \exp([-t_{\pi(1)}] A) \mu_1^{n_1}(dt_{1}, \ldots, dt_{n_1}) \times \cdots \times \\
\times \mu_k^{n_k}(dt_{n_1 + \cdots + n_{k-1} + 1}, \ldots, dt_{n})
\]

where \(n = n_1 + \cdots + n_k\) as before. This is equation (3.14) in Section 3 of [2].

The stability theorem we will state in detail concerns the simplest case, namely the case where the operator-valued functions \(B_i(\cdot)\) are constant (in \(L(H)\)); i.e. the time independent case. However, aspects of other stability theorems obtained in [9] will be discussed below. In the theorem to be stated below, the stability to be considered is with respect to the measures. In this situation, we will choose a weakly convergent sequence of measures associated with each of the operator-valued functions \(B_i(\cdot)\). The result that will be stated will assert that the exponential disentangling series obtained for each element of the sequence will converge in the strong operator topology to the disentangling series corresponding to the limit measures.

The reader will note that, even though the statement of the theorem assumes that the measures are probability measures on \([0, t]\), one could instead use measures that are general finite continuous Borel measures and make the same proof after normalizing the measures.
In the statement of the theorem we use the following notation:

\[
L_t = \sum_{n_1, \ldots, n_k=0}^{\infty} L_{t, n_1, \ldots, n_k},
\]

\[
L_{t, n_1, \ldots, n_k} = \sum_{\pi \in P_{n_1, \ldots, n_k}} \int (\Delta_{n_1} \times \cdots \times \Delta_{n_k})(\pi) \exp(-(t - t_\pi(n_1))A) \times C_{t_\pi(n_1)}(t_\pi(n_1)) \cdots C_{t_\pi(n_k)}(t_\pi(n_k)) \exp(-(t_\pi(n_2) - t_\pi(n_1))A) \times \cdots \times \mu_{n_1}(dt_1, \ldots, dt_{n_1}) \times \cdots \times \mu_{n_k}(dt_{n_1+\cdots+n_k-1+1}, \ldots, dt_{n_k}).
\]

We will also use the notion of weak convergence of measures. To remind the reader of this type of convergence we recall that a sequence \( \{\nu_m\} \) of probability measures on a separable metric space \( X \) converges weakly to the probability measure \( \nu \) provided that

\[
\int_X f d\nu_m \to \int_X f d\nu
\]

for all bounded continuous functions \( f \). This convergence is denoted by \( \nu_m \rightharpoonup \nu \).

**Theorem 5.1.** Let \( B_1, \ldots, B_k \in \mathcal{L}(\mathcal{H}) \). We will consider these operators as constant functions from \( [0, t] \) to \( \mathcal{L}(\mathcal{H}) \) for all \( t > 0 \). Fix a \( t > 0 \) and let \( \mu_1, \ldots, \mu_k \) be continuous Borel probability measures on \( [0, t] \). Let \( \{\mu_{ip}\}_{p=1}^{\infty} \) for \( i = 1, \ldots, k \) be sequences of continuous Borel probability measures on \( [0, t] \) such that \( \mu_{ip} \rightharpoonup \mu_i \) for \( i = 1, \ldots, k \). Write \( L_{t, n_1, \ldots, n_k} \) and \( L_{t,p} \) to denote the same expressions as defined in equation (44) but written using the \( p^{th} \) term of each sequence of measures.

It follows that \( \lim_{p \to \infty} L_{t,p} = L_t \) in the strong operator topology on \( \mathcal{L}(\mathcal{H}) \).

We include a proof of this theorem in an interesting case which was not included in [9]. Suppose that \( p_{jm}(x), p_j, j = 1, \ldots, k, m \in \mathbb{N} \), are probability densities on \( [0, t] \); i.e. \( p_{jm}, p_j \geq 0, \int_0^t p_{jm}(x) \, dx = \int_0^t p_j(x) \, dx = 1 \) for all \( m \in \mathbb{N} \). Let \( \nu_{jm}, \nu_j \) denote the probability measures obtained from these densities. If \( p_{jm} \to p_j \) Lebesgue almost everywhere on \( [0, t] \), Scheffe’s Theorem [1, page 224] implies that \( \nu_{jm} \rightharpoonup \nu_j \).
We may write, using these measures in the expression for \( L_{t,m} \) above, (45)
\[
L_{t,m} = \sum_{n_1, \ldots, n_k = 0}^{\infty} \sum_{\pi \in \mathcal{P}_{n_1, \ldots, n_k}} \int_{(\Delta_{n_1} \times \cdots \times \Delta_{n_k}) \pi} \exp(-(t - t_{\pi(n)}) A) \times \\
\times C_{\pi(n)}(t_{\pi(n)}) \cdots C_{\pi(2)}(t_{\pi(2)}) \exp(-(t_{\pi(2)} - t_{\pi(1)}) A) \times \\
\times C_{\pi(1)}(t_{\pi(1)}) \exp(-t_{\pi(1)} A) p_{1m}(t_1) \cdots p_{1m}(t_{n_1}) p_{2m}(t_{n_1+1}) \cdots \\
p_{2m}(t_{n_1+n_2}) \cdots p_{km}(t_{n_1+\cdots+n_{k-1}+1}) \cdots p_{km}(t_{n}) \\
l^{n_1}(dt_1, \ldots, dt_{n_1}) \times \cdots \times l^{n_k}(dt_{n_1+\cdots+n_{k-1}+1}, \ldots, dt_{n}),
\]
where \( l \) denotes Lebesgue measure. Using this expression with \( \phi \in \mathcal{H} \) we find at once that (46)
\[
\|L_{t,m} \phi - L_t \phi\| \leq \|\phi\| \sum_{n_1, \ldots, n_k = 0}^{\infty} \sum_{\pi \in \mathcal{P}_{n_1, \ldots, n_k}} \int_{(\Delta_{n_1} \times \cdots \times \Delta_{n_k}) \pi} \|B_1\|^{n_1} \cdots \|B_k\|^{n_k} \\
\times |p_{1m}(t_1) \cdots p_{1m}(t_{n_1}) p_{2m}(t_{n_1+1}) \cdots p_{2m}(t_{n_1+n_2}) \cdots p_{km}(t_{n_1+\cdots+n_{k-1}+1}) \\
\cdots p_{k}(t_{n_1+\cdots+n_{k-1}+1}) \cdots p_{k}(t_{n})| l^n(dt_1, \ldots, dt_{n}).
\]
But for each \( k \in \mathbb{N} \) the product of the \( p_{jm} \)'s is a probability density function on \([0, t]^n\) as is the product of the \( p_j \)'s. Hence, because we have convergence of the sequence of densities on \([0, t]^n\) Lebesgue almost everywhere we can apply Scheffe’s Theorem again and, provided we can pass the limit on \( m \) through the sum over \( n_1, \ldots, n_k \) (which can, of course be done), we obtain our result.

The proof of Theorem 5.1 is quite different than the argument sketched out above. The rather strong conclusion stated in the theorem comes from the fact that we are working in the time independent case; i.e. the functions \( B_1, \ldots, B_k \) are constant operators.

If we allow time dependent functions in our exponential function, stability theorems become more difficult to obtain. However, by making appropriate norm-boundedness assumptions or integrability assumptions on the functions one can prove theorems similar in spirit to Theorem 5.1 under weak convergence or total variation convergence assumptions on the sequences of measures. The convergence of \( L_{t,m} \) will not generally be as strong as in the time independent case.

We further remark that stability theorems can be deduced by holding the measures fixed and considering sequences of operators or operator-valued functions. The most straightforward theorem to prove in this
situation is the one we obtain in the time independent case. If we move to
the time dependent case, we can still obtain the same type of convergence
as in the time independent case but the proofs are more complex.

6. Effects of commutativity

In this section we consider the effects of commutativity on the action
of the disentangling map. The time independent case is considered in
Section 3 of [4] and the time dependent case is considered in Section 2
of Chapter 2 of [9]. Since a number of the results in [4] have identical
proofs in the present time dependent setting, we will, for the most part,
simply state the propositions in the time dependent situation.

6.1. Assumptions

Here we list the following assumptions which will be made throughout
the remainder of the paper.

(A.1a) For each \( i = 1, \ldots, k \), let \( \mu_i \) be a \( \mathbb{C} \)-valued measure on \( B([0, T]) \)
such that \( |\mu_i([0, T])| < \infty \).

The reader should think mainly in terms of nonnegative measures but
signed or \( \mathbb{C} \)-valued measures are allowable.

(A.1b) Each \( \mu_i \) is a continuous measure; that is, \( \mu_i(\{s\}) = 0 \) for every
s \( \in [0, T] \).

(A.2a) Each \( B_i : [0, T] \to \mathcal{L}(X) \) has the property that \( B_i^{-1}(E) \in B([0, T]) \)
for every strong operator open subset \( E \) of \( \mathcal{L}(X) \) where \( X \)
is a Banach space.

(A.2b) For each \( i = 1, \ldots, k \),

\[
\int_{[0,T]} \|B_i(s)\| |\mu_i|(ds) < \infty.
\]

In the following proposition, the proof differs enough from the proof
of Proposition 3.1 of [4] that we will include it here.

**Proposition 6.1.** Let (A.1)-(A.2) hold and further suppose that
\( B_i(s)B_j(t) = B_j(t)B_i(s) \) whenever the products are defined. Then for
all choices of the measures \( \mu_1, \ldots, \mu_n \) on \([0, T] \) we have

\[
T_{\mu_1, \ldots, \mu_n} P^{m_1, \ldots, m_n} (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)
\]

\[
\left( \int_{[0,T]} B_1(s) \mu_1(ds) \right)^{m_1} \cdots \left( \int_{[0,T]} B_n(s) \mu_n(ds) \right)^{m_n}
\]
where the product on the right hand side of (48) may be written in any order. Further, for all \( f \in \mathbb{D}_T (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim) \),
\[
T_{\mu_1,\ldots,\mu_n} f (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)
\]
\[
= f \left( \int_{[0,T]} B_1(s) \mu_1(ds), \ldots, \int_{[0,T]} B_n(s) \mu_n(ds) \right)
\]
where \( f \) given by
\[
f(z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^\infty c_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n}
\]
is an element of \( \mathbb{A}_T(r_1, \ldots, r_n) \) where
\[
r_i = \int_{[0,T]} \| B_i(s) \| |\mu_i|(ds), i = 1, \ldots, n.
\]

Proof. For given measures \( \mu_1, \ldots, \mu_n \) the operator \( T_{\mu_1,\ldots,\mu_n} f (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim) \) is given in terms of \( T_{\mu_1,\ldots,\mu_n} P_{m_1,\ldots,m_n} (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim) \) and so it suffices to establish equation (48).

Let \( m_1, \ldots, m_n \in \mathbb{N} \). We can write, using Proposition 2.5 of [4] and the calculation done in (10) above in the second equality below,
\[
T_{\mu_1,\ldots,\mu_n} P_{m_1,\ldots,m_n} (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)
\]
\[
= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \cdot
\]
\[
(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_n)
\]
\[
= m_1! \cdots m_n! \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} \{ B_1(s_{1,1}) \cdots B_1(s_{1,m_1}) \} \cdots
\]
\[
\cdots \{ B_n(s_{n,1}) \cdots B_n(s_{n,m_n}) \} (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n})(ds_1, \ldots, ds_n)
\]
\[
= \frac{m_1!}{m_1!} \left\{ \int_{[0,T]} B_1(s) \mu_1(ds) \right\}^{m_1} \cdots \frac{m_n!}{m_n!} \left\{ \int_{[0,T]} B_n(s) \mu_n(ds) \right\}^{m_n}
\]
and this finishes the proof. \( \square \)

We see that, in this result, the maps \( B_i(\cdot) \) appear in separate integrals in the final expression for the disentangling (equation (49)). It is not surprising that we obtain such a result since the disentangling map
involves integration and the commutativity lets us split the integrals apart.

The next result is a lemma concerning what happens when one of the \( B_i \)'s commutes with all of the others.

**Lemma 6.1.** Let (A.1)-(A.2) hold and suppose that \( B_n(s)B_i(t) = B_i(t)B_n(s) \) for \( i = 1, \ldots, n - 1 \) whenever the product is defined. Then for any nonnegative integers \( m_1, \ldots, m_n \) we have

\[
P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n+1} (B_1(\cdot), \ldots, B_n(\cdot))
\]

\[
= \left( \int_{[0,T]} B_n(s)\mu_n(ds) \right) P_{\mu_1, \ldots, \mu_{n-1}, \mu_n}^{m_1, \ldots, m_{n-1}, m_n} (B_1(\cdot), \ldots, B_n(\cdot)).
\]

**Proof.** We take \( m = m_1 + \cdots + m_n \). There is a unique correspondence between permutations \( \tau \in S_m \) and pairs \((\pi, j)\) where \( \pi \in S_m \) and \( j \in \{1, \ldots, m+1\} \). This correspondence is established in the proof of Lemma 3.2 of [4] and we state the justification of this correspondence for clarity. The permutation \( \pi \) determines one of the \( m! \) possible strict orderings of the variables \((s_1, \ldots, s_m)\) in \([0, T]^m\): \( 0 < s_{\pi(1)} < s_{\pi(2)} < \cdots < s_{\pi(m)} < T \); the integer \( j \) picks out one of the \( m+1 \) intervals \((0, s_{\pi(1)}), \ldots, (s_{\pi(m)}, T)\) in which to insert \( s_{m+1} \). Thus a strict ordering of the variables \((s_1, \ldots, s_m, s_{m+1})\) in \([0, T]^{m+1}\), and hence, an associated permutation \( \tau \in S_{m+1} \) is determined.

Using this correspondence and our assumed commutativity, we obtain (abbreviating the argument presented in [4])

\[
P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n+1} (B_1(\cdot), \ldots, B_n(\cdot))
\]

\[
= \sum_{\tau \in S_{m+1}} \int_{\Delta_{m+1}^{(\tau)}} C_{\tau^{(m+1)}}(s_{\tau^{(m+1)}}) \cdots C_{\tau^{(1)}}(s_{\tau^{(1)}}) \cdot
\]

\[
(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n+1}) (ds_1, \ldots, ds_{m+1})
\]

\[
= \sum_{\pi} \sum_{j=1}^{m+1} \int_{\Delta_{m}^{(\pi)}} \left\{ \int_{(s_{\pi(j-1)}, s_{\pi(j)})} C_{\pi^{(m)}}(s_{\pi^{(m)}}) \cdots C_{\pi^{(j)}}(s_{\pi^{(j)}}) B_n(s_{m+1}) \cdot
\]

\[
C_{\pi^{(j-1)}}(s_{\pi^{(j-1)}}) \cdots C_{\pi^{(1)}}(s_{\pi^{(1)})}) \mu_n(ds_{m+1}) \right\} \cdot
\]

\[
(\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots, ds_m)
\]

\[
= \left( \int_{[0,T]} B_n(s)\mu_n(ds) \right) \sum_{\pi} \int_{\Delta_{m}^{(\pi)}} C_{\pi^{(m)}}(s_{\pi^{(m)}}) \cdots
\]

\[
\cdots C_{\pi^{(1)}}(s_{\pi^{(1)}}) (\mu_1^{m_1} \times \cdots \times \mu_n^{m_n}) (ds_1, \ldots, ds_m).
\]
This is the required result.

The next proposition is a generalization of the preceding lemma.

**Proposition 6.2.** Let (A.1)-(A.2) hold and suppose that $B_n(s)$ commutes with $B_i(s)$ for $i = 1, \ldots, n - 1$ whenever both are defined. Then given measures $\mu_1, \ldots, \mu_n$ on $[0, T]$ and $m_1, \ldots, m_n \in \mathbb{N}$ we have

$$P_{\mu_1, \ldots, \mu_{n-1}, \mu_n}^{m_1, \ldots, m_{n-1}, m_n} (B_1(\cdot), \ldots, B_{n-1}(\cdot), B_n(\cdot))$$

$$= \left( \int_{[0, T]} B_n(s) \mu_n(ds) \right)^{m_n} P_{\mu_1, \ldots, \mu_{n-1}}^{m_1, \ldots, m_{n-1}} (B_1(\cdot), \ldots, B_{n-1}(\cdot)).$$

**Proof.** Apply lemma 6.1 $m_n$ times and apply Proposition 2.9 of [4].

The next proposition tells us what occurs when a block of the $B_i$’s commute with all $n$ of the $B_i$. We remark here that the proof of this result is essentially identical to the proof of Proposition 3.4 of [4].

**Proposition 6.3.** Assume that (A.1)-(A.2) hold. Let $j \in \{0, \ldots, n-1\}$ and suppose each $B_{j+1}(\cdot), \ldots, B_n(\cdot)$ commutes with all of $B_1(\cdot), \ldots, B_n(\cdot)$ when the products are defined. Then given any finite continuous Borel measures $\mu_1, \ldots, \mu_n$ on $[0, T]$ and nonnegative integers $m_1, \ldots, m_n$ we have

$$P_{\mu_1, \ldots, \mu_{j+1}, \mu_{j+2}, \ldots, \mu_n}^{m_1, \ldots, m_{j+1}, m_{j+2}, \ldots, m_n} (B_1(\cdot), \ldots, B_{j+1}(\cdot), B_{j+2}(\cdot), \ldots, B_n(\cdot))$$

$$= \left( \int_{[0, T]} B_{j+1}(s) \mu_{j+1}(ds) \right)^{m_{j+1}} \cdots \left( \int_{[0, T]} B_n(s) \mu_n(ds) \right)^{m_n} P_{\mu_1, \ldots, \mu_j}^{m_1, \ldots, m_j} (B_1(\cdot), \ldots, B_j(\cdot)).$$

Next we move on to the analogue of Proposition 3.7 of [4]. the proof is extremely simple in both settings and will be omitted. We state it as follows.

**Proposition 6.4.** Suppose that (A.1)-(A.2) hold and that $B_1(s) = kI$ for all $s \in [0, T]$ where $k \in \mathbb{C}$. Then

$$P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n} (B_1(\cdot), \ldots, B_n(\cdot))$$

$$= (\mu_1([0, T])k)^{m_1} P_{\mu_2, \ldots, \mu_n}^{m_2, \ldots, m_n} (B_2(\cdot), \ldots, B_n(\cdot)).$$
and so
\begin{equation}
    f_{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot)) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \left( \mu_1([0,T])k \right)^{m_1} P_{m_2, \ldots, m_n}^m (B_2(\cdot), \ldots, B_n(\cdot))
\end{equation}
where $f \in \mathbb{D}_T$ has the usual power series expansion.

For the analogue to Proposition 3.8 of [4], we consider the case where $\mu_2 = \mu_1$ and $B_2(s) = B_1(s)$ $\mu_2$-a.e. Note that even though these two functions are equal a.e., we consider the corresponding formal objects as being distinct.

**Proposition 6.5.** Assume that (A.1)-(A.2) hold. Suppose that $\mu_2 = \mu_1$ and $B_2(s) = B_1(s)$ $\mu_2$-a.e. Then
\begin{equation}
    P_{m_1, m_2, m_3, \ldots, m_n}^{m_1, m_2, m_3, \ldots, m_n} (B_1(\cdot), B_2(\cdot), B_3(\cdot), \ldots, B_n(\cdot)) = P_{m_1 + m_2, m_3, \ldots, m_n}^{m_1 + m_2, m_3, \ldots, m_n} (B_1(\cdot), B_3(\cdot), \ldots, B_n(\cdot)).
\end{equation}
Also, if $f \in \mathbb{A}_T (r_1, \ldots, r_n)$ has the usual power series expansion, and we let
\begin{equation}
    g(z_1, z_3, \ldots, z_n) = \sum_{m_1', m_3, \ldots, m_n = 0}^{\infty} d_{m_1', m_3, \ldots, m_n} \left( \frac{z_1}{z_3} \right)^{m_1'} z_3^{m_3} \cdots z_n^{m_n}
\end{equation}
where
\begin{equation}
    d_{m_1', m_3, \ldots, m_n} = \sum_{m_1 + m_2 = m_1'} c_{m_1, \ldots, m_n},
\end{equation}
then $g \in \mathbb{A}_T (r_1, r_3, \ldots, r_n)$ and
\begin{equation}
    f_{\mu_1, \mu_2, \mu_3, \ldots, \mu_n} (B_1(\cdot), B_2(\cdot), B_3(\cdot), \ldots, B_n(\cdot)) = g_{\mu_1, \mu_3, \ldots, \mu_n} (B_1(\cdot), B_3(\cdot), \ldots, B_n(\cdot)).
\end{equation}

**Proof.** The proof of this proposition is the same as the proof of Proposition 3.8 in [4] keeping in mind that the equality of $B_1$ and $B_2$ is true almost everywhere. It is easy to see that this doesn’t affect any of the calculations in the proof of Proposition 3.8 of [4] since these calculations involve integration, and, as is well known, sets of measure zero do not change the value of an integral. The other calculations in Proposition 3.8 of [4] all involve norms, and these do not change since the norm in the time-dependent case has the same properties. \qed
Our last result in this section is a straightforward extension of Proposition 3.9 of [4].

**Proposition 6.6.** Suppose that $Y$ and $Z$ are Banach spaces and that $X = Y \oplus Z$ is the direct sum of $Y$ and $Z$ with $\|x\|_X := \|y\|_Y + \|z\|_Z$ for $x = (y, z) \in X$. Further suppose that $B_j : [0, T] \to \mathcal{L}(Y)$ and $C_j : [0, T] \to \mathcal{L}(Z)$, $j = 1, \ldots, n$, satisfy assumption (A.1) and let $\mu_1, \ldots, \mu_n$ be measures satisfying (A.2) for $B_1, \ldots, B_n$ as well as for $C_1, \ldots, C_n$. Then $A_j(s) := B_j(s) \oplus C_j(s)$ is an integrable operator-valued function on $X$ with respect to $\mu_j$ with values in $\mathcal{L}(X)$ for $j = 1, \ldots, n$. Further

\[
P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n} (A_1(\cdot), \ldots, A_n(\cdot)) = P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n} (B_1(\cdot), \ldots, B_n(\cdot)) \oplus P_{\mu_1, \ldots, \mu_n}^{m_1, \ldots, m_n} (C_1(\cdot), \ldots, C_n(\cdot)).
\]

Also, if $f \in \mathcal{A}_T (r_1^A, \ldots, r_n^A)$, then $f \in \mathcal{A}_T (r_1^B, \ldots, r_n^B) \cap \mathcal{A}_T (r_1^C, \ldots, r_n^C)$ where

\[
r_j^A = \int_{[0, T]} \|A_j(s)\| \, |\mu_j|(ds),
\]

\[
r_j^B = \int_{[0, T]} \|B_j(s)\| \, |\mu_j|(ds),
\]

and

\[
r_j^C = \int_{[0, T]} \|C_j(s)\| \, |\mu_j|(ds)
\]

for $j = 1, \ldots, n$ and

\[
f_{\mu_1, \ldots, \mu_n} (A_1(\cdot), \ldots, A_n(\cdot)) = f_{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot)) \oplus f_{\mu_1, \ldots, \mu_n} (C_1(\cdot), \ldots, C_n(\cdot)).
\]

**7. Symmetry**

We consider in this brief section conditions under which the expression $f_{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot))$ possesses symmetry properties. This section parallels Section 4 of [4]. We state the following proposition which corresponds to Proposition 4.1 of [4].

**Proposition 7.1.** Let $B_i : [0, T] \to \mathcal{L}(X)$, $i = 1, \ldots, n$, and $\mu_i$, $i = 1, \ldots, n$ satisfy (A.1)-(A.2).

Let $f \in \mathcal{A}_T (r_1, \ldots, r_n)$ with the usual power series expansion. Suppose that there is some subsequence $i_1, \ldots, i_l$ of $1, \ldots, n$ such that the function $f$ is symmetric in the variables $z_{i_1}, \ldots, z_{i_l}$.
Then the function $f$ belongs to $\mathbb{A}_T(v_1, \ldots, v_n)$ where $v_i = r_i$ if $i \notin \{i_1, \ldots, i_l\}$ and $v_i = \max(r_i_1, \ldots, r_i_l)$ if $i \in \{i_1, \ldots, i_l\}$. Further, any permutation of the operator-valued functions $B_1(\cdot), \ldots, B_n(\cdot)$ accompanied by the same permutation of the associated measures $\mu_1, \ldots, \mu_n$ leaves the expression $f_{\mu_1, \ldots, \mu_n}(B_1(\cdot), \ldots, B_n(\cdot))$ unchanged. Finally, if $\mu_1 = \cdots = \mu_l$, then the function $f_{\mu_1, \ldots, \mu_n}(B_1(\cdot), \ldots, B_n(\cdot))$ is a symmetric function of $B_1(\cdot), \ldots, B_n(\cdot)$.

Proof. The proof of this proposition is the same as the proof of Proposition 4.1 of [4] since the proof in [4] uses only the definition the disentangling map and this definition is the same in the time dependent case as in the time independent case.

8. Series and tensors in $\mathbb{D}_T$

This section parallels Section 5 of [4]. Here we consider, as indicated in the title of the section, disentangling of functions that are series or tensor products of functions in $\mathbb{D}_T$. In this paper we will address the time dependent version of the results in Section 5 of [4].

**Proposition 8.1.** Assume (A.1)-(A.2). Suppose also that
\[\sum_{j=1}^{\infty} f_j(B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)\] is a convergent series in the disentangling algebra $\mathbb{D}_T(B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)$, where for each $j$,
\[(65)\]
\[f_j(B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim) = \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1, \ldots, m_n}^{(j)} (B_1(\cdot)^\sim)^{m_1} \cdots (B_n(\cdot)^\sim)^{m_n}.\]

Then, letting
\[(66)\]
\[f(B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim) = \sum_{j=1}^{\infty} f_j(B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)\]

denote the sum of our series with $f$ given by the usual power series expansion (50) as an element of $\mathbb{A}_T(r_1, \ldots, r_n)$, we have that
\[(67)\]
\[c_{m_1, \ldots, m_n} = \sum_{j=1}^{\infty} c_{m_1, \ldots, m_n}^{(j)},\]

and
\[(68)\]
\[\|f(B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim)\| = \left\|\sum_{j=1}^{\infty} \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1, \ldots, m_n}^{(j)} r_1^{m_1} \cdots r_n^{m_n}\right\|.\]
Further, if
\[
\sum_{m_1, \ldots, m_n=0}^{\infty} \left( \sum_{j=1}^{\infty} |c_{m_1, \ldots, m_n}^{(j)}| \right) r_1^{m_1} \cdots r_n^{m_n} < \infty,
\]
then the series in (66) is absolutely convergent in $\mathbb{D}_T$, and
\[
\|f (B_1 (\cdot)^\sim, \ldots, B_n (\cdot)^\sim)\| \leq \sum_{j=1}^{\infty} \|f_j (B_1 (\cdot)^\sim, \ldots, B_n (\cdot)^\sim)\|
\]
\[
= \sum_{m_1, \ldots, m_n=0}^{\infty} \left( \sum_{j=1}^{\infty} |c_{m_1, \ldots, m_n}^{(j)}| \right) r_1^{m_1} \cdots r_n^{m_n}.
\]

Proof. Again, the proof is exactly the same as in [4], since only the norms come into play.

Analytic functions of $n$ variables which happen to be the tensor product of $n$ analytic functions of one variable play a role in Feynman's operational calculi. In particular, the function
\[
\exp (z_1 + \cdots + z_n) = \exp (z_1) \cdots \exp (z_n)
\]
plays a role in this paper as was seen above in connection with our evolution equation (41). The following proposition is the time dependent analogue of Proposition 5.2 of [4].

**Proposition 8.2.** Suppose that for $i = 1, \ldots, n$, $g_i \in \mathcal{A}_T(r_i)$ where
\[
g_i(z_i) = \sum_{m_i=0}^{\infty} c_{m_i}^{(i)} z_i^{m_i}.
\]
Then $g_1 \otimes \cdots \otimes g_n \in \mathcal{A}_T(r_1, \ldots, r_n)$ with
\[
(g_1 \otimes \cdots \otimes g_n) (z_1, \ldots, z_n) := g_1(z_1) \cdots g_n(z_n)
\]
\[
= \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1}^{(1)} \cdots c_{m_n}^{(n)} z_1^{m_1} \cdots z_n^{m_n}.
\]
Of course, it then follows that
\[
(g_1 \otimes \cdots \otimes g_n) (B_1 (\cdot)^\sim, \ldots, B_n (\cdot)^\sim) = g_1 (B_1 (\cdot)^\sim) \cdots g_n (B_n (\cdot)^\sim)
\]
\[
= \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1}^{(1)} \cdots c_{m_n}^{(n)} (B_1 (\cdot)^\sim)^{m_1} \cdots (B_n (\cdot)^\sim)^{m_n}.
\]
Finally,

\[ (g_1 \otimes \cdots \otimes g_n) (B_1(\cdot)^\sim, \ldots, B_n(\cdot)^\sim) = \|g_1 (B_1(\cdot)^\sim) \| \cdots \|g_n (B_n(\cdot)^\sim) \| \]

(75)

Proof. Since the time indices are suppressed when we form the formal commuting objects in the time dependent case, the proofs of equations (73) and (74) are exactly as in [4]. For the norm equality in equation (75), the proof is also the same as the proof in [4] with appropriate changes in the notation for the weights.

9. The effect of ordered supports on disentangling

The results of this section have proofs that are nearly identical to the corresponding theorems in Section 7 of [4] and consequently no proofs are presented here. However, time dependence does effect the statement of the theorems in some ways. Denote the support of the measure \( \mu \) by \( S(\mu) \).

The first theorem corresponds to Theorem 7.1 of [4]

**THEOREM 9.1.** Suppose (A.1)-(A.2) hold. If there exists \( \xi \in (0, T) \) such that \( S(\mu_1) \subset [0, \xi] \) and \( S(\mu_i) \subset [\xi, T] \) for \( i = 2, \ldots, n \), then

\[
P_{\mu_1, \mu_2, \ldots, \mu_n}^{m_1, m_2, \ldots, m_n} (B_1(\cdot), B_2(\cdot), \ldots, B_n(\cdot)) = P_{\mu_2, \ldots, \mu_n}^{m_2, \ldots, m_n} (B_2(\cdot), \ldots, B_n(\cdot)) \left( \int_{[0,T]} B_1(s) \mu_1(ds) \right)^{m_1}
\]

(76)

The following corollary corresponding to Corollary 7.3 of [4] is obtained from Theorem 9.1.

**COROLLARY 9.1.** Let (A.1)-(A.2) hold. Let \( \{i_1, \ldots, i_k\} \) be a subset of \( \{1, \ldots, n\} \) and suppose that the supports of the corresponding measures are ordered as follows:

\[
S(\mu_{i_1}) \leq S(\mu_{i_2}) \leq \cdots \leq S(\mu_{i_k}).
\]

(77)

Using the notation

\[
\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\} = \{j_1, \ldots, j_{n-k}\}, \quad j_1 < \cdots < j_{n-k},
\]

(78)

we finally assume that

\[
S(\mu_{i_k}) \leq S(\mu_{j_l}), \quad l = 1, \ldots, n-k.
\]

(79)
Then we have
\[
P_{m_1, \ldots, m_n}^{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot)) = P_{j_1, \ldots, j_{n-k}}^{m_{j_1}, \ldots, m_{j_{n-k}}} (B_{j_1}(\cdot), \ldots, B_{j_{n-k}}(\cdot)) \cdot \left( \int_{[0,T]} B_{i_k}(s) \mu_{i_k}(ds) \right)^{m_{i_k}} \cdots \left( \int_{[0,T]} B_{i_1}(s) \mu_{i_1}(ds) \right)^{m_{i_1}}.
\]

In particular, if \( k = n \), we have
\[
P_{m_1, \ldots, m_n}^{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot)) = \left( \int_{[0,T]} B_{i_n}(s) \mu_{i_n}(ds) \right)^{m_{i_n}} \cdots \left( \int_{[0,T]} B_{i_1}(s) \mu_{i_1}(ds) \right)^{m_{i_1}}.
\]

If the inequalities involved in the ordering of the supports in (77) and (79) are exactly reversed, then the ordering of the operators on the right-hand side of (80) is exactly reversed.

The preceding corollary has implications for the functions \( f_{\mu_1, \ldots, \mu_n} \) of the operator-valued functions \( B_1(\cdot), \ldots, B_n(\cdot) \).

**Corollary 9.2.** Let the assumptions of the first part of Corollary 9.1 be satisfied including the assumptions (77) and (79) involving the ordering of the supports. Then for any \( f \in \mathcal{D}_T(B_1(\cdot)\sim, \ldots, B_n(\cdot)\sim) \) given by
\[
f(z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n},
\]
we have
\[
T_{\mu_1, \ldots, \mu_n} f (B_1(\cdot)\sim, \ldots, B_n(\cdot)\sim) = \sum_{m_1, \ldots, m_n=0}^{\infty} c_{m_1, \ldots, m_n} B_{j_1, \ldots, j_{n-k}}^{m_{j_1}, \ldots, m_{j_{n-k}}} (B_{j_1}(\cdot), \ldots, B_{j_{n-k}}(\cdot)) \cdot \left( \int_{[0,1]} B_{i_k}(s) \mu_{i_k}(ds) \right)^{m_{i_k}} \cdots \left( \int_{[0,1]} B_{i_1}(s) \mu_{i_1}(ds) \right)^{m_{i_1}}.
\]
In particular, if \(k = n\) so that the supports are completely ordered and (81) holds, then

\[
\mathcal{T}_{\mu_1, \ldots, \mu_n} f (B_1(\cdot), \ldots, B_n(\cdot)) = \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1, \ldots, m_n} \left( \int_{[0,1]} B_{i_n}(s) \mu_{i_n}(ds) \right)^{m_{i_n}} \cdots \left( \int_{[0,1]} B_{i_1}(s) \mu_{i_1}(ds) \right)^{m_{i_1}}.
\]

Finally, if the inequalities involved in the ordering of the supports are exactly reversed, then the ordering of the operators in each term on the right-hand side of (83) is exactly reversed.

If the function \(f\) from (82) is the tensor product of \(n\) functions of a single complex variable, then the results of Corollary 9.2 take a particularly nice form. This is especially true when we consider the case \(n = k\) from the corollary above.

**Proposition 9.1.** Let (A.1)-(A.2) hold. Suppose that the supports of \(\mu_1, \ldots, \mu_n\) are totally ordered, that is

\[
S(\mu_{i_1}) \leq S(\mu_{i_2}) \leq \cdots \leq S(\mu_{i_n})
\]

where \(i_1, \ldots, i_n\) is some permutation of \(1, \ldots, n\). Further suppose that each \(g_i\) belongs to \(\mathcal{K}(r_i)\), where \(g_i\) is given by (72) for \(i = 1, \ldots, n\).

Then

\[
\mathcal{T}_{\mu_1, \ldots, \mu_n} (g_1 \otimes \cdots \otimes g_n) (B_1(\cdot), \ldots, B_n(\cdot)) = g_n \left( \int_{[0,T]} B_{i_n}(s) \mu_{i_n}(ds) \right) \cdots g_1 \left( \int_{[0,T]} B_{i_1}(s) \mu_{i_1}(ds) \right).
\]

In particular,

\[
\exp_{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot)) = \exp \left( \int_{[0,T]} B_{i_n}(s) \mu_{i_n}(ds) \right) \cdots \exp \left( \int_{[0,T]} B_{i_1}(s) \mu_{i_1}(ds) \right).
\]

Finally, we extend Proposition 9.1 to functions which are finite or infinite sums of tensors.

**Proposition 9.2.** Suppose that the measures \(\mu_1, \ldots, \mu_n\) satisfy the assumptions of the preceding proposition. Also for each \(j = 1, 2, \ldots \) and
\( i = 1, \ldots, n \), let \( g_{i,j} \in \mathcal{A}(r_i) \) so that \( g_{1,j} \otimes \cdots \otimes g_{n,j} \in \mathcal{A}(r_1, \ldots, r_n) \) for each \( j \). Finally, suppose that the series

\[
\sum_{j=1}^{\infty} g_{1,j} \otimes \cdots \otimes g_{n,j}
\]

(88)

converges in \( \mathcal{A}(r_1, \ldots, r_n) \).

Then

\[
\mathcal{T}_{\mu_1, \ldots, \mu_n} \left[ \sum_{j=1}^{\infty} (g_{1,j} \otimes \cdots \otimes g_{n,j}) (B_1(\cdot), \ldots, B_n(\cdot)) \right]
\]

(89)

\[
= \left( \sum_{j=1}^{\infty} g_{1,j} \otimes \cdots \otimes g_{n,j} \right)_{\mu_1, \ldots, \mu_n} (B_1(\cdot), \ldots, B_n(\cdot))
\]

\[
= \sum_{j=1}^{\infty} g_{n,j} \left( \int_{[0,T]} B_{i_n}(s) \mu_{i_n} (ds) \right) \cdots g_{i,j} \left( \int_{[0,T]} B_{i_1}(s) \mu_{i_1} (ds) \right).
\]

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