ON 3-ADDITIVE MAPPINGS AND COMMUTATIVITY IN CERTAIN RINGS

KYOO-HONG PARK AND YONG-SOO JUNG

Reprinted from the
Communications of the Korean Mathematical Society
Vol. 22, No. 1, January 2007

©2007 The Korean Mathematical Society
ON 3-ADDITIVE MAPPINGS AND COMMUTATIVITY IN CERTAIN RINGS

KYOO-HONG PARK AND YONG-SOO JUNG

Abstract. Let $R$ be a ring with left identity $e$ and suitably-restricted additive torsion, and $Z(R)$ its center. Let $H : R \times R \times R \to R$ be a symmetric 3-additive mapping, and let $h$ be the trace of $H$. In this paper we show that (i) if for each $x \in R$, 
\[ h(x), x \rceil_n = \langle \cdots \langle h(x), x \rceil, x \rceil, \ldots, x \rangle \in Z(R) \]
with $n \geq 1$ fixed, then $h$ is commuting on $R$. Moreover, $h$ is of the form 
\[ h(x) = \lambda_0 x^3 + \lambda_1(x)x^2 + \lambda_2(x)x + \lambda_3(x) \]
for all $x \in R$, where $\lambda_0 \in Z(R)$, $\lambda_1 : R \to R$ is an additive commuting mapping, $\lambda_2 : R \to R$ is the commuting trace of a bi-additive mapping and the mapping $\lambda_3 : R \to Z(R)$ is the trace of a symmetric 3-additive mapping; (ii) for each $x \in R$, either $h(x), x \rceil_n = 0$ or $\langle h(x), x \rceil, x \rceil = 0$ with $n \geq 0$, $m \geq 1$ fixed, then $h = 0$ on $R$, where $(y, x)$ denotes the product $yx + xy$ and $Z(R)$ is the center of $R$. We also present the conditions which implies commutativity in rings with identity as motivated by the above result.

1. Introduction

Throughout, $R$ will represent an associative ring, and $Z(R)$ will be its center. Let $x, y \in R$. The commutator $yx - xy$ will be denoted by $[y, x]$. We define the $(n + 1)$-tuple $(y, x_1, \ldots, x_n)$ as follows: $(y, x_1) := yx_1 + x_1y$ and $(y, x_1, \ldots, x_{n-1}, x_n) := \langle \langle y, x_1, \ldots, x_{n-1}, x_n \rangle \rangle$. In particular, in the case $x_1 = x_2 = \cdots = x_n = x$, $(y, x) \rangle_n$ will stand for the $(n + 1)$-tuple $(y, x, \ldots, x)$ and let $(y, x) = y$. We will also make extensive use of the following basic properties: for any $x, y, z \in R$, 
\[ [xy, z] = x[y, z] + [x, z]y, \quad [y, x] = \langle [y, x] \rangle \]
A mapping $f : R \to R$ is said to be commuting on $R$ if $[f(x), x] = 0$ for all $x \in R$. Similarly $f$ is called skew-commuting (resp. skew-centralizing) on $R$ if $f(x), x = 0$ (resp. $\langle f(x), x \rangle \in Z(R)$) for all $x \in R$. By analogy with the definition of $n$-commutativity introduced in [3], for $n \geq 2$ we define a mapping $f : R \to R$ to be $n$-skew-commuting (resp. $n$-skew-centralizing) on $R$ if

Received July 4, 2006.
2000 Mathematics Subject Classification. 16W20, 16U80, 16W25.
Key words and phrases. skew-commuting mappings, skew-centralizing mappings, commuting mappings, derivations.
Let \( \langle f(x), x^n \rangle = 0 \) (resp. \( \langle f(x), x^n \rangle \in Z(R) \)) for all \( x \in R \). An 1-skew-commuting mapping (resp. 1-skew-centralizing) is called simply a skew-commuting mapping (resp. skew-centralizing).

A map \( H : R \times R \times R \rightarrow R \) is said to be symmetric if \( H(x_1, x_2, x_3) = H(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) \) for all \( x_1, x_2, x_3 \in R \) for every permutation \( \{\pi(1), \pi(2), \pi(3)\} \). A map \( h : R \rightarrow R \) defined by \( h(x) = H(x, x, x) \) for all \( x \in R \), where \( H : R \times R \times R \rightarrow R \) is a symmetric map, is called the trace of \( H \). It is obvious that, in case when \( H : R \times R \times R \rightarrow R \) is a symmetric map which is also 3-additive (i.e., additive in each argument), the trace \( h \) of \( H \) satisfies the relation

\[
\begin{align*}
\text{h}(x + y) &= \text{h}(x) + \text{h}(y) + 3\text{H}(x, x, y) + 3\text{H}(x, y, y) \quad \text{for all } x, y \in R.
\end{align*}
\]

Bell and Lucier [1] obtained some results for skew-commuting and skew-centralizing additive maps in rings with left identity and recently, in [4] we obtained the similar results for bi-additive maps in rings with left identity.

The main purpose of this paper is to investigate 3-additive mappings in rings with left identity under some conditions and is to obtain the conditions which implies commutativity in rings with identity by using them.

2. Main results

Let \( R \) be a ring with left identity \( e \) and let \( n \) be any positive integer. The resulting tuple after the substitutions \( x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{j-1} = x_{j+1} = \cdots = x_{k-1} = x_{k+1} = \cdots = x_n = e \) and \( x_i = x_j = x_k = x \) in the \((n+1)\)-tuple \( \langle y, x_1, \ldots, x_n \rangle \) will be denoted by \( T_{i,j,k}(y, x, e) \) for all \( x_i, x_j, x_k, y \in R \), where \( i, j, k = 1, 2, \ldots, n \) with \( i \neq j \neq k \).

Similarly, the one after the substitutions \( x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_{j-1} = x_{j+1} = \cdots = x_n = e \) and \( x_i = x_j = x \) in the \((n+1)\)-tuple \( \langle y, x_1, \ldots, x_n \rangle \) will be denoted by \( T_{i,j}(y, x, e) \) for all \( x_i, x_j, y \in R \), where \( i, j = 1, 2, \ldots, n \) with \( i \neq j \). If \( i = j \), then \( T_{i,j}(y, x, e) \) stands for the tuple \( \langle y, x_1, \ldots, x_n \rangle \) such that \( x_i = x \) and \( x_l = e \) for all \( l \neq i \) and all \( x_i, y \in R \), where \( i, l = 1, 2, \ldots, n \).

We begin with the following result which is motivated by [1] and [2].

**Theorem 1.** Let \( n \geq 1 \). Let \( R \) be a \((n+2)!\)-torsion-free ring with left identity \( e \). Let \( H : R \times R \times R \rightarrow R \) be a symmetric 3-additive mapping and let \( h \) be the trace of \( H \). If \( \langle h(x), x, x \rangle \in Z(R) \) for all \( x \in R \), then \( h \) is commuting on \( R \). Moreover, \( h \) is of the form

\[
\text{h}(x) = \lambda_0 x^3 + \lambda_1(x) x^2 + \lambda_2(x) x + \lambda_3(x)
\]

for all \( x \in R \), where \( \lambda_0 \in Z(R) \), \( \lambda_1 : R \rightarrow R \) is an additive commuting mapping, \( \lambda_2 : R \rightarrow R \) is the commuting trace of a bi-additive mapping and the mapping \( \lambda_3 : R \rightarrow Z(R) \) is the trace of a 3-additive mapping.

**Proof.** We first remark that the relation \( [x, e]y = 0 \) holds for all \( x, y \in R \) since \( e \) is a left identity. Let \( n \geq 1 \). Since our assumption is

\[
(1) \quad \langle h(x), x, x \rangle = \langle h(x), x, x \rangle \in Z(R) \quad \text{for all } x \in R,
\]

we have

\begin{equation}
\langle h(e), e \rangle_n = \langle h(e), e \rangle_{n-1} + \langle h(e), e \rangle_{n-1} \in Z(R).
\end{equation}

Commuting with \( e \) gives \( \langle h(e), e \rangle_{n-1} = \langle h(e), e \rangle_{n-1} \). It follows from (1) that \( 2\langle h(e), e \rangle_{n-1} \in Z(R) \), hence \( \langle h(e), e \rangle_{n-1} \in Z(R) \). Continuing in the same manner with this expression, we arrive at \( \langle h(e), e \rangle = \langle h(e), e \rangle_1 \in Z(R) \), that is,

\begin{equation}
h(e) + h(e) \in Z(R).
\end{equation}

Commuting with \( e \) yields \( h(e) = h(e) \); and by (3), \( 2h(e) \in Z(R) \), and so \( h(e) \in Z(R) \).

Let \( t \) be any positive integer. Replacing \( x \) by \( x + te \) in (1) and noting that \( h(x + te) = h(x) + th(e) + 3tH(x, x, e) + 3t^2H(x, e, e) \) for all \( x \in R \), we obtain

\[ tp_1(x, e) + t^2p_2(x, e) + \cdots + t^{n+2}p_{n+2}(x, e) \in Z(R) \]

for all \( x \in R \), where \( p_k(x, e) \) is the sum of terms involving \( x \) and \( e \) such that \( p_k(x, te) = tkp_k(x, e) \), \( k = 1, 2, \ldots, n+2 \).

Replacing \( t \) by \( 1, 2, \ldots, n+2 \) in turn, and expressing the resulting system of \( n+1 \) non-homogeneous equations with the variables \( p_1(x, e), p_2(x, e), \ldots, p_{n+2}(x, e) \), we see that the coefficient matrix of the system is a van der Monde matrix

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
n+2 & (n+2)^2 & \cdots & (n+2)^{n+2}
\end{pmatrix}.
\]

Since the determinant of the matrix is equal to a product of positive integers, each of which is less or equal to \( n+1 \), and since \( R \) is \((n+2)!\)-torsion free, it follows immediately that for each \( k = 1, 2, \ldots, n+2 \),

\[ p_k(x, e) \in Z(R) \quad \text{for all} \quad x \in R. \]

In particular, we have for all \( x \in R \),

\begin{equation}
Z(R) \ni p_{n+2}(x, e) = \sum_{i=1}^{n} T_{i,i}(h(e), x, e) + 3(H(x, x, e), e, \ldots, e), \quad \text{n times}
\end{equation}

\begin{equation}
Z(R) \ni p_{n+1}(x, e) = 3(H(x, x, e), e, \ldots, e) + \sum_{i=1}^{n-1} T_{i,(i+1)}(h(e), x, e) \quad \text{n times}
\end{equation}

\begin{equation}
+ \sum_{i=2}^{n-1} T_{i,(i+1)}(h(e), x, e) + \sum_{i=2}^{n-2} T_{i,(i+2)}(h(e), x, e) + \sum_{i=1}^{n} 3T_{i,i}(H(x, e, e), x, e). \quad \text{n times}
\end{equation}
and

\[
Z(R) \ni P_n(x, e) = \langle h(x), e, \ldots, e \rangle + \sum_{k=2}^{n-2} \sum_{i=2}^{n-(k-2)} T_{1,i,[i+(k-2)]}(h(e), x, e)
\]

\[
+ \sum_{i=2}^{n-3} T_{2,(i+1),(i+2)}(h(e), x, e)
\]

\[
+ \sum_{i=2}^{n-3} T_{2,(i+1),(i+3)}(h(e), x, e)
\]

\[
+ \sum_{i=2}^{n-3} T_{3,(i+2),(i+3)}(h(e), x, e)
\]

\[
+ \sum_{k=2}^{n} \sum_{i=1}^{n-(k-1)} 3T_{(k-1),(i+(k-1))}(H(x, e, e), x, e)
\]

\[
+ \sum_{i=1}^{3} 3T_{1,i}(H(x, e, e), x, e).
\]

(6)

Since \( h(e) \in Z(R) \), the first sum in (4) becomes \( n2^n x h(e) \). A simple calculation shows that the second term in (4) makes \( 3 \{ (2^n - 1) H(x, e, e) e + H(x, e, e) \} \).

Hence we conclude that for all \( x \in R \),

(7) \( P_{n+2}(x, e) = n2^n x h(e) + 3 \{ (2^n - 1) H(x, e, e) e + H(x, e, e) \} \in Z(R) \).

By using \( [x, e]y = 0 \) for all \( x, y \in R \), commuting with \( e \) gives

(8) \[ [H(x, e, e), e] = 0 \quad \text{for all } x \in R; \]

that is, \( H(x, e, e) = H(x, e, e) e \) for all \( x \in R \).

Now it follows from (7) that

(9) \[ n2^n x h(e) + 3 \cdot 2^n H(x, e, e) \in Z(R) \quad \text{for all } x \in R; \]

and commuting with \( x \) in (9) yields

(10) \[ 3 \cdot 2^n [H(x, e, e), x] = 0 = [H(x, e, e), x] \quad \text{for all } x \in R. \]

On the other hand, it follows from an easy calculation that the first term in (5) becomes \( 3 \cdot (2^n - 1) H(x, e, e) e + H(x, e, e) \). Note that the total number of all the terms in \( P_{n+1}(x, e) \) is \( \frac{n^2 + n + 2}{2} \) if \( n > 1 \) and is 2 if \( n = 1 \). Since the number of terms of the second sum in (5) is \( n - 1 \) if \( n > 1 \), and the total number of terms of the third sum and the fourth sum in (5) is \( \frac{(n-2)(n-1)}{2} \), we see that the total sum of terms of the second sum, the third sum and the fourth sum in (5), by considering \( h(e) \in Z(R) \), amounts to \( \frac{(n-1)n}{2} 2^n x^2 h(e) \). The number of terms of the fifth sum in (5) is \( n \), and hence it follows from (8) and (10) that the sum of the terms comes to \( 3n 2^n x H(x, e, e) \).
Therefore we conclude that for all \( x \in R \),

\[
P_{n+1}(x, e) = 3\{(2^n - 1)H(x, x, e) + H(x, x, e)\} + \frac{(n-1)n}{2} 2^n x^2 h(e) + n 2^n x H(x, e, e) \in Z(R).
\]

By again using \([x, e]y = 0\) for all \( x, y \in R \), commuting with \( e \) gives

\[
[H(x, x, e), e] = 0 \quad \text{for all } x \in R,
\]

that is, \( H(x, x, e) = H(x, x, e)e \) for all \( x \in R \).

Therefore we can rewrite (11) in the form

\[
P_{n+1}(x, e) = 3 \cdot 2^n H(x, x, e) + \frac{(n-1)n}{2} 2^n x^2 h(e)
+ n 2^n x H(x, e, e) \in Z(R)
\]

for all \( x \in R \); thus commuting with \( x \) and using (10) give

\[
[H(x, x, e), x] = 0 \quad \text{for all } x \in R.
\]

Finally, the first term in (6) is \((2^n - 1)h(x)e + h(x)\). Using \(h(e) \in Z(R)\), we see that the total sum of terms of the second sum, the third sum, the fourth sum and the fifth sum in (6) is \(\alpha_n x^3 h(e)\), where \(\alpha_n = \frac{n^2 + 3n - 20}{2} \) if \( n \geq 4 \), \(\alpha_n = 1 \) if \( n = 3 \) and \(\alpha_n = 0 \) if \( n = 1, 2 \). From (8), (10), (12) and (13), it follows that the sixth sum in (6) is \(3\beta_n x^2 H(x, e, e)\), where \(\beta_n = \frac{(n-1)n}{2}\) and that the final sum in (6) is \(n 2^n x H(x, x, e)\).

Hence we have

\[
P_n(x, e) = (2^n - 1)h(x)e + h(x) + \alpha_n 2^n x^3 h(e)
+ 3\beta_n 2^n x^2 H(x, e, e) + 3n 2^n x H(x, x, e) \in Z(R).
\]

By using \([x, e]y = 0\) for all \( x, y \in R \), commuting with \( e \) yields

\[
[h(x), e] = 0 \quad \text{for all } x \in R,
\]

that is, \( h(x) = h(x)e \) for all \( x \in R \).

We now can rewrite (14) in the form

\[
P_n(x, e) = 2^n h(x) + \alpha_n 2^n x^3 h(e) + 3\beta_n 2^n x^2 H(x, e, e)
+ 3n 2^n x H(x, x, e) \in Z(R)
\]

for all \( x \in R \). Thus commuting with \( x \) and using (10) and (13) give

\[
[h(x), x] = 0 \quad \text{for all } x \in R.
\]

Moreover, (15) implies that

\[
P_n(x, e) = h(x) + \alpha_n h(e)x^3 + 3\beta_n H(x, x, e)x^2
+ 3n H(x, x, e)x \in Z(R).
\]

Let \(\lambda_0 = -\alpha_n h(e)\). Then we have \(\lambda_0 \in Z(R)\) since \(h(e) \in Z(R)\).
Let \( \lambda_1 : R \to R \) defined by \( \lambda_1(x) = -3\beta_n H(x, e, e) \) for all \( x \in R \) is additive and commuting by the additivity of \( H(x, e, e) \) and \([H(x, e, e), x] = 0 \) for all \( x \in R \), respectively.

Setting \( \lambda_2(x) = -3nH(x, x, e) \) for all \( x \in R \), a mapping \( \lambda_2 : R \to R \) is the commuting trace of a bi-additive mapping \( G : R \times R \to R \) defined by \( G(x, y) = -3nH(x, y, e) \) for all \( x, y \in R \) since \([H(x, x, e), x] = 0 \) for all \( x \in R \).

Hence we can rewrite (16) in the desired structure

\[
h(x) = \lambda_0 x^3 + \lambda_1(x)x^2 + \lambda_2(x)x + \lambda_3(x) \quad \text{for all } x \in R,
\]

where \( \lambda_3(x) \in Z(R) \), that is, we define a mapping \( \lambda_3 : R \to Z(R) \) which is the trace of a 3-additive mapping \( C : R \times R \to R \) defined by

\[
C(x, y, z) = H(x, y, z) + \alpha_n h(e)xyz + 3\beta_n H(x, e, e)yz + 3nH(x, y, e)z
\]

for all \( x, y, z \in R \). The proof is complete.

\( \square \)

**Theorem 2.** Let \( n \geq 1 \). Let \( R \) be a \((n+2)!\)-torsion-free ring with left identity \( e \). Let \( H : R \times R \times R \to R \) be a symmetric 3-additive mapping and let \( h \) be the trace of \( H \). If \( \langle h(x), x \rangle_n = 0 \) for all \( x \in R \), then we have \( H = 0 \).

**Proof.** We follow the same argument as in the proof of Theorem 1. In the proof of Theorem 1, letting \( Z(R) = \{0\} \) and then using (1), (2) and (3), we obtain \( h(e) = 0 \). Thus (7), in conjunction with (4), shows that

\[
(17) \quad 3\langle H(x, e, e), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle = 3\{(2^n - 1)H(x, e, e)e + H(x, e, e)\} = 0
\]

for all \( x \in R \). Right-multiply by \( e \), obtaining

\[
2^n H(x, e, e)e = 0 = H(x, e, e)e
\]

for all \( x \in R \), and so, by (17), \( H(x, e, e) = 0 \) for all \( x \in R \). Consequently, (5) becomes

\[
(18) \quad 3\langle H(x, e, e), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle = 3\{(2^n - 1)H(x, x, e)e + H(x, x, e)\} = 0
\]

for all \( x \in R \). Right-multiplying by \( e \) gives

\[
2^n H(x, x, e)e = 0 = H(x, x, e)e = 0
\]

which means that, in view of (18), \( H(x, x, e) = 0 \) for all \( x \in R \). Since we know that \( h(e) = 0 \), \( H(x, e, e) = 0 \) and \( H(x, x, e) = 0 \) for all \( x \in R \), it follows that

\[
(19) \quad \langle h(x), \overbrace{e, \ldots, e}^{n \text{ times}} \rangle = (2^n - 1)h(x)e + h(x) = 0 \quad \text{for all } x \in R.
\]

Again by right-multiplying by \( e \), we obtain \( 2^n h(x)e = 0 = h(x)e \) for all \( x \in R \), and so, by (19), \( h(x) = 0 \) for all \( x \in R \) which implies that \( H = 0 \). \( \square \)

The following is a result concerning \( m \)-skew-commuting mappings.
Theorem 3. Let \( n \geq 0 \) and \( m \geq 1 \). Let \( R \) be a \((n + m + 2)\)-torsion-free ring with left identity \( e \). Let \( H : R \times R \times R \to R \) be a symmetric 3-additive mapping and let \( h \) be the trace of \( H \). If the mapping \( x \mapsto \langle h(x), x \rangle \) is \( m \)-skew-commuting on \( R \), then we have \( H = 0 \).

Proof. Suppose that

\[
\langle \langle h(x), x \rangle, x^n \rangle = 0 \quad \text{for all} \quad x \in R.
\]

Then we get

\[
\langle \langle h(e), e \rangle, e^m \rangle = \langle h(e), e \rangle_n e + \langle h(e), e \rangle_{n+1} = 0;
\]

and right-multiplying by \( e \) gives \( 2\langle h(e), e \rangle_n e = 0 = \langle h(e), e \rangle_n e \). Hence (21) yields \( \langle h(e), e \rangle_n = 0 \). Using similar approach as in the proof of Theorem 1, we obtain \( h(e) = 0 \).

Let \( t \) be any positive integer. Replacing \( x \) by \( x + te \) in (20) and considering \( h(x + te) = h(x) + t^2 h(e) + 3tH(x, x, e) + 3t^2 H(x, e, e) \) for all \( x \in R \), we obtain

\[
tP_1(x, e) + t^2 P_2(x, e) + \cdots + t^{n+2} P_{n+m+2}(x, e) = 0 \quad \text{for all} \quad x \in H,
\]

where \( P_k(x, e) \) is the sum of terms involving \( x \) and \( e \) such that \( P_k(x, te) = P_k(x, e) \), \( k = 1, 2, \ldots, n + m + 2 \). Replacing \( t \) by \( 1, 2, \ldots, n + m + 2 \) in turn, and expressing the resulting system of \( n + m + 2 \) homogeneous equations with the variables \( P_1(x, e), P_2(x, e), \ldots, P_{n+m+2}(x, e) \), we see that the coefficient matrix of the system is a van der Monde matrix

\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 2^2 & \cdots & 2^{n+m+2} \\
\vdots & \vdots & \ddots & \vdots \\
n + m + 2 & (n + m + 2)^2 & \cdots & (n + m + 2)^{n+m+2}
\end{pmatrix}.
\]

Since the determinant of the matrix is equal to a product of positive integers, each of which is less or equal to \( n + m + 2 \), and since \( R \) is \((n + m + 2)!\)-torsion free, it follows immediately that for each \( k = 1, 2, \ldots, n + m + 2 \),

\[
P_k(x, e) = 0 \quad \text{for all} \quad x \in R.
\]

In particular, we have, by utilizing \( h(e) = 0 \) and \( e^m = e \), for all \( x \in R \),

\[
0 = P_{n+m+2}(x, e) = 3\langle H(x, e, e), e, \ldots, e \rangle \quad \text{for all} \quad x \in R,
\]

\[
0 = P_{n+m+1}(x, e)
\]

\[
= 3\langle H(x, e, e), e, \ldots, e \rangle + Q_{n+m+1} \quad \text{for all} \quad x \in R,
\]

where \( Q_{n+m+1} \) is the sum of all terms containing \( H(x, e, e) \) in \( P_{n+m+1}(x, e) \) and

\[
0 = P_{n+m}(x, e) = \langle h(x), e, \ldots, e \rangle + Q_{n+m} + R_{n+m},
\]
where \( Q_{n+m} \) and \( R_{n+m} \) are the sums of all terms containing \( H(x, e, e) \) and \( H(x, x, e) \), respectively, in \( P_{n+m}(x, e) \).

We now obtain from (22) that
\[
3\{(2^{n+1} - 1)H(x, e, e)e + H(x, e, e)e\} = 0 \quad \text{for all} \quad x \in R;
\]
and right-multiplying by \( e \) gives \( 3 \cdot 2^{n+2}H(x, e, e)e = 0 = H(x, e, e)e \) for all \( x \in R \), therefore, by (25), \( H(x, e, e) = 0 \) for all \( x \in R \). This forces (23) to
\[
\langle H(x, e, e), e, \ldots, e \rangle = 0 \quad \text{for all} \quad x \in R.
\]

By calculating (26), we get
\[
3\{(2^{n+1} - 1)H(x, x, e)e + H(x, x, e)e\} = 0 \quad \text{for all} \quad x \in R;
\]
and the right-multiplication by \( e \) yields \( 3 \cdot 2^{n+1}H(x, x, e)e = 0 = H(x, x, e)e \) for all \( x \in R \), hence, by (27), \( H(x, x, e) = 0 \) for all \( x \in R \). Since \( H(x, e, e) = 0 \) and \( H(x, x, e) = 0 \) holds for all \( x \in R \), it follows from (24) that
\[
\langle h(x), e, \ldots, e \rangle = 0
\]
which implies that \( (2^{n+1} - 1)b(x)e + b(x)e = 0 \) for all \( x \in R \). As above, we obtain \( b(x) = 0 \) for all \( x \in R \). This completes the proof of the theorem.

\(\square\)

**Example 1.** Let
\[
R = \left\{ \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} : w, x, y, z \in \mathbb{C} \right\},
\]
where \( \mathbb{C} \) is the set of complex numbers. Then \( R \) is a noncommutative associative ring with left identity, i.e., the unit matrix under the usual matrix addition and multiplication. We define a mapping \( f : R \to R \) by
\[
f(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & z & 0 \end{pmatrix} \quad \text{for all} \quad X \in R.
\]
It is obvious that \( f \) is additive.

On the other hand, putting
\[
Z(R) = \left\{ \begin{pmatrix} w & 0 & 0 \\ 0 & w & 0 \\ y & 0 & w \end{pmatrix} : w, y \in \mathbb{C} \right\},
\]
it is immediate to see that \( Z(R) \) is the center of \( R \).

Now, defining a mapping \( H : R \times R \times R \to R \) by
\[
H(X, Y, Z) = [f(X), Y] + [f(Y), X] + [f(Y), Z] + [f(Z), Y] + [f(Z), X] + [f(X), Z]
\]
Let $X, Y, Z \in R$, we can easily check that $H$ is 3-additive, and that the mapping $h$ on $R$ defined by $h(X) = H(X, X, X)$ for all $X \in R$ is the trace of $H$ such that $\langle h(X), X \rangle_n \in Z(R)$ for all $X \in R$.

3. Some results concerning the commutativity of rings with identity

Let $R$ be a ring. An additive mapping $d : R \to R$ is called a derivation if the Leibniz rule $d(xy) = d(x)y + xd(y)$ is valid for all $x, y \in R$.

In this section, we use the results in Section 2 to establish some results concerning the commutativity of rings with identity.

First, we need the following well-known lemma [5].

**Lemma 4.** Let $R$ be a prime ring. Let $d : R \to R$ be a nonzero derivation such that $d$ is commuting on $R$. Then $R$ is commutative.

**Theorem 5.** Let $n \geq 0$ and $m \geq 1$. Let $R$ be a $(n + m + 2)!$-torsion-free prime ring with identity. If there is a nonzero derivation $d : R \to R$ such that the mapping $x \mapsto \langle d(x), x \rangle_n$ is $m$-skew-centralizing on $R$, then $R$ is commutative.

**Proof.** We define a mapping $H : R \times R \times R \to R$ by

$$H(x, y, z) = [d(x), y] + [d(y), x] + [d(y), z] + [d(z), y] + [d(z), x] + [d(x), z]$$

for all $x, y, z \in R$. Then it is clear that $H$ is symmetric and 3-additive, and that the mapping $h : R \to R$ defined by $h(x) = H(x, x, x) = 6[d(x), x]$ for all $x \in R$ is the trace of $H$.

Since it follows from the hypothesis that $\langle \langle d(x), x \rangle_n, x^m \rangle \in Z(R)$ for all $x \in R$, we have, by recalling $\langle [y, x], x \rangle = \langle [y, x], x \rangle$,

$$[\langle d(x), x \rangle_n x^m + x^m \langle d(x), x \rangle_n, x] = 0 \text{ for all } x \in R,$$

which implies that $\langle [d(x), x]_n, x^m + x^m [\langle d(x), x \rangle_n, x] = 0 \text{ for all } x \in R$. This reduces to $\langle [d(x), x]_n x^m + x^m [\langle d(x), x \rangle_n, x] = 0 \text{ for all } x \in R$, that is, $\langle h(x), x \rangle_n x^m + x^m \langle h(x), x \rangle_n = 0 \text{ for all } x \in R$. Hence it follows from Theorem 3 that $h = 0$ on $R$ and so $d$ is commuting on $R$. In view of Lemma 4, this means that $R$ is commutative. \hfill \Box

**Theorem 6.** Let $n \geq 1$. Let $R$ be a $(n + 2)!$-torsion-free prime ring with identity. If there is a nonzero derivation $d : R \to R$ such that the mapping $x \mapsto \langle d(x), x \rangle_n$ is commuting on $R$, then $R$ is commutative.

**Proof.** Let us define the symmetric 3-additive mapping $H : R \times R \times R \to R$ and the trace $h : R \to R$ as in Theorem 5.

By hypothesis, we have $\langle [d(x), x]_n, x \rangle = 0 \text{ for all } x \in R$, and so we get $\langle h(x), x \rangle_n = \langle [d(x), x]_n, x \rangle = \langle [d(x), x]_n, x \rangle = 0 \text{ for all } x \in R$. Thus we obtain from Theorem 2 that $h = 0$ on $R$ and so $d$ is commuting on $R$. Lemma 4 yields that $R$ is commutative. \hfill \Box
Theorem 7. Let $n \geq 2$. Let $R$ be a 2-torsion-free ring with identity such that

$$\langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle \in Z(R)$$

holds for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$. Then $R$ is commutative.

Proof. Replacing $x_n$ by $x_n + 1$ in $\langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle \in Z(R)$, we get

$$Z(R) \ni \langle x_1 x_2 \cdots x_{n-1}(x_n + 1), (x_n + 1) x_n x_{n-1} \cdots x_2 x_1 \rangle$$

$$= \langle x_1 x_2 \cdots x_{n-1} x_n + x_1 x_2 \cdots x_{n-1}, x_n x_{n-1} \cdots x_2 x_1 + x_n x_{n-1} \cdots x_2 x_1 \rangle$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$. That is,

$$Z(R) \ni \langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle + \langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$. Since we see that

$$\langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle \in Z(R),$$

we therefore have

$$Z(R) \ni \langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle + \langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in R$.

Putting $-x_n$ instead of $x_n$ now gives $\langle x_1 x_2 \cdots x_{n-1} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle \in Z(R)$ for all $x_1, x_2, \ldots, x_{n-1} \in R$. Similarly, replacing $x_{n-1}$ by $x_{n-1} + 1$ in the just above relation and then letting $x_{n-1} := -x_{n-1}$ in the result, we obtain that

$$\langle x_1 x_2 \cdots x_{n-2} x_n, x_n x_{n-1} \cdots x_2 x_1 \rangle \in Z(R)$$

for all $x_1, x_2, \ldots, x_{n-2}, x_n \in R$. Continuing in the similar processing with this relation, we finally arrive at $\langle x_1, x_1 \rangle \in Z(R)$ for all $x_1 \in R$. This implies that $[y, x_1]^2 = 0$ for all $x_1$, $y \in R$ and hence we see that $\langle [y, x_1], x_1 \rangle = 0$ for all $x_1$, $y \in R$. Since the mapping $x_1 \mapsto [y, x_1]$ for any fixed $y \in R$ is additive on $R$, it follows from [1, Theorem 2] that $[y, x_1] = 0$ for all $x_1 \in R$. Since $y$ is arbitrary, thus $R$ is commutative. \qed

References


Kyoo-Hong Park  
Department of Mathematics Education,  
Seowon University  
Chungbuk 361-742, Korea  
E-mail address: parkkh@seowon.ac.kr

Yong-Soo Jung  
Department of Mathematics  
Sun Moon University  
Chungnam 336-708, Korea  
E-mail address: ysjung@sunmoon.ac.kr