ON INJECTIVITY AND $P$-INJECTIVITY

GUANGSHI XIAO AND WENTING TONG

Abstract. The following results are extended from $P$-injective rings to $AP$-injective rings: (1) $R$ is left self-injective regular if and only if $R$ is a right (resp. left) $AP$-injective ring such that for every finitely generated left $R$-module $M$, $R(M/Z(M))$ is projective, where $Z(M)$ is the left singular submodule of $RM$; (2) if $R$ is a left nonsingular left $AP$-injective ring such that every maximal left ideal of $R$ is either injective or a two-sided ideal of $R$, then $R$ is either left self-injective regular or strongly regular. In addition, we answer a question of Roger Yue Chi Ming [13] in the positive. Let $R$ be a ring whose every simple singular left $R$-module is $YJ$-injective. If $R$ is a right $MI$-ring whose every essential right ideal is an essential left ideal, then $R$ is a left and right self-injective regular, left and right $V$-ring of bounded index.

1. Introduction

Throughout this paper, a ring $R$ denotes an associative ring with identity and all modules are unitary. A ring $R$ is called (von Neumann) regular if for any $a \in R$, there exists $b \in R$ such that $a = aba$; $R$ is called strongly regular if for any $a \in R$, there exists $b \in R$ such that $a = a^2b$. We write $J$ and $Z(R)$ for the Jacobson radical of $R$ and the left singular ideal of $R$ respectively. $l(X), r(X)$ denote respectively the left and the right annihilator of $X$ in $R$. If $X = \{a\}$, we will write it for $l(a), r(a)$. For a left $R$-module $M$, $Z(M) = \{z \in M \mid l(z)$ is an essential left ideal of $R\}$ is called the left singular submodule of $M$. $M$ is called left nonsingular (resp. singular) if $Z(M) = 0$ (resp. $Z(M) = M$).

The concept of $P$-injective modules was introduced in 1974 to study von Neumann regular rings, $V$-rings, self-injective rings and their generalizations (see [6], [7]). This was generalized to $YJ$-injective modules and $AP$-injective modules. It is well-known that von Neumann regular
rings are $P$-injective since all left (right) modules are $P$-injective (see [6]). We are thus motivated to study $P$-injective modules over rings which are not necessarily von Neumann regular.

Let $R$ be a ring. A right $R$-module $M$ is called $P$-injective [6] if every right $R$-homomorphism from any principal right ideal $aR$ to $M$ extends to one from $R_R$ to $M$; $R$ is called right $P$-injective if the right $R$-module $R_R$ is $P$-injective. A right $R$-module $M$ is called right $P$-injective if the right $R$-module $R_R$ is $P$-injective; Similarly, we may define left $YJ$-injective rings. A module $M_R$ is said to be almost principally injective (or $AP$-injective for short) [3] if, for any $a \in R$, there exists an $S$-submodule $X_a$ of $M$ such that $l_M(r_R(a)) = Ma \oplus X_a$ as left $S$-modules, where $S = \text{End}(M)$. If $R_R$ is an $AP$-injective module, then $R$ is called a right $AP$-injective ring. Thus, $R$ is right $AP$-injective if and only if, for any $a \in R$, there exists a left ideal $L_a$ of $R$ such that $l(r(a)) = Ra \oplus L_a$. It is well known that $R$ is right $P$-injective if and only if $l(r(a)) = Ra$ for any $a \in R$ (see [2]), which implies that $P$-injective rings are $AP$-injective. But $AP$-injective rings need not be neither $P$-injective nor $YJ$-injective by [3, Example 1.5].

In this paper, we extend the following results from $P$-injective rings to $AP$-injective rings:

1. $R$ is left continuous regular if and only if $R$ is a right (resp. left) $AP$-injective ring such that for every cyclic left $R$-module $M$, $r(M/Z(M))$ is projective;

2. $R$ is left self-injective regular if and only if $R$ is a right (resp. left) $AP$-injective ring such that for every finitely generated left $R$-module $M$, $r(M/Z(M))$ is projective;

3. If $R$ is a left nonsingular left $AP$-injective ring such that every maximal left ideal of $R$ is either injective or a two-sided ideal of $R$, then $R$ is either left self-injective regular or strongly regular. We answer a question of Roger Yue Chi Ming [13] in the positive and prove the following facts:

1. If $R$ is a ring whose every simple singular left $R$-module is $YJ$-injective and every essential right ideal is an essential left ideal, then $R$ is von Neumann regular;

2. If $R$ is right $MI$-ring whose every simple singular left $R$-module is $YJ$-injective and every essential right ideal is an essential left ideal,
then \( R \) is a left and right self-injective regular, left and right \( V \)-ring of bounded index.

2. Main results

It has been demonstrated in [13, Page 230] that if \( R \) is left nonsingular, then (1) \( Z(M) \) is injective for every injective left \( R \)-module \( M \) and (2) for any complement left ideal \( C \) of \( R \), \( Z(R/C) = 0 \). A ring \( R \) is said to be left continuous (Y. Utumi [4]) if (1) every left ideal of \( R \) isomorphic to a direct summand of \( R \) is a direct summand of \( R \) and (2) every complement left ideal of \( R \) is a direct summand of \( R \). Thus, if \( R \) is left continuous, then \( J = Z(R) \) and \( R/Z(R) \) is von Neumann regular. By [5, Proposition 3.3], we have the following fact which will be needed.

**Lemma 2.1.** If \( R \) is left AP-injective, then any left ideal isomorphic to a direct summand of \( R \) is a direct summand of \( R \).

**Proposition 2.2.** The following are equivalent for a ring \( R 

(1) \( R \) is left continuous regular;
(2) \( R \) is a right AP-injective ring such that for every cyclic left \( R \)-module \( M \), \( R(M/Z(M)) \) is projective;
(3) \( R \) is a left AP-injective ring such that for every cyclic left \( R \)-module \( M \), \( R(M/Z(M)) \) is projective.

**Proof.** (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (3). By [13, Theorem 13] and all regular rings are both left and right AP-injective.

(2) \( \Rightarrow \) (1). By assumption, \( R/Z(R) \) is projective which implies \( Z(R) \) is a direct summand of \( R \), whence \( Z(R) = 0 \) (since \( Z(R) \) contains no non-zero idempotent elements). Then for every complement left ideal \( K \) of \( R \), \( Z(R/K) = 0 \). By assumption again, the left \( R \)-module \( R/K \) is projective which implies \( R \) is a direct summand of \( R \). Since \( R \) is right AP-injective, there exists a left ideal \( L \) of \( R \) such that \( l(r(a)) = Ra \oplus L \). Note that \( Z(R) = 0 \), which implies that \( l(r(a)) \) is a complement left ideal of \( R \) and hence \( l(r(a)) \) is a direct summand of \( R \). This shows that \( Ra \) is a direct summand of \( R \). Therefore, \( R \) is left continuous regular.

(3) \( \Rightarrow \) (1). By Lemma 2.1 and apply the proof in “(2) \( \Rightarrow \) (1)”.

The following result give a characterization of left self-injective regular rings and extends [13, Theorem 14].
Theorem 2.3. The following are equivalent for a ring $R$:

1. $R$ is left self-injective regular;
2. $R$ is a right $AP$-injective ring such that for every finitely generated left $R$-module $M$, $R(M/Z(M))$ is projective;
3. $R$ is a left $AP$-injective ring such that for every finitely generated left $R$-module $M$, $R(M/Z(M))$ is projective.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). By [13, Theorem 14] and all regular rings are both left and right $AP$-injective.

(2) $\Rightarrow$ (1). By assumption and Proposition 2.2, $R$ is left continuous regular. Denote $RE$ as the injective hull of $RR$. For any $u \in E$, $B = R + Ru$ is a finitely generated nonsingular left $R$-module. By our assumption, $RB$ is projective. Since the left annihilator of any proper finitely generated right ideal of $R$ is nonzero, by a well-known theorem of H. Bass, $RB$ is a direct summand of $RE$. But $RB$ is essential in $RE$ which implies $R = B$. This proves that $R = E$ is left self-injective regular.

(3) $\Rightarrow$ (1). As the proof in “(2) $\Rightarrow$ (1)” and by Proposition 2.2, we may complete our proof.

Recall that a ring $R$ is called reduced if it contains no non-zero nilpotent elements. It is well-known that a reduced left $P$-injective ring is strongly regular. By [10, Proposition 1(2)], if $R$ is a reduced left $YJ$-injective ring, then $R$ is strongly regular. Now, we have the same result for left $AP$-injective rings.

Lemma 2.4. If $R$ is a reduced left $AP$-injective ring, then $R$ is strongly regular.

Proof. For any $0 \neq a \in R$, $l(a) = l(a^2)$ by assumption. Thus there exists a right ideal $L$ of $R$ such that

$a \in r(l(a)) = r(l(a^2)) = a^2R \oplus L.$

Then $a = a^2r + x$ for some $r \in R$, $x \in L$, which implies $a^2 - a^2ra = xa \in a^2R \cap L = 0$. Hence $a^2 = a^2ra$, so $a = a^2r$ since $R$ is reduced. This proves that $R$ is strongly regular.

By [3, Corollary 2.3], the following result is immediate.

Lemma 2.5. If $R$ is a left $AP$-injective ring, then $J = Z(R)$.

A ring $R$ is called a left (resp. right) $MI$-ring [12] if $R$ contains an injective left (resp. right) ideal. In [12], R. Yue Chi Ming gave an example, in which $R$ is an $MI$-ring and not left self-injective. The following result extends [13, Proposition 11].
Proposition 2.6. Let $R$ be a left nonsingular left $AP$-injective ring such that every maximal left ideal is either injective or an ideal of $R$. Then $R$ is either left self-injective or strongly regular.

Proof. Since $R$ is a left $AP$-injective ring, by Lemma 2.5, $J = Z(R) = 0$. First suppose that every maximal left ideal of $R$ is an ideal of $R$. Since $J = 0$, by [13, Lemma 2], $R$ is reduced. Thus $R$ is strongly regular by Lemma 2.4. Now suppose there exists a maximal left ideal $M$ of $R$ which is not an ideal. By assumption, $M$ is left injective and $R$ is a left $MI$-ring. Note that $R$ is semiprime, thus $eR$ is a minimal right ideal of $R$ if and only if $Re$ is a minimal left ideal of $R$. By [13, Lemma 10], $R$ is left self-injective. Thus $R$ is regular since $J = 0$.

In general, if $R$ is a ring whose every simple singular left $R$-module is $YJ$-injective, then $J \cap Z(R) = 0$ (cf. [8, Proposition 3]). In [13], Yue Chi Ming raised a question: is it true that $J \cap Z(R) = 0$ if every simple singular left $R$-module is $YJ$-injective? Now, we answer it in the positive. Let’s start with the following lemma.

Lemma 2.7. If $R$ is a ring whose every simple singular left $R$-module is $YJ$-injective, then $J \cap Z(R)$ contains no nonzero nilpotent elements.

Proof. Take any $b \in J \cap Z(R)$ with $b^2 = 0$. If $b \neq 0$, then $l(b) + RbR$ is an essential left ideal of $R$. We will prove that $l(b) + RbR = R$. If not, there exists a maximal essential left ideal $M$ of $R$ containing $l(b) + RbR$. By assumption, the simple singular left $R$-module $R/M$ is $YJ$-injective, thus there exists a positive $n$ such that $b^n \neq 0$ and every left $R$-homomorphism from $Rb^n$ to $R/M$ extends to one from $Rb^n$ to $R/M$. Therefore $n = 1$ and every left $R$-homomorphism from $Rb$ to $R/M$ extends to one from $Rb$ to $R/M$. Since $l(b) \subseteq M$, the left $R$-homomorphism $f : Rb \to R/M$ by $f(rb) = r + M$ is well defined. Since $R/M$ is left $YJ$-injective, there exists $c \in R$ such that $1 + M = bc + M$. Note that $bc \in RbR \subseteq M$, which implies $1 \in M$, it is a contradiction. This proves that $l(b) + RbR = R$ and hence $b = db$ for some $d \in RbR \subseteq J$. This implies $b = 0$, which is required contradiction.

Lemma 2.8. If $R$ is a ring whose every simple singular left $R$-module is $YJ$-injective, then for any $b \in J \cap Z(R)$, $l(b^n) = l(b)$ for any positive integer $n > 1$.

Proof. Take any $x \in l(b^n)$. Then $xb^n = 0$, and thus $(b^{n-1}xb^{n-1})^2 = 0$. Note that $b^{n-1}xb^{n-1} \in J \cap Z(R)$, so $b^{n-1}xb^{n-1} = 0$ by Lemma 2.7. This implies $(xb^{n-1})^2 = 0$, by Lemma 2.7, and thus $xb^{n-1} = 0$ since
$xb_n^{-1} \in J \cap Z(R)$. This proves that \( x \in l(b_n^{-1}) \) and \( l(b^n) = l(b_n^{-1}) \). By
induction, \( l(b^n) = l(b) \) for any positive integer \( n > 1 \).

**Theorem 2.9.** If \( R \) is a ring whose every simple singular left \( R \)-module is \( YJ \)-injective, then \( J \cap Z(R) = 0 \).

**Proof.** Take any \( b \in J \cap Z(R) \). If \( b \neq 0 \), then \( l(b) \neq R \) and \( l(b) + RbR \) is an essential left ideal of \( R \). We will prove \( l(b) + RbR = R \). If not, as the proof in Lemma 2.7, there exist a maximal essential left ideal \( M \) of \( R \) containing \( l(b) + RbR \) and a positive integer \( n \) such that \( b^n \neq 0 \) and every left \( R \)-homomorphism from \( Rb^n \) to \( R/M \) extends to one from \( R \) to \( R/M \). Since \( l(b) \subseteq M \), so \( l(b^n) \subseteq M \) by Lemma 2.8. Thus the left \( R \)-homomorphism \( f : Rb^n \to R/M \) by \( f(rb^n) = r + M \) is well defined. Since the simple singular left \( R \)-module \( R/M \) is \( YJ \)-injective, there exists \( c \in R \) such that \( 1 + M = b^n c + M \). Note that \( b^n c \in RbR \subseteq M \), which implies \( 1 \in M \), it is a contradiction. This proves that \( l(b) + RbR = R \) and hence \( b = db \) for some \( d \in RbR \subseteq J \). This implies \( b = 0 \), which is required contradiction.

If \( R \) is a ring whose every simple left \( R \)-module is \( YJ \)-injective, then \( J = 0 \). But the result need not be true for a ring whose every simple singular left \( R \)-module is \( P \)-injective.

The following example shows that there exists a ring whose every simple singular left \( R \)-module is \( P \)-injective with \( J \neq 0 \), and thus the ring \( R \) is a ring whose every simple singular left \( R \)-module is \( YJ \)-injective with \( J \neq 0 \).

**Example 2.10.** Let \( R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix} \), where \( F \) is a field. Then \( J(R) \neq 0 \). Note that \( M = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \) is the unique maximal essential left ideal. Since \( M \) is an ideal and the right \( R \)-module \( R/M \) is flat, so the simple singular left \( R \)-module \( R/M \) is \( P \)-injective. Thus \( R \) is a ring whose every simple singular left \( R \)-module is \( P \)-injective with \( J \neq 0 \).

**Proposition 2.11.** If \( R \) is a ring whose every simple singular left \( R \)-module is \( YJ \)-injective, then \( J \) contains no nonzero nilpotent elements if and only if \( J = 0 \).

**Proof.** Assume \( J \) contains no nonzero nilpotent elements. Write \( L = Rb + l(b) \) for any \( b \in J \). If \( L = R \), then there exists \( a \in R, c \in l(b) \) such that \( 1 = ab + c \). Hence \( a = ab^2 \) and \( (b - bab)^2 = 0 \), which implies \( b = bab \) since \( b - bab \in J \). Let \( e = ab \), then \( e^2 = e \in J \), which implies \( e = 0 \) and \( b = 0 \). If \( L \neq R \), then there exists a left ideal \( K \) of \( R \) such
that \( L \oplus K \) is an essential left ideal of \( R \). We claim that \( L \oplus K = R \).
If not, there is a maximal essential left ideal \( M \) of \( R \) containing \( L \oplus K \).
By assumption, the simple singular left \( R \)-module \( R/M \) is \( YJ \)-injective.
Since \( J \) contains no nonzero nilpotent elements and \( b \in J \), a left \( R \)-homomorphism \( f : Rb^n \to R/M \) by \( f(rb^n) = r + M \) is well defined.
Thus there exists \( c \in R \) such that \( 1 - b^n c \in M \). Note that \( b^n c \in J \subseteq M \),
which implies that \( 1 \in M \), contradicting with \( M \) is maximal. This
shows that \( L \oplus K = R \). Then \( Rb + l(b) = Re \) with \( e^2 = e \in R \), so
\( b^2 = beb = bab^2 \) for some \( a \in R \). But \( b \in J \), thus \( b = 0 \) by the preceding proof.
This gives that \( J = 0 \). The converse is obvious.

It is well-known that if \( R \) is semiprime, then every essential right ideal of \( R \) which is an ideal of \( R \) must be left essential. But the converse is obviously not true.

**Theorem 2.12.** If \( R \) is a ring whose every simple singular left \( R \)-module is \( YJ \)-injective and every essential right ideal of \( R \) is an essential left ideal, then \( R \) is von Neumann regular.

**Proof.** We first prove that \( R \) is a semiprime ring. If not, then there exists \( 0 \neq b \in R \) such that \( RbRb = 0 \). Then \( b \in J \). Let \( K \) be a complement right ideal of \( R \) such that \( RbR \oplus K \) is an essential right ideal of \( R \). Then \( Kb \subseteq K \cap RbR = 0 \) which implies \( RbR \oplus K \subseteq l(b) \). By assumption, \( RbR \oplus K \) is an essential left ideal of \( R \), thus \( b \in Z(R) \). This implies \( b \in J \cap Z(R) \), so \( b = 0 \) by Theorem 2.9. It is a contradiction. This proves that \( R \) is a semiprime ring. By [8, Proposition 6], \( R \) is fully left idempotent. By [9, Proposition 9], \( R \) is von Neumann regular since \( R \) is a ring whose every essential right ideal is an ideal.

**Lemma 2.13.** Let \( R \) be a ring whose every essential right ideal is an essential left ideal with \( J \cap Z(R) = 0 \). If \( J = Z(R_R) \), where \( Z(R_R) \) is the right singular ideal of \( R \), then \( J = 0 \).

**Proof.** Assume \( J \neq 0 \). Then there exists \( 0 \neq b \in J = Z(R_R) \), so \( r(b) \) is an essential right ideal of \( R \). By assumption, \( r(b) \) is an essential left ideal. Thus \( r(b) \cap Rb \neq 0 \), and hence there exists \( a \in R \) such that \( 0 \neq ab \in J \) and \( bab = 0 \). Note that \( r(b) \) is an ideal, so \( bRab = 0 \). This implies \( abRab = 0 \). Let \( K \) be a complement right ideal of \( R \) such that \( RabR \oplus K \) is an essential right ideal. As the proof in Theorem 2.12, we have \( ab = 0 \) since \( J \cap Z(R) = 0 \). It is a contradiction. Therefore \( J = 0 \).

**Theorem 2.14.** Let \( R \) be a right MI-ring whose every essential right ideal is an essential left ideal. If \( R \) is a ring whose every simple singular
left $R$-module is $YJ$-injective, then $R$ is a left and right self-injective regular, left and right $V$-ring, and with bounded index.

Proof. By Theorem 2.9, $J \cap Z(R) = 0$. As the proof in Theorem 2.12, $R$ is semiprime since $R$ is a ring whose every essential right ideal is an essential left ideal. Thus $eR$ is a minimal right ideal of $R$ if and only if $Re$ is a minimal left ideal of $R$. As the proof in [13, Lemma 10], $R$ is right self-injective. This proves $J = Z(R_R)$ and $R/J$ is regular. By Lemma 2.13, $J = 0$. Therefore $R$ is right self-injective regular.

Let $M$ be a maximal ideal. Then $R/M$ is simple, thus $R/M$ is regular simple since $R$ is regular. Moreover, $R/M$ is a ring whose every maximal essential right ideal is an ideal and every maximal essential left ideal is an ideal. Write $B = R/M$. We will prove $Soc(BB) = B$, where $Soc(BB)$ denote the socle of $RB$. If not, there exists an maximal left ideal $L$ of $B$ containing $Soc(BB)$ and $L$ is an essential left ideal of $B$. Thus $L$ is an ideal. But $B$ is a simple ring, so $L = 0$ or $L = B$. It is a contradiction. This proves $Soc(BB) = B$ and hence $R/M$ is Artinian semisimple. By [1, Page 79], $R/M$ is a ring of bounded index. Thus $R$ is left self-injective by [1, Theorem 6.20] and [1, Corollary 6.22].

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GUANGSHI XIAO, DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY OF AERONAUTICS AND ASTRONAUTICS, NANJING 210016, P. R. CHINA
E-mail: xgs01cn@yahoo.com.cn

WENTING TONG, DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, P. R. CHINA
E-mail: wttong@nju.edu.cn