SELECTION PRINCIPLES AND HYPERSPACE TOPOLOGIES IN CLOSURE SPACES

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Abstract. Relations between closure-type properties of hyperspaces over a Čech closure space \((X, u)\) and covering properties of \((X, u)\) are investigated.

Introduction

1. An operator \(u : \mathcal{P}(X) \to \mathcal{P}(X)\) defined on the power set \(\mathcal{P}(X)\) of a set \(X\) satisfying the axioms:
   \begin{align*}
   (C1) \quad & u(\emptyset) = \emptyset, \\
   (C2) \quad & A \subset u(A) \text{ for every } A \subset X, \\
   (C3) \quad & u(A \cup B) = u(A) \cup u(B) \text{ for all } A, B \subset X,
   \end{align*}

   is called a Čech closure operator and the pair \((X, u)\) is a Čech closure space. For short, \((X, u)\) will be noted by \(X\) as well, and called a closure space or a space.

   A subset \(A\) is closed in \((X, u)\) if \(u(A) = A\) holds. It is open if its complement is closed.

   The interior operator \(\text{int}_u : \mathcal{P}(X) \to \mathcal{P}(X)\) is defined by means of the closure operator in the usual way: \(\text{int}_u = c \circ u \circ c\), where \(c : \mathcal{P}(X) \to \mathcal{P}(X)\) is the complement operator. A subset \(U\) is a neighborhood of a point \(x\) in \(X\) if \(x \in \text{int}_u U\) holds.

   A closure space \((X, u)\) is T1 if for each two distinct points in \(X\) the following holds: \(\{x\} \cap u(\{y\}) \cup \{y\} \cap u(\{x\}) = \emptyset\) whenever \(x \neq y\). It is equivalent to: every one-point subset of \(X\) is closed in \((X, u)\).

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A space \((X, u)\) is \(T_2\) (Hausdorff) if each two distinct points in \(X\) have disjoint neighborhoods.

All considered spaces are \(T_1\).

2. A collection \(\{G_\alpha\}\) is an interior cover of a set \(A\) in \((X, u)\) if the collection \(\{\text{int}_u G_\alpha\}\) covers \(A\). We suppose that the interior of every element of an interior cover is non-empty and that each cover is non-trivial, i.e., that the set \(X\) does not belong to the cover.

A subset \(A\) in a space \((X, u)\) is compact if every interior cover of \(A\) has a finite subcover.

Let \(C\) be a collection of subsets of \(X\). An interior cover \(U\) is a \(C\)-cover of \(X\) if for every \(C \in C \setminus \{X\}\) there is a \(U \in U\) such that \(C \subset \text{int}_u U\) holds.

The collection of all interior covers \(U\) of \((X, u)\) will be denoted by \(I\) and of all interior \(C\)-covers by \(IC\).

The following notations are used:

- \(H = \{u(A) \mid A \subset X\}\), \(J = \{\text{int}_u(A) \mid A \subset X\}\),
- \(\Phi_x = \{A \subset X \mid x \in u(A) \setminus A\}\),
- \(F(X)\) the family of all finite subsets of \(X\),
- \(K(X)\) the family of all compact subsets of \(X\),
- \(Q(X)\) the family of all closed subsets of \(X\).

\(F(X)^+\), \(K(X)^+\) and \(Q(X)^+\)-interior covers will be called \(\omega\)-, \(\kappa\)- and \(\zeta\)-covers of \(X\), respectively. \(I\Omega\) stands for the collection of all \(\omega\)-covers of \(X\), \(IK\) for the collection of all \(k\)-covers, and \(IQ\) for the collection of all \(\zeta\)-covers.

For \(A \subset X\) the usual notation is \(A^+ = \{H \in H \mid H \subset A\}\).

To the end \(\Delta\) and \(\Sigma\) are subcollections of \(H\) closed for finite unions and containing all singletons. The upper \(\Delta\)-topology for \(H\), denoted by \(\Delta^+\), has for a base the collection \(\{(D^+)^+ \mid D \in \Delta\} \cup \{H\}\).

Following [4], the \(F(X)^+\)-topology will be denoted by \(Z^+\) and the \(K(X)^+\)-topology, the upper Fell topology (or the co-compact topology), by \(F^+\). Also, \(V^+\) will stand for the \(Q(X)^+\)-topology, the upper Vietoris topology.

Let \(A\) be a subcollection of \(H\). Each \(A \in A\) is of the form \(A = u(B)\). We pick one such \(B = B(A)\). The collection \(U = \{B(A)^+\}\) will be denoted by \(AC\).

The following is our key lemma and the constructions and notations introduced in it are used throughout the paper.
Lemma. (i) Let \((X, u)\) be a space, a subset \(Y \in \mathcal{J}\), and \(\mathcal{U}\) be an interior \(\Delta\)-cover of \(Y\). Set \(A = \{(\text{int}_u U)^c \mid U \in \mathcal{U}\} = \{u(U^c) \mid U \in \mathcal{U}\}\).

Then \(A \subset \mathcal{H}\) holds and \(Y^c \in \text{cl}_{\Delta^+} A\).

(ii) Conversely, having a collection \(A \subset \mathcal{H}\) and a set \(H \in \mathcal{H}\) such that \(H \in \text{cl}_{\Delta^+} A\), the collection \(U = A^c\) is an interior \(\Delta\)-cover of \(H^c\).

Proof. (i) Let \(U\) be an interior \(\Delta\)-cover of \(Y\) and \((D^+)\) be a basic \(\Delta^+\)-neighborhood of \(Y^c\). Since \(D \subset Y\), there is a \(U \in \mathcal{U}\) such that \(D \subset \text{int}_u U\) holds. Then \(A = (\text{int}_u U)^c \in A\) and \(A \subset D^c\) imply \(Y^c \in \text{cl}_{\Delta^+} A\).

(ii) Let \(H \in \text{cl}_{\Delta^+} A\) and \(D \in \Delta\) such that \(D \subset H^c\). The family \((D^+)\) is a \(\Delta^+\)-neighborhood of \(H\). There is an \(A \in A\), such that \(A \subset D^c\). For the corresponding \(U \in \mathcal{U} = A^c\), \(\text{int}_u U = A^c \supset D\) holds, so \(U\) is an interior \(\Delta\)-cover of \(Y = H^c\).

In particular, when \(\Delta = F(X)\) (respectively, \(\Delta = K(X), Q(X)\)), then \(Y^c \in \text{cl}_Z A\) (resp. \(Y^c \in \text{cl}_F A, Y^c \in \text{cl}_V A\)) for the corresponding interior \(\omega\)-cover (resp. interior \(\kappa\)-cover, interior \(\zeta\)-cover) of \(Y \in \mathcal{J}\).

Let \(A\) and \(B\) be sets whose members are families of subsets of an infinite set \(X\). Then ([8], [3]):

\(S_1(A, B)\) denotes the selection principle: For each sequence \((A_n)\) of elements of \(A\) there is a sequence \((b_n)\) such that \(b_n \in A_n\) for each \(n \in \mathbb{N}\) and \(\{b_n \mid n \in \mathbb{N}\}\) is an element of \(B\).

\(S_{\text{fin}}(A, B)\) denotes the selection principle: For each sequence \((A_n)\) of elements of \(A\) there is a sequence \((B_n)\) of finite sets such that \(B_n \subset A_n\) for each \(n \in \mathbb{N}\) and \(\bigcup_{n \in \mathbb{N}} B_n \in B\).

Varying the collections \(A\) and \(B\) in the above defined selection principles, characterizations of the space \((X, u)\) and its hyperspaces are defined. When \((X, u)\) is a topological space, all definitions considered in this paper coincide with the corresponding topological ones. Interior covers are replaced with open covers denoted by \(\mathcal{O}\), and \(\mathcal{H}\) coincides with \(Q(X)\), the family of all closed subsets of \(X\).

We assume that the space \((X, u)\) is not compact.

All notions not explained here concerning selection principles can be found in [3], while those concerning Čech closure spaces in [1] and [6].

1. The Rothberger-like selection principles

The property \(S_1(\mathcal{O}, \mathcal{O})\) for topological spaces was introduced by F. Rothberger in 1938 and it is called nowadays the Rothberger property.
A topological space $X$ has countable strong fan tightness [7] if for each $x \in X$ the selection principle $S_1(\Phi_x, \Phi_x)$ holds.

We generalize these notions to closure spaces in the following way:

**Definition 1.** A space $(X, u)$ has the Rothberger property if for every sequence $(U_n)$ of interior covers of $X$ there is a sequence $(U_n)$ such that $U_n \in U_n$ for each $n \in \mathbb{N}$ and the collection $\{U_n \mid n \in \mathbb{N}\}$ is an interior cover of $X$.

**Definition 2.** A space $(X, u)$ has countable strong fan tightness if for every $x \in X$ and each sequence $(A_n)$ of subsets of $X$ such that $x \in \cap_{n \in \mathbb{N}} u(A_n)$, there is a sequence $(x_n)$, $x_n \in A_n$, such that $x \in u(\{x_n \mid n \in \mathbb{N}\})$.

**Theorem 1.** For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $\mathcal{H}$ satisfies the selection principle $S_1(\Phi^+_{\mathcal{H}}, \Phi^+_{\mathcal{H}})$ for each $H \in \mathcal{H}$.
2. Each $Y \in \mathcal{J}$ satisfies $S_1(\mathcal{I}\Delta, \mathcal{I}\Sigma)$.

**Proof.** (1) $\Rightarrow$ (2): Let $(U_n)$ be a sequence of interior $\Delta$-covers of $Y$. Then for the sequence $(A_n)$ of elements of $\mathcal{H}$, $A_n = \{\text{int}_u U \} \cap U_n = \{u(U) \cap U \in U_n\}$, $Y^c = \text{cl}_{\Delta^+} A_n$ for each $n \in \mathbb{N}$ by Lemma. By assumption, there is a sequence $(A_n)$ such that $A_n \in A_n$, for every $n \in \mathbb{N}$ and $Y^c \in \text{cl}_{\Sigma^+} \{A_n \mid n \in \mathbb{N}\}$. The corresponding collection $\{U_n \mid n \in \mathbb{N}\}$, $A_n = u(U^c_n)$, is an interior $\Sigma$-cover of $Y$. Indeed, for every $S \in \Sigma$ such that $S \subseteq Y$, $(S^c)^+ \subseteq Y$. There is an $A_m$ such that $A_m \subseteq (S^c)^+$, i.e., $A_m \subseteq S^c$ implies $S \subseteq A_m^c = (c \circ u)(U_n^c) = \text{int}_u U_n$. (2) $\Rightarrow$ (1): Let $(A_n)$ be a sequence of elements of $\mathcal{H}$ such that $H \in \mathcal{H}$ belongs to $\text{cl}_{\Delta^+} A_n$ for each $n \in \mathbb{N}$. Then by Lemma, the corresponding sequence $(U_n)$, $U_n = A_n^c$ for $n \in \mathbb{N}$, is a sequence of interior $\Delta$-covers of $H^c = Y$. Since $H^c$ satisfies $S_1(\mathcal{I}\Delta, \mathcal{I}\Sigma)$, there is a sequence $(U_n)$, $U_n \in U_n$, such that $\{U_n \mid n \in \mathbb{N}\}$ is an interior $\Sigma$-cover of $H^c$. By applying Lemma again, $H \in \text{cl}_{\Sigma^+} \{A_n \mid n \in \mathbb{N}\}$ which proves that (1) holds.

In particular, when $\Sigma = \Delta$, the next statement is true.

**Theorem 2.** For a space $(X, u)$ and a collection $\Delta$ the following are equivalent:

1. $(\mathcal{H}, \Delta^+)$ has countable strong fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_1(\mathcal{I}\Delta, \mathcal{I}\Delta)$.

Setting $\Delta = F(X)$, resp. $K(X)$, $Q(X)$, in Theorem 2, we get generalizations of the results for topological spaces.
Corollary 1. (cf. [2]) For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, Z^+)$ has countable strong fan tightness.
2. Each $Y \in \mathcal{F}$ satisfies $S_1(I\Omega, I\Omega)$.

Corollary 2. (cf. [2]) For a Hausdorff space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, F^+)$ has countable strong fan tightness.
2. Each $Y \in \mathcal{F}$ satisfies $S_1(I\Omega, I\Omega)$.

Proof. Note that in a $T_2$ space $(X, u)$ a compact subset is closed. Thus the family $K$ of compact subsets is closed for finite unions, contains all singletons and is a subfamily of $\mathcal{H}$.

Corollary 3. For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, V^+)$ has countable strong fan tightness.
2. Each $Y \in \mathcal{F}$ satisfies $S_1(I\Omega, I\Omega)$.

Setting in Theorem 1: $\Delta = K(X)$ and $\Sigma = F(X)$, (resp. $\Delta = Q(X)$ and $\Sigma = K(X)$) we get

Corollary 4. (cf. [2]) For a $T_2$ space $(X, u)$ the following are equivalent:

1. $\mathcal{H}$ satisfies $S_1(\Phi F^+, \Phi Z^+)$ for each $H \in \mathcal{H}$.
2. Each $Y \in \mathcal{F}$ satisfies $S_1(I\Omega, I\Omega)$.

Corollary 5. For a $T_2$ space $(X, u)$ the following are equivalent:

1. $\mathcal{H}$ satisfies $S_1(\Phi V^+, \Phi F^+)$ for each $H \in \mathcal{H}$.
2. Each $Y \in \mathcal{F}$ satisfies $S_1(I\Omega, I\Omega)$.

We denote by $\mathcal{D}$ the family of dense subsets of a space. When necessary to distinguish between two topologies on the same set we use extra notations; for example: $\mathcal{D}_{\Delta^+}$ (resp. $\mathcal{D}_{\Sigma^+}$) stands for the family of dense collections in the space $(\mathcal{H}, \Delta^+)$ (resp. $(\mathcal{H}, \Sigma^+)$).

Theorem 3. For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $\mathcal{H}$ satisfies $S_1(D_{\Delta^+}, D_{\Sigma^+})$.
2. $(X, u)$ satisfies $S_1(I\Delta, I\Sigma)$.

Proof. $(1) \Rightarrow (2)$: Let $(U_n)$ be a sequence of interior $\Delta$-covers of $X$. For each $n \in \mathbb{N}$ the corresponding $A_n$ is dense in the space $(\mathcal{H}, \Delta^+)$ since for every non-empty basic open set $(D^c)^+$ there is an $U_n \in U_n$ such that $D \subset \text{int}_n(U_n)$. Hence $A_n = u(U_n^c) \subset D^c$ and $A_n \in A_n \cap (D^c)^+$ implies $A_n$ is dense in $(\mathcal{H}, \Delta^+)$.

By applying (1), there is a sequence $(A_n)$, $A_n \in A_n$, such that $\text{cl}_{\Sigma^+}\{A_n \mid n \in \mathbb{N}\} = \mathcal{H}$. For
each \( n \in \mathbb{N} \) choose \( U_n \in \mathcal{U}_n \) such that \( u(U_n^c) = A_n \). The collection \( \{ U_n \mid n \in \mathbb{N} \} \) is an interior \( \Sigma \)-covers of \( X \). Indeed, for every \( S \in \Sigma \) there is an \( A_m \in (S^c)^+ \); hence for the corresponding \( U_m, S \subset \text{int}_u(U_m) \) holds. Thus \( (X, u) \) satisfies \( S_1(I\Delta, I\Sigma) \).

(2) \( \Rightarrow \) (1): Let \( (\mathcal{D}_n) \) be a sequence of dense subsets in \( (\mathcal{H}, \Delta^+) \). For each \( n \in \mathbb{N} \) the collection \( \mathcal{U}_n = \mathcal{D}_n^c \) is an interior \( \Delta \)-cover of \( X \). It is true since for every \( G \in \Delta \) and \( D \subset \text{int}_u(A^c) \subset G^c \), \( D \subset \text{int}_u(U_m) \) for the corresponding \( U_m \in \mathcal{U}_n \). By assumption, there is a sequence \( (\mathcal{U}_n) \), \( \mathcal{U}_n \in \mathcal{U}_n \), such that the collection \( \{ \mathcal{U}_n \mid n \in \mathbb{N} \} \) is an interior \( \Sigma \)-cover of \( X \). The collection \( \{ \mathcal{D}_n \mid n \in \mathbb{N} \} \), \( \mathcal{D}_n = u(U_n^c) \), is dense in \( (\mathcal{H}, \Sigma^+) \). For every \( S \in \Sigma \) there is an \( m \in \mathbb{N} \) such that \( S \subset \text{int}_u(U_m) \) implies \( D_m \subset S^c \), i.e., \( \{ \mathcal{D}_n \mid n \in \mathbb{N} \} \cap (S^c)^+ \neq \emptyset \).

Again, when \( \Delta = F(X) \), (resp. \( K(X), Q(X) \)), or \( \Delta = K(X) \) and \( \Sigma = F(X) \), (resp. \( \Delta = Q(X) \) and \( \Sigma = K(X) \)) we get

**Corollary 6.** (cf. [2]) For a space \( (X, u) \) the following are equivalent:

1. \( (\mathcal{H}, Z^+) \) satisfies \( S_1(D, D) \).
2. \( (X, u) \) satisfies \( S_1(I\Omega, I\Omega) \).

**Corollary 7.** (cf. [2]) For a \( T_2 \) space \( (X, u) \) the following are equivalent:

1. \( (\mathcal{H}, F^+) \) satisfies \( S_1(D, D) \).
2. \( (X, u) \) satisfies \( S_1(IK, IK) \).

**Corollary 8.** For a space \( (X, u) \) the following are equivalent:

1. \( (\mathcal{H}, V^+) \) satisfies \( S_1(D, D) \).
2. \( (X, u) \) satisfies \( S_1(IQ, IQ) \).

**Corollary 9.** (cf. [2]) For a \( T_2 \) space \( (X, u) \) the following are equivalent:

1. \( \mathcal{H} \) satisfies \( S_1(D_{F^+}, D_{Z^+}) \).
2. \( (X, u) \) satisfies \( S_1(I\Omega, I\Omega) \).

**Corollary 10.** For a \( T_2 \) space \( (X, u) \) the following are equivalent:

1. \( \mathcal{H} \) satisfies \( S_1(D_{V^+}, D_{F^+}) \).
2. \( (X, u) \) satisfies \( S_1(IQ, IK) \).

A family \( \mathcal{N} \) of subsets of \( X \) is a \( \pi \)-network for a topological space \( X \) if every open set in \( X \) contains some element of \( \mathcal{N} \).

We introduce the following definition.
Definition 3. A family $\mathcal{N} = \{N_\lambda\}_{\lambda \in \Lambda}$ of subsets of a closure space $(X, u)$ is a $\pi$-network for $(X, u)$ if every non-empty interior of a subset of $X$ contains some $N_\lambda$, i.e., for every $A \subset X$ such that $\text{int}_u A \neq \emptyset$, there is $N_\lambda \subset \text{int}_u A$.

By $\Pi_\Delta$ (respectively $\Pi_\omega, \Pi_\kappa, \Pi_\zeta$) we denote the family of $\pi$-networks consisting of elements from $\Delta \subset \mathcal{H}$ (resp. $F(X), K(X), Q(X)$).

Theorem 4. For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $\mathcal{H}$ satisfies $S_1(\mathcal{O}\Delta^+, \mathcal{O}\Sigma^+)$.  
2. $(X, u)$ satisfies $S_1(\Pi_\Delta, \Pi_\Sigma)$.

Proof. (1) $\Rightarrow$ (2): Let $(\Delta_n)$ be a sequence from $\Pi_\Delta$. Then for each $n \in \mathbb{N}$ the collection $\{(D_n^c)^+ \mid D \in \Delta_n\}$ is a $\Delta^+$-open cover of $\mathcal{H}$. Indeed, for a fixed $n \in \mathbb{N}$ and any $H \in \mathcal{H}$, $H^c = \text{int}_u (A^c)$ for some $A \subset X$. There is a $D \in \Delta_n$ such that $D \subset H^c$. Thus $H \in (D_n^c)^+$. By (1), there is a sequence $(D_n)$, $D_n \in \Delta_n$, such that the collection $\{(D_n^c)^+ \mid n \in \mathbb{N}\}$ is a $\Sigma^+$-open cover of $\mathcal{H}$. Then the collection $\{D_n \mid n \in \mathbb{N}\}$ is a $\Sigma - \pi$-network for $(X, u)$. For, let $A \subset X$ such that $\text{int}_u A \neq \emptyset$. Then for $u(A^c)$ there is a $D_n$ such that $u(A^c) \in (D_n^c)^+$ holds, that is $u(A^c) \subset D_n^c$, i.e., $D_n \subset \text{int}_u A$.

(2) $\Rightarrow$ (1): Let $(U_n)$ be a sequence of $\Delta^+$-open covers of $\mathcal{H}$. We may assume that each cover consists of basic open sets, that is $U_n = \{(D_n^c)^+ \mid \lambda \in \Lambda\}$. Then for each $n$ the collection $\Delta_n = \{D_{n,\lambda} \mid \lambda \in \Lambda\}$ is a $\Delta - \pi$-network for $(X, u)$. Indeed, for $A \subset X$ such that $\text{int}_u A \neq \emptyset$, there is a $D_{n,\lambda}$ such that $u(A^c) \in (D_{n,\lambda}^c)^+$, that is $u(A^c) \subset D_{n,\lambda}^c$, i.e., $D_{n,\lambda} \subset \text{int}_u A$. Applying (2), there is a sequence $(D_n)$, $D_n \in \Delta_n$, such that the collection $\{D_n \mid n \in \mathbb{N}\}$ is a $\Sigma - \pi$-network for $(X, u)$. Then the collection $\{(D_n^c)^+ \mid n \in \mathbb{N}\}$ is a $\Sigma^+$-open cover of $\mathcal{H}$.

By taking $\Sigma = \Delta$ we get

Theorem 5. For a space $(X, u)$ and a collection $\Delta$ the following are equivalent:

1. $(\mathcal{H}, \Delta^+)$ has the Rothberger property.  
2. $(X, u)$ satisfies $S_1(\Pi_\Delta, \Pi_\Delta)$.  

Again, by specifying the family $\Delta$ we get special cases.

Corollary 11. (cf. [2]) For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, \mathcal{Z}^+)$ has the Rothberger property.  
2. $(X, u)$ satisfies $S_1(\Pi_\omega, \Pi_\omega)$.  

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Corollary 12. (cf. [2]) For a $T_2$ space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, F^+$) has the Rothberger property.
2. $(X, u)$ satisfies $S_1(\Pi_k, \Pi_k)$.

Corollary 13. For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, V^+)$ has the Rothberger property.
2. $(X, u)$ satisfies $S_1(\Pi_\zeta, \Pi_\zeta)$.

We end this section by proving a result concerning Rothberger-type selection principles.

In [7, Lemma] a relation between the property $S_1(O, O)$ for each finite product of a topological space $X$ and the property $S_1(\Omega, \Omega)$ for $X$ was given. For closure spaces we prove the next statement.

Theorem 6. If each finite product of $(X, u)$ satisfies $S_1(I, I)$, then each finite product of $(X, u)$ has the property $S_1(I\Omega, I\Omega)$.

Proof. (i) First we prove that if each finite product of $X$ satisfies $S_1(I, I)$, then $X$ has the property $S_1(I\Omega, I\Omega)$. Let $(\mathcal{U}_n)$ be a sequence of interior $\omega$-covers of $X$. Let $\mathcal{N} = \cup_{m\in\mathbb{N}}N_m$ be a partition of $\mathbb{N}$ into infinite sets. For each $m \in \mathbb{N}$ and each $k \in N_m$ let $\mathcal{V}_k = \{U^m \mid U \in \mathcal{U}_k\}$. Since the interior operator in the product space $(X^m, v)$ the equality $\text{int}_v(U^m) = (\text{int}_uU)^m$ holds, $(\mathcal{V}_k)$ for $k \in N_m$, is a sequence of interior covers of $X^m$ as for each $x = (x_1, \ldots, x_m) \in X^m$ and each $k \in N_m$ there is a $U \in \mathcal{U}_k$ such that $\{x_1, \ldots, x_m\} \subseteq \text{int}_uU$, so that $x \in (\text{int}_uU)^m$. Applying to the sequence $(\mathcal{V}_k)$, $k \in N_m$, the assumption that each finite product of $X$ satisfies $S_1(I, I)$, there is for each $m \in \mathbb{N}$ a sequence $(\mathcal{V}_k)$, $\mathcal{V}_k \in \mathcal{V}_k$ for each $k \in N_m$, such that the collection $\{\mathcal{V}_k \mid k \in N_m\}$ is an interior cover of $X^m$. For each $k \in N_m$ let $U_k$ be an element in $\mathcal{U}_k$ with $V_k = U_k^m$. Then the collection $\{U_k \mid k \in N_m, m \in \mathbb{N}\}$ is an interior $\omega$-cover of $(X, u)$ which witnesses for $(\mathcal{U}_n)$ that $S_1(I\Omega, I\Omega)$ holds. Indeed, for every finite set $F = \{x_1, \ldots, x_m\} \subseteq X$ there is a $V_k = U_k^m \in \mathcal{V}_k$ with $(x_1, \ldots, x_m) \in \text{int}_uU_k^m$, hence $F \subseteq \text{int}_uU_k$.

(ii) Now we show that if $X$ has the property $S_1(I\Omega, I\Omega)$, then all finite powers of $X$ satisfy $S_1(I\Omega, I\Omega)$. Fix $m$. Let $(\mathcal{U}_n)$ be a sequence of interior $\omega$-covers of $X^m$. For each $n \in \mathbb{N}$ set $\mathcal{V}_n = \{V \subseteq X \mid V^m \subseteq U$ for some $U \in \mathcal{U}_n\}$. The collection $\mathcal{V}_n$ is an interior $\omega$-cover of $X$ for each $n$. Indeed, for every finite set $F = \{x_1, \ldots, x_k\} \subseteq X$ there is a $U \in \mathcal{U}_n$ such that $F^m \subseteq \text{int}_uU$. We choose a $V \subseteq X$ so that $F^m \subseteq \text{int}_uV^m = (\text{int}_uV)^m \subseteq V^m \subseteq U$. The set $V$ satisfies the required condition, that is $F \subseteq V \in \mathcal{V}_n$. Applying the assumption to
the sequence $V_n$ there exist $V_n \in V_n$, $n \in \mathbb{N}$, such that the collection
\{\{V_n \mid n \in \mathbb{N}\}\} is an interior $\omega$-cover of $X$. For each $n$ we pick $U_n \in U_n$, so that $V_n \supset U_n$. Since the collection \{\{V_n \mid n \in \mathbb{N}\}\} is an interior $\omega$-cover of $X^m$, so is \{\{U_n \mid n \in \mathbb{N}\}\}.

\[2. \text{ The Menger-like selection principles}\]

In topological spaces the property $S_{fin}(\mathcal{O}, \mathcal{O})$ is known as the Menger property. A topological space $X$ has countable fan tightness if for each $x \in X$ the selection principle $S_{fin}(\Phi_x, \Phi_x)$ holds.

In closure spaces these definitions read as follows.

**Definition 4.** A space $(X, u)$ has the Menger property if for every sequence $(U_n)$ of interior covers of $X$ there is a sequence $(V_n)$ of finite collections such that $V_n \subset U_n$ for each $n \in \mathbb{N}$ and the collection $\bigcup_{n \in \mathbb{N}} V_n$ is an interior cover of $X$.

**Definition 5.** A space $(X, u)$ has countable fan tightness if for every $x \in X$ and for each sequence $(A_n)$ of subsets of $X$ such that $x \in \bigcap_{n \in \mathbb{N}} u(A_n)$, there exist finite sets $B_n \subset A_n$ such that $x \in u(\bigcup_{n \in \mathbb{N}} B_n)$.

The proofs of the next theorems are analogous to those in the previous section. The selection principle $S_1$ is replaced with $S_{fin}$.

**Theorem 7.** For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $H$ satisfies the selection principle $S_{fin}(\Phi_H^\Delta, \Phi_H^\Sigma)$ for each $H \in \mathcal{H}$.
2. Each $Y \in \mathcal{J}$ satisfies $S_{fin}(I_{\Delta}, I_{\Sigma})$.

**Theorem 8.** For a space $(X, u)$ and a collection $\Delta$ the following are equivalent:

1. $(H, \Delta^\Delta)$ has countable fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_{fin}(I_{\Delta}, I_{\Delta})$.

**Theorem 9.** For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $H$ satisfies $S_{fin}(\mathcal{D}_\Delta^+, \mathcal{D}_\Sigma^+)$.
2. $(X, u)$ satisfies $S_{fin}(I_{\Delta}, I_{\Delta})$.

**Theorem 10.** For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $H$ satisfies $S_{fin}(\mathcal{O}_\Delta^+, \mathcal{O}_\Sigma^+)$.
2. $(X, u)$ satisfies $S_{fin}(\Pi_{\Delta}, \Pi_{\Sigma})$. 
In particular, when specifying \( \Delta \) and \( \Sigma \) the statements corresponding to Corollaries 1-10 are obtained.

By replacing the principle \( S_1 \) with \( S_{\text{fin}} \) in Theorem 6, we get by a similar proof

**Theorem 11.** If each finite product of \((X,u)\) satisfies \( S_{\text{fin}}(I,I) \), then each finite product of \((X,u)\) has the property \( S_{\text{fin}}(I\Omega, I\Omega) \).

### 3. Set-tightness

Recall that the *set-tightness* \( t_s \) of a space \( X \) is countable if for each \( A \subset X \) and each \( x \in A \) there is a sequence \((A_n)\) of subsets of \( A \) such that \( x \in \bigcup\{A_n \mid n \in \mathbb{N}\} \) but \( x \not\in A_n \) for each \( n \in \mathbb{N} \).

**Theorem 12.** For a space \((X,u)\) and a collection \( \Delta \) the following are equivalent:

1. \((\mathcal{H}, \Delta^+)\) has countable set-tightness.
2. For each \( Y \in \mathcal{J} \) and each interior \( \Delta \)-cover \( \mathcal{U} \) of \( Y \) there is a countable collection \( \{U_n \mid n \in \mathbb{N}\} \) of subsets of \( \mathcal{U} \) such that no \( U_n \) is an interior \( \Delta \)-cover of \( Y \) and \( \bigcup_{n \in \mathbb{N}} U_n \) is an interior \( \Delta \)-cover of \( Y \).

**Proof.** Follows from definitions and Lemma.

In particular, setting \( \Delta = F(X) \) (resp. \( K(X), Q(X) \)), we get

**Corollary 14.** (cf. [2]) For a space \((X,u)\) the following are equivalent:

1. \((\mathcal{H}, F^+)\) has countable set-tightness.
2. For each \( Y \in \mathcal{J} \) and each interior \( \omega \)-cover \( \mathcal{U} \) of \( Y \) there is a countable collection \( \{U_n \mid n \in \mathbb{N}\} \) of subsets of \( \mathcal{U} \) such that no \( U_n \) is an interior \( \omega \)-cover of \( Y \) and \( \bigcup_{n \in \mathbb{N}} U_n \) is an interior \( \omega \)-cover of \( Y \).

**Corollary 15.** (cf. [2]) For a \( T_2 \) space \((X,u)\) the following are equivalent:

1. \((\mathcal{H}, F^+)\) has countable set-tightness.
2. For each \( Y \in \mathcal{J} \) and each interior \( \kappa \)-cover \( \mathcal{U} \) of \( Y \) there is a countable collection \( \{U_n \mid n \in \mathbb{N}\} \) of subsets of \( \mathcal{U} \) such that no \( U_n \) is an interior \( \kappa \)-cover of \( Y \) and \( \bigcup_{n \in \mathbb{N}} U_n \) is an interior \( \kappa \)-cover of \( Y \).

**Corollary 16.** For a space \((X,u)\) the following are equivalent:

1. \((\mathcal{H}, F^+)\) has countable set-tightness.
2. For each \( Y \in \mathcal{J} \) and each interior \( \zeta \)-cover \( \mathcal{U} \) of \( Y \) there is a countable collection \( \{U_n \mid n \in \mathbb{N}\} \) of subsets of \( \mathcal{U} \) such that no \( U_n \) is an interior \( \zeta \)-cover of \( Y \) and \( \bigcup_{n \in \mathbb{N}} U_n \) is an interior \( \zeta \)-cover of \( Y \).
4. The Pytkeev property

A topological space $X$ has the Pytkeev property if for every $A \subset X$ and each $x \in A \setminus \{x\}$ there is a countable collection $\{A_n \mid n \in \mathbb{N}\}$ of countable infinite subsets of $A$ which is a $\pi$-network at $x$, i.e., each neighborhood of $x$ contains some $A_n$.

Let $\tau$ and $\sigma$ be two topologies on a set $X$. Then $X$ has the $(\tau, \sigma)$-Pytkeev property if for every $A \subset X$ and each $x \in \text{cl}_\tau(A \setminus \{x\})$ there is a countable collection $\{A_n \mid n \in \mathbb{N}\}$ of countable infinite subsets of $A$ which is a $\pi$-network at $x$ with respect to the $\sigma$ topology.

In the next theorem we suppose that for every interior $\Delta$-cover $U$ of $Y$, $Y \neq \text{int}_u U$ for each $U \in \mathcal{U}$ holds.

**Theorem 13.** For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $\mathcal{H}$ has the $(\Delta^+, \Sigma^+)$-Pytkeev property.
2. For each $Y \in \mathcal{J}$ and each interior $\Delta$-cover $U$ of $Y$ there is a sequence $(U_n)$ of infinite subsets of $U$ such that $\{\text{int}_u U_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) $\Sigma$-cover of $Y$. \[\cap\text{int}_u U_n = \cap\{\text{int}_u U \mid U \in U_n\}\]

**Proof.** As in [4, Theorem 8] by using Lemma. \qed

In particular, setting $\Delta = \Sigma = F(X)$ (resp. $\Delta = \Sigma = K(X), Q(X)$), we get

**Corollary 17.** (cf. [4]) For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, Z^+)$ has the Pytkeev property.
2. For each $Y \in \mathcal{J}$ and each interior $\omega$-cover $U$ of $Y$ there is a sequence $(U_n)$ of countable infinite subsets of $U$ such that $\{\text{int}_u U_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) $\omega$-cover of $Y$.

**Corollary 18.** (cf. [4]) For a $T_2$ space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, F^+)$ has the Pytkeev property.
2. For each $Y \in \mathcal{J}$ and each interior $\kappa$-cover $U$ of $Y$ there is a sequence $(U_n)$ of countable infinite subsets of $U$ such that $\{\text{int}_u U_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) $\kappa$-cover of $Y$.

**Corollary 19.** For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, V^+)$ has the Pytkeev property.
2. For each $Y \in \mathcal{J}$ and each interior $\zeta$-cover $U$ of $Y$ there is a sequence $(U_n)$ of countable infinite subsets of $U$ such that $\{\text{int}_u U_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) $\zeta$-cover of $Y$. 
The notion of groupability was introduced in [5]. A countable interior $C$-cover $\mathcal{U}$ of a space $(X,u)$ is \textit{groupable} if there is a partition $\mathcal{U} = \bigcup_{n \in \mathbb{N}} U_n$ into finite sets such that for each $C \in C$ and for all but finitely many $n$ there is a $U \in U_n$ such that $C \subset \text{int}_u U$ holds. Let $\mathcal{IC}^{gp}$ denote the family of groupable interior $C$-covers of the space.

A countable set $A \in \Phi_x$ is \textit{groupable} if there is a partition $A = \bigcup_{n \in \mathbb{N}} A_n$ into finite sets such that each neighborhood of $x$ has non-empty intersection with all but finitely many $A_n$.

Recall that a space $X$ has the Reznichenko property [\text{is weakly Fréchet-Urysohn}] if for every $A \subset X$ and each $x \in A \setminus A$ there is a countable infinite family $A$ of finite pairwise disjoint subsets of $A$ such that each neighborhood of $x$ meets all but finitely many elements of $A$.

\textbf{Theorem 14.} For a space $(X,u)$ and a collection $\Delta$ the following are equivalent:

1. $(\mathcal{H}, \Delta^+)$ has the Reznichenko property.
2. For each $Y \in \mathcal{J}$ and each interior $\Delta$-cover $\mathcal{U}$ of $Y$, there is a sequence $(U_n)$ of finite pairwise disjoint subsets of $\mathcal{U}$ such that each $D \in \Delta$ belongs to $\text{int}_u U_n$ for some $U_n \in \mathcal{U}$ for all but finitely many $n$.

\textit{Proof.} (1) $\Rightarrow$ (2): Let $Y \in \mathcal{J}$ and $\mathcal{U}$ be an interior $\Delta$-cover of $Y$. By Lemma, $Y^c \in \text{cl}_{\Delta^+} A$ where $A = \{u(U^c) \mid U \in \mathcal{U}\}$. By assumption, there is a sequence $(A_n)$ of finite pairwise disjoint subsets of $A$ such that each $\Delta^+$-neighborhood of $Y^c$ intersects $A_n$ for all but finitely many $n$. The sequence $(U_n)$ defined by $U_n = A_n^c$ satisfies the required conditions. Indeed, $U_n$ are finite, pairwise disjoint and for $D \in \Delta$ such that $D \subset Y$, the collection $(D^c)^+$ is a $\Delta^+$-neighborhood of $Y^c$. There is $n_0$ such that $(D^c)^+ \cap A_n \neq \emptyset$ for each $n > n_0$. Hence, for every $n > n_0$ there is a set $A_n \in A_n$ such that $A_n \subset D^c$, that is $D \subset A_n^c = \text{int}_u U_n$ and $U_n \in \mathcal{U}$ holds.

(2) $\Rightarrow$ (1): Let $A \subset \mathcal{H}$ and $H \in \mathcal{H}$ so that $H \in \text{cl}_{\Delta^+} A \setminus A$. Then $\mathcal{U} = A^c$ is an interior $\Delta$-cover of $H^c = Y$. Applying (2) to $Y$ and $\mathcal{U}$, there is a sequence $(U_n)$ of finite pairwise disjoint subsets of $\mathcal{U}$ such that for each $D \subset Y, D \in \Delta$, for all but finitely many $n$ there is a $U_n \in \mathcal{U}_n$ such that $D \subset \text{int}_u U_n$. For each $n$ we set $G_n = \{u(U^c) \mid U \in U_n\}$. The collection $\{G_n \mid n \in \mathbb{N}\}$ satisfies (1). \hfill $\square$
By combining Theorem 14 with Theorems 2 and 8 we get the following statements.

**Theorem 15.** For a space $(X, u)$ and a collection $\Delta$ the following are equivalent:

1. $(\mathcal{H}, \Delta^+)$ has the Reznichenko property and countable strong fan tightness.
2. Each $Y \in \mathcal{I}$ satisfies $S_1(\mathcal{I} \Delta, \mathcal{I} \Delta^{op})$.

**Theorem 16.** For a space $(X, u)$ and a collection $\Delta$ the following are equivalent:

1. $(\mathcal{H}, \Delta^+)$ has the Reznichenko property and countable fan tightness.
2. Each $Y \in \mathcal{I}$ satisfies $S_{fin}(\mathcal{I} \Delta, \mathcal{I} \Delta^{gp})$.

In particular, setting $\Delta = \mathcal{F}(X)$ (resp. $\Delta = \mathcal{K}(X), \mathcal{Q}(X)$), we get

**Corollary 20.** (cf. [4]) For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, \mathcal{Z}^+)$ has the Reznichenko property and countable strong fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_1(\mathcal{I} \Omega, \mathcal{I} \Omega^{op})$.

**Corollary 21.** (cf. [4]) For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, \mathcal{Z}^+)$ has the Reznichenko property and countable fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_{fin}(\mathcal{I} \Omega, \mathcal{I} \Omega^{gp})$.

**Corollary 22.** (cf. [4]) For a $T_2$ space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, \mathcal{F}^+)$ has the Reznichenko property and countable strong fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_1(\mathcal{I} \mathcal{K}, \mathcal{I} \mathcal{K}^{op})$.

**Corollary 23.** (cf. [4]) For a $T_2$ space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, \mathcal{F}^+)$ has the Reznichenko property and countable fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_{fin}(\mathcal{I} \mathcal{K}, \mathcal{I} \mathcal{K}^{gp})$.

**Corollary 24.** For a space $(X, u)$ the following are equivalent:

1. $(\mathcal{H}, \mathcal{V}^+)$ has the Reznichenko property and countable strong fan tightness.
2. Each $Y \in \mathcal{J}$ satisfies $S_1(\mathcal{I} \mathcal{Q}, \mathcal{I} \mathcal{Q}^{op})$.
COROLLARY 25. For a space $(X,u)$ the following are equivalent:

1. $(\mathcal{H},V^+)\) has the Reznichenko property and countable fan tightness.

2. Each $Y \in \mathcal{J}$ satisfies $S_{fin}(\mathcal{I}_Q,\mathcal{I}_Q^gp)$.

Using appropriate modifications in the proof of Theorem 6 we get the statement which includes groupability.

THEOREM 17. If each finite product of $(X,u)$ satisfies $S_1(\mathcal{I},\mathcal{I}^{gp})$, then each finite product of $(X,u)$ has the property $S_1(\mathcal{I}_Q,\mathcal{I}_Q^{gp})$.

Proof. (i) First we prove that if each finite product of $X$ satisfies $S_1(\mathcal{I},\mathcal{I}^{gp})$, then $X$ has the property $S_1(\mathcal{I}_Q,\mathcal{I}_Q^{gp})$. Following the proof of Theorem 6 and applying the assumption to the sequence $(\nu_k)$, $k \in \mathbb{N}$, there is for each $m \in \mathbb{N}$ a sequence $(\nu_k)$, $V_k \in \nu_k$ for each $k \in \mathbb{N}$, such that the collection $W_m = \{V_k \mid k \in \mathbb{N}\}$ is a groupable interior cover of $X^m$. There is a partition $W_m = \cup_{\nu \in \mathbb{N}} W_{m\nu}$ into finite sets such that for each $x \in X^m$ and for all but finitely many $\nu$ there is a $V \in W_{m\nu}$ such that $x \in \text{int}_V$ in the product space $(X^m,\nu)$ holds. Each $W_{m\nu} = \{V_{k_1(\nu)}, \ldots, V_{k_s(\nu)}\}$ for some $k_1(\nu), \ldots, k_s(\nu) \in \mathbb{N}$. For each $k \in \mathbb{N}$ let $U_k$ be an element in $U_k$ with $V_k = U_k^m$ and $\gamma_{m\nu} = \{U_{k_1(\nu)}, \ldots, U_{k_s(\nu)}\}$. The collection $\{\gamma_{\mu} \mid \mu \in \mathbb{N}\}$ where $\gamma_{\mu} = \cup\{\gamma_{m\nu} \mid \mu = m + \nu - 1; n, \nu \in \mathbb{N}\}$ witnesses the groupability of the interior $\omega$-cover $\gamma = \cup_{\mu \in \mathbb{N}} \gamma_{\mu} = \{U_k \mid k \in \mathbb{N}, m \in \mathbb{N}\}$ of $(X,u)$.

Indeed, for every finite subset $F = \{x_1, \ldots, x_m\} \subset X$ there is a $\nu_0$ such that for all $\nu \in \mathbb{N}$, $\nu \geq \nu_0$ implies there is an $m \in \mathbb{N}$ and a $V \in W_{m\nu}$ such that $x = (x_1, \ldots, x_m) \in \text{int}_V$ holds. Hence, there is a $\mu_0 = m + \nu_0$ such that for all $\mu \in \mathbb{N}$, $\mu \geq \mu_0$ implies there is a $U \in \gamma_{\mu}$ such that $F \subset \text{int}_U$.

(ii) Now we show that if $X$ has the property $S_1(\mathcal{I}_Q,\mathcal{I}_Q^{gp})$, then all finite powers of $X$ satisfy $S_1(\mathcal{I}_Q,\mathcal{I}_Q^{gp})$. As in Theorem 6, fix $m$. Given a sequence $(\mathcal{U}_n)$ of interior $\omega$-covers of $X^m$ consider a sequence $(\nu_n)$ of interior $\omega$-covers of $X$ where $\mathcal{V}_n = \{V \subset X \mid V^m \subset U\}$ for some $U \in \mathcal{U}_n$. By the assumption there exist $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, such that the collection $\mathcal{W} = \{V_n \mid n \in \mathbb{N}\}$ is a groupable interior $\omega$-cover of $X$. For each $n$ we pick $U_n \in \mathcal{U}_n$ so that $V_n^m \subset U_n$. Since the collection $\{V_n^m \mid n \in \mathbb{N}\}$ is a groupable interior $\omega$-cover of $X^m$, so is $\{U_n \mid n \in \mathbb{N}\}$.

\[\square\]

In a similar way we prove

THEOREM 18. If each finite product of $(X,u)$ satisfies $S_{fin}(\mathcal{I},\mathcal{I}^{gp})$, then each finite product of $(X,u)$ has the property $S_{fin}(\mathcal{I}_Q,\mathcal{I}_Q^{gp})$. 

We conclude this paper with the next statements.

**Theorem 19.** (see [4]) For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $\mathcal{H}$ satisfies $S_1(\Phi_H^\Delta, (\Phi_H^\Sigma)^{gp})$ for each $H \in \mathcal{H}$.
2. Each $Y \in \mathcal{J}$ satisfies $S_1(I\Delta, I\Sigma^{gp})$.

**Proof.** The constructions are analogous to those in the previous theorems.

By replacing the selection principle $S_1$ with $S_{fin}$ we get

**Theorem 20.** (see [4]) For a space $(X, u)$ and collections $\Delta$ and $\Sigma$ the following are equivalent:

1. $\mathcal{H}$ satisfies $S_{fin}(\Phi_H^\Delta, (\Phi_H^\Sigma)^{gp})$ for each $H \in \mathcal{H}$.
2. Each $Y \in \mathcal{J}$ satisfies $S_{fin}(I\Delta, I\Sigma^{gp})$.

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