MAXIMAL SPACE-LIKE HYPERSURFACES IN $H^4_1(-1)$ WITH ZERO GAUSS-KRONECKER CURVATURE

QING-MING CHENG AND YOUNG JIN SUH

Abstract. In this paper, we study complete maximal space-like hypersurfaces with constant Gauss-Kronecker curvature in an anti-de Sitter space $H^4_1(-1)$. It is proved that complete maximal space-like hypersurfaces with constant Gauss-Kronecker curvature in an anti-de Sitter space $H^4_1(-1)$ are isometric to the hyperbolic cylinder $H^2(c_1) \times H^1(c_2)$ with $S = 3$ or they satisfy $S \leq 2$, where $S$ denotes the squared norm of the second fundamental form.

1. Introduction

Let $M^n_s(c)$ be an $n$-dimensional connected semi-Riemannian manifold of index $s(\geq 0)$ and of constant curvature $c$. It is called a semi-definite space form of index $s$. When $s = 1$, $M^n_1(c)$ is said to be a Lorentz space form. Such Lorentz space forms $M^n_1(c)$ can be divided into three kinds of semi-definite space forms: the de Sitter space $S^n_1(c)$, the Minkowski space $R^n_1$, or the anti-de Sitter space $H^n_1(c)$, according to the sign of its sectional curvature $c > 0$, $c = 0$, or $c < 0$ respectively.

In connection with the negative settlement of the Bernstein problem due to Calabi [4] and Cheng-Yau [8], Chouque-Bruhat et al. [9] proved the following theorem independently.

Theorem A. Let $M$ be a complete space-like hypersurface in an $(n+1)$-dimensional Lorentz space form $M^{n+1}_1(c)$, $c \geq 0$. If $M$ is maximal, then it is totally geodesic.
As a generalization of this result, complete space-like hypersurfaces with constant mean curvature in a Lorentz manifold have been investigated by Akutagawa [1], Li [11], Montiel [12], Nishikawa [13], Baek and the present authors [3], and Choi, Yang and the second author [16].

On the other hand, some generalizations of Theorem A for submanifolds with codimension $p \geq 1$ were given by Ishihara [10], Nakagawa and the first author [7], and the first author [5]. Among them Ishihara [10] proved that an $n$-dimensional complete maximal space-like submanifolds with codimension $p$ in an $(n + p)$-dimensional semi-definite space form $M^{n+p}_p(c)$, $c \geq 0$ is totally geodesic.

Now let us consider a complete maximal space-like hypersurface in an anti-de Sitter space $H^{n+1}_{n+1}(-1)$ and denote by $S$ the squared norm of the second fundamental form of this hypersurface. Then Ishihara [10] has also proved that the squared norm $S$ satisfies $0 \leq S \leq n$ and the hyperbolic cylinders $H^{n-k}(c_1) \times H^k(c_2)$, $k = 1, 2, \ldots, n-1$ are the only complete maximal space-like hypersurfaces in an anti-de Sitter space $H^{n+1}_{n+1}(-1)$ satisfying $S \equiv n$.

Then it could be natural to investigate complete maximal space-like hypersurfaces in $H^{n+1}_{n+1}(-1)$, which do not satisfy $S \equiv n$. When $n = 3$, the first author [6] gave several characterizations for such hypersurfaces and it was proved that hyperbolic cylinders $H^2(c_1) \times H^1(c_2)$ are the only complete maximal space-like hypersurfaces in $H^3_{3}(-1)$ with nonzero constant Gauss-Kronecker curvature.

For the case that Gauss-Kronecker curvature is zero we have no result until now. Since totally geodesic maximal space-like hypersurfaces were known to have zero Gauss-Kronecker curvature, the following problem was proposed by the first author [5].

Problem. (6) Is it true that every complete maximal space-like hypersurface in $H^4_{4}(-1)$ with zero Gauss-Kronecker curvature is totally geodesic?

In this paper, we shall give two characterizations of such hypersurfaces, which imply the above problem may be solved affirmatively.

**Theorem 1.** Let $M^3$ be a complete maximal space-like hypersurface in an anti-de Sitter space $H^4_{4}(-1)$ with zero Gauss-Kronecker curvature. Then, $M^3$ satisfies $S \leq 2$, where $S$ denotes the squared norm of the second fundamental form.

From Theorem 1 and the result due to the first author [6], we obtain
**Corollary.** Let $M^3$ be a complete maximal space-like hypersurface in an anti-de Sitter space $H^4_1(-1)$ with constant Gauss-Kronecker curvature. Then, $M^3$ is isometric to the hyperbolic cylinder $H^2(c_1) \times H^1(c_2)$ with $S = 3$ or $M^3$ satisfies $S \leq 2$.

If a maximal space-like hypersurface in $H^4_1(-1)$ is not assumed to be complete, we can assert the following:

**Theorem 2.** Let $M^3$ be a maximal space-like hypersurface in an anti-de Sitter space $H^4_1(-1)$ with zero Gauss-Kronecker curvature. If the principal curvature functions are constant along the curvature line corresponding to the zero principal curvature, then $M^3$ is totally geodesic.

## 2. Preliminaries

We consider Minkowski space $\mathbb{R}^{n+2}$ as the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given by

\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1} - x_{n+2} y_{n+2}
\]

for $x, y \in \mathbb{R}^{n+2}$. Then, for $c > 0$, the anti-de Sitter space $H^{n+1}_1(-c)$ can be defined as the following hyperquadric of $\mathbb{R}^{n+2}$

\[
H^{n+1}_1(-c) = \left\{ x \in \mathbb{R}^{n+2} : |x|^2 = -\frac{1}{c} \right\}.
\]

In this way, the anti-de Sitter space $H^{n+1}_1(-c)$ inherits from $\langle \cdot, \cdot \rangle$ a metric which makes it an indefinite Riemannian manifold of constant sectional curvature $-c$. For indefinite Riemannian manifolds, refer to B. O’Neill [15].

Moreover, if $x \in H^{n+1}_1(-c)$, we can put

\[
T_x H^{n+1}_1(-c) = \{ v \in \mathbb{R}^{n+2} : \langle v, x \rangle = 0 \}.
\]

If $\nabla^L$ and $\bar{\nabla}$ denote the metric connections of $\mathbb{R}^{n+2}$ and $H^{n+1}_1(-c)$ respectively, we have

\[
\nabla^L_v w - \bar{\nabla}_v w = c(\langle v, w \rangle x
\]
for all vector fields $v, w$ which are tangent to $H^{n+1}_1(-c)$. Let

$$\phi : M^n \rightarrow H^{n+1}_1(-c)$$

be a connected space-like hypersurface immersed in $H^{n+1}_1(-c)$ and let $\nabla$ be the Levi-Civita connection corresponding to the Riemannian metric $g$ induced on $M^n$ from $\langle , \rangle$. Then the second fundamental form $\tilde{h}$ and the Weingarten endomorphism $A$ of $\phi$ are given by

$$\nabla_v w - \nabla_v w = \tilde{h}(v, w),$$

where $v, w$ are vector fields tangent to $M^n$ and $N$ is a unit timelike vector field normal to $M^n$. So, the mean curvature $H$ of the immersion $\phi$ is given by $nH = \text{trace} A$.

Let us denote by $R$ the curvature tensor field of $M$. The Gauss equation is given by

$$R(v, w)u = -c\{g(w, u)v - g(v, u)w\} - \{g(Aw, u)Av - g(Av, u)Aw\},$$

where $v, w$ and $u$ are vector fields tangent to $M^n$. The Codazzi equation is expressed by

$$(\nabla_v A)w = (\nabla_w A)v.$$
Next we consider the case of $n = 3$. Since $\nabla e_i e_j$ are tangent to $M^3$ and $e_1, e_2, e_3$ is a local field of orthonormal differentiable frames, we know that there are 9 functions $a_1, a_2, \ldots, a_9$ such that
\begin{align}
\nabla e_1 e_1 &= a_1 e_2 + a_2 e_3, \quad \nabla e_1 e_2 = -a_1 e_1 + a_3 e_3, \quad \nabla e_1 e_3 = -a_2 e_1 - a_3 e_2, \\
\nabla e_2 e_1 &= -a_4 e_2 + a_6 e_3, \quad \nabla e_2 e_2 = a_4 e_1 + a_5 e_3, \quad \nabla e_2 e_3 = -a_6 e_1 - a_5 e_2, \\
\nabla e_3 e_1 &= a_9 e_2 - a_7 e_3, \quad \nabla e_3 e_2 = -a_9 e_1 - a_8 e_3, \quad \nabla e_3 e_3 = a_7 e_1 + a_8 e_2.
\end{align}

The following Generalized Maximum Principle due to Omori and Yau will be used in order to prove our theorems.

**Generalized Maximum Principle.** (Omori [14] and Yau [17]) Let $M^n$ be a complete Riemannian manifold whose Ricci curvature is bounded from below and $f \in C^2(M)$ a function bounded from above on $M^n$. Then for any $\epsilon > 0$, there exists a point $p \in M^n$ such that
\[ f(p) \geq \sup f - \epsilon, \quad \|\text{grad}f\|(p) < \epsilon, \quad \nabla_i \nabla_i f(p) < \epsilon, \]
for $i = 1, 2, \ldots, n$.

3. Proofs of theorems

In order to prove our theorems, we shall prepare two lemmas, firstly.

**Lemma 1.** Let $M^3$ be a space-like hypersurface in an anti-de Sitter space $\mathbb{H}_4^1(-1)$. If the principal curvatures $\lambda_i$’s are different from each others on an open subset $\mathcal{U}$ of $M^3$, then on $\mathcal{U}$, we have the following:
\begin{align*}
e_1(\lambda_2) &= a_4(\lambda_2 - \lambda_1), \quad e_1(\lambda_3) = a_7(\lambda_3 - \lambda_1), \\
e_2(\lambda_1) &= a_1(\lambda_1 - \lambda_2), \quad e_2(\lambda_3) = a_8(\lambda_3 - \lambda_2), \\
e_3(\lambda_1) &= a_2(\lambda_1 - \lambda_3), \quad e_3(\lambda_2) = a_5(\lambda_2 - \lambda_3), \\
a_9(\lambda_1 - \lambda_2) &= a_3(\lambda_2 - \lambda_3) = a_6(\lambda_1 - \lambda_3),
\end{align*}
where the above functions $a_i$, $i = 1, \ldots, 9$ are given in section 2.
Proof. Since these principal curvatures $\lambda_i$'s are different from each other on the open subset $\mathcal{U}$ of $M$, then on $\mathcal{U}$, $\lambda_i$'s are differentiable functions. From Codazzi equation (2.7), we have

$$(\nabla_{e_1} A)e_2 = (\nabla_{e_2} A)e_1.$$ 

From (2.9), we obtain

$$\nabla_{e_1}(\lambda_2 e_2) - A\nabla_{e_1} e_2 = \nabla_{e_2}(\lambda_1 e_1) - A\nabla_{e_2} e_1,$$

$$e_1(\lambda_2)e_2 + \lambda_2 \nabla_{e_1} e_2 - A\nabla_{e_1} e_2 = e_2(\lambda_1)e_1 + \lambda_1 \nabla_{e_2} e_1 - A\nabla_{e_2} e_1.$$ 

From (2.10) and (2.11), we infer

$$e_1(\lambda_2)e_2 + \lambda_2(-a_1e_1 + a_3e_3) + a_1\lambda_1e_1 - a_3\lambda_3e_3 = e_2(\lambda_1)e_1 + \lambda_1(-a_4e_2 + a_6e_3) + a_4\lambda_2e_2 - a_6\lambda_3e_3.$$ 

Hence, we have

$$e_1(\lambda_2) = a_4(\lambda_2 - \lambda_1), \quad e_2(\lambda_1) = a_1(\lambda_1 - \lambda_2), \quad a_3(\lambda_2 - \lambda_3) = a_6(\lambda_1 - \lambda_3).$$

Similarly, we can prove the other also holds. Now we complete the proof of Lemma 1. \qed

Since $M^3$ is maximal and the Gauss-Kronecker curvature is zero, we can assume $\lambda_1 = \lambda = -\lambda_2, \lambda_3 = 0$. Then we are able to state the following:

**Lemma 2.** Let $M^3$ be a maximal space-like hypersurface with zero Gauss-Kronecker curvature in an anti-de Sitter space $H^4_{1}(-1)$. If $S$ is not zero on an open subset $\mathcal{U}$ of $M^3$, then on $\mathcal{U}$, we have

\begin{align*}
(3.1) \quad & e_1(a_4) + e_2(a_1) = \lambda^2 - 1 + a_1^2 + a_2^2 + 2a_3^2 + a_4^2, \\
(3.2) \quad & e_3(a_1) + \frac{1}{2}e_1(a_3) = a_1a_2 - \frac{1}{2}a_3a_4, \\
(3.3) \quad & e_3(a_4) - \frac{1}{2}e_2(a_3) = a_2a_4 + \frac{1}{2}a_1a_3, \\
(3.4) \quad & e_3(a_2) = -1 + a_2^2 - a_3^2, \\
(3.5) \quad & e_1(a_2) = e_2(a_3), \quad e_1(a_3) = -e_2(a_2), \quad e_3(a_3) = 2a_2a_3,
\end{align*}

where $\lambda = \lambda_1 \neq 0$. 

Proof. Since $M^3$ is maximal and the Gauss-Kronecker curvature is zero, we may assume $\lambda_1 = \lambda = -\lambda_2 \neq 0$, $\lambda_3 = 0$. According to Lemma 1, we have

\begin{equation}
(3.6) \quad e_1(\lambda) = 2a_4 \lambda, \quad e_2(\lambda) = 2a_1 \lambda, \quad e_3(\lambda) = a_2 \lambda,
\end{equation}

and

\begin{equation}
(3.7) \quad a_5 = a_2, \quad 2a_9 = -a_3 = a_6, \quad a_7 = a_8 = 0.
\end{equation}

From (2.10), (2.11), and (2.12), we can obtain the following formulas

\begin{equation}
(3.8) \quad [e_1, e_2] = -a_1 e_1 + a_4 e_2 + 2a_3 e_3,
\end{equation}

\begin{equation}
(3.9) \quad [e_1, e_3] = -a_2 e_1 - \frac{1}{2} a_3 e_2,
\end{equation}

\begin{equation}
(3.10) \quad [e_2, e_3] = \frac{1}{2} a_3 e_1 - a_2 e_2.
\end{equation}

From the definition of the curvature tensor and the Gauss equation (2.6), we have

\begin{equation}
(3.11) \quad \nabla e_1 \nabla e_2 e_2 - \nabla e_2 \nabla e_1 e_2 - \nabla [e_1, e_2] e_2 = R(e_1, e_2) e_2 = (\lambda^2 - 1) e_1.
\end{equation}

From (2.10) and (2.11), we have

\begin{equation}
(3.12) \quad \nabla e_1 \nabla e_2 e_2 - \nabla e_2 \nabla e_1 e_2 - \nabla [e_1, e_2] e_2 \\
= \nabla e_1 (a_4 e_1 + a_2 e_3) - \nabla e_2 (-a_1 e_1 + a_3 e_3) - \nabla (-a_1 e_1 + a_4 e_2 + 2a_3 e_3) e_2 \\
= \{e_1(a_4) + e_2(a_1) - a_1^2 - a_2^2 - 2a_3^2 - a_4^2\} e_1 + \{e_1(a_2) - e_2(a_3)\} e_3.
\end{equation}

From (3.11) and (3.12), we infer

\begin{equation}
\begin{aligned}
e_1(a_4) + e_2(a_1) &= \lambda^2 - 1 + a_1^2 + a_2^2 + 2a_3^2 + a_4^2, \quad e_1(a_2) = e_2(a_3).
\end{aligned}
\end{equation}

Making use of a similar proof, we can obtain

\begin{equation}
\begin{aligned}
e_1(a_3) &= -e_2(a_2), \\
e_3(a_1) + \frac{1}{2} e_1(a_3) &= a_1 a_2 - \frac{1}{2} a_3 a_4, \\
e_3(a_2) &= -1 + a_2^2 - a_3^2.
\end{aligned}
\end{equation}
\begin{align*}
e_3(a_4) - \frac{1}{2}e_2(a_3) &= a_2a_4 + \frac{1}{2}a_1a_3, \\
e_3(a_3) &= 2a_2a_3,
\end{align*}
from
\begin{align*}
\nabla e_1 \nabla e_2 e_1 - \nabla e_2 \nabla e_1 e_1 - \nabla_{[e_1, e_2]} e_1 &= R(e_1, e_2)e_1 = (1 - \lambda^2)e_2, \\
\nabla e_1 \nabla e_3 e_1 - \nabla e_3 \nabla e_1 e_1 - \nabla_{[e_1, e_3]} e_1 &= R(e_1, e_3)e_1 = e_3, \\
\nabla e_2 \nabla e_3 e_2 - \nabla e_3 \nabla e_2 e_2 - \nabla_{[e_2, e_3]} e_2 &= R(e_2, e_3)e_2 = e_3
\end{align*}
and
\begin{align*}
\nabla e_3 \nabla e_1 e_3 - \nabla e_1 \nabla e_3 e_3 - \nabla_{[e_1, e_3]} e_1 &= R(e_3, e_1)e_3 = e_1.
\end{align*}
Thus, the proof is completed. \(\square\)

**Proof of Theorem 1.** From a result due to Ishihara [10], we know \(S \leq 3\). If \(\sup S = 0\), then our theorem is true. Next we consider the case of \(\sup S > 0\). Then let us construct an open subset \(U\) of \(M^3\) in such a way that
\[U = \{p \in M^3; S(p) > 0\}.
\]
Since the Gauss-Kronecker curvature is zero and \(M^3\) is maximal, we can assume
\[\lambda_1 = \lambda, \quad \lambda_2 = -\lambda \quad \text{and} \quad \lambda_3 = 0.
\]
Thus, on such an open subset \(U\), these principal curvatures \(\lambda_1, \lambda_2\) and \(\lambda_3\) are different from each other. Hence, they are differentiable on \(U\).

Now we are able to assume that \(\lambda > 0\) on \(U\). From the Gauss equation, we know that the sectional curvature is bounded from below by \(-1\). Applying the Generalized Maximum Principle due to Omori [14] and Yau [17] in section 2 to the function \(S\), we know that there exists a sequence \(\{p_k\} \subset M^3\) such that
\begin{align*}
(3.13) & \lim_{k \to \infty} S(p_k) = \sup S, \quad \lim_{k \to \infty} \|\text{grad}\ S\|(p_k) = 0, \\
(3.14) & \lim_{k \to \infty} \sup_{\nabla_i \nabla_i S(p_k) \leq 0, \ for \ i = 1, 2, 3.}
\end{align*}
Since \(\sup S > 0\), we can assume \(\{p_k\} \subset U\). On \(U\), \(S = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 2\lambda^2\). Hence, we have
\begin{align*}
(3.15) & \sup S = \lim_{k \to \infty} S(p_k) = 2 \lim_{k \to \infty} \lambda(p_k)^2.
\end{align*}
From (3.13) and
\[ (3.16) \quad e_1(\lambda) = 2a_4 \lambda, \quad e_2(\lambda) = 2a_1 \lambda, \quad e_3(\lambda) = a_2 \lambda, \]
we have
\[ (3.17) \quad \lim_{k \to \infty} a_1(p_k) = 0, \quad \lim_{k \to \infty} a_2(p_k) = 0, \quad \lim_{k \to \infty} a_4(p_k) = 0. \]

From (3.14) and \( S = 2\lambda^2, \lambda > 0 \), we have
\[ (3.18) \quad \lim_{k \to \infty} \sup_{k} e_1(\lambda)(p_k) \leq 0, \quad \lim_{k \to \infty} \sup_{k} e_2(\lambda)(p_k) \leq 0, \quad \lim_{k \to \infty} \sup_{k} e_3(\lambda)(p_k) \leq 0. \]

From (3.16), we have
\[ e_1 e_1(\lambda) = 2e_1(a_4) \lambda + 2a_4 e_1(\lambda), \]
\[ e_2 e_2(\lambda) = 2e_2(a_1) \lambda + 2a_1 e_2(\lambda), \]
\[ e_3 e_3(\lambda) = e_3(a_2) \lambda + a_2 e_3(\lambda). \]

Thus, we obtain
\[ \lim_{k \to \infty} \sup_{k} e_1(a_4)(p_k) \leq 0 \quad \text{and} \quad \lim_{k \to \infty} \sup_{k} e_2(a_1)(p_k) \leq 0. \]

From the formula (3.1) in Lemma 2, we have
\[ \lim_{k \to \infty} \lambda(p_k)^2 \leq 1. \]

Hence, we infer \( \sup S \leq 2 \). Now we complete the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** If there exists a point \( p \in M^3 \) such that \( S(p) > 0 \), then by using the similar assertion as in the proof of Theorem 1, we have that on an open subset \( \Omega \), \( S(p) > 0 \) and these principal curvatures are differentiable. From the assumption of Theorem 2, we have \( a_2 = 0 \) according to (3.6). From (3.4), we infer \( -1 - a_3^2 = 0 \). This is impossible. Hence, \( S \equiv 0 \) on \( M^3 \), that is, \( M^3 \) is totally geodesic. Thus, Theorem 2 is proved. \( \square \)

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References


Qing-Ming Cheng
Department of Mathematics
Faculty of Science and Engineering
Saga University
Saga 840-8502, Japan
E-mail: cheng@ms.saga-u.ac.jp
Maximal space-like hypersurfaces

Young Jin Suh
Department of Mathematics
Kyungpook National University
Taegu, 702-701, Korea
E-mail: yjsuh@mail.knu.ac.kr