THE LATTICE OF INTUITIONISTIC FUZZY IDEALS OF A RING

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Abstract. We investigate the lattice structure of various sublattices of the lattice of intuitionistic fuzzy subrings of a given ring. We prove that a special class of intuitionistic fuzzy ideals forms a modular sublattice of the lattice of intuitionistic fuzzy ideals of a ring. Finally, we show that the lattice of intuitionistic fuzzy ideals of \( R \) is not complemented [resp. has no atoms (dual atoms)].

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0. Introduction

The concept of a fuzzy set was introduced by Zadeh[18], and it is now a rigorous area of research with manifold applications ranging from engineering and computer science to medical diagnosis and social behavior studies. In particular, some researchers [1, 17,19] applied the notion of fuzzy sets to ideals of a ring.

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov[2] in 1986. After that time, Coker and his colleagues [6,7,8], Lee and Lee[16], and Hur and his colleagues [13] applied the concept of intuitionistic fuzzy sets to topology. In particular, Hur and his colleagues [12] applied the notion of intuitionistic fuzzy sets to topological group. Also, several researchers [3,5,9-11,14,15] applied one to group theory.

In this paper, we investigate the lattice of various sublattices of the lattice of intuitionistic fuzzy subrings of a given ring. We prove that a special class of intuitionistic fuzzy ideals forms a modular sublattice of the lattice of intuitionistic fuzzy ideals of a ring. Finally, we show that the lattice of intuitionistic fuzzy ideals of \( R \) is not complemented [resp. has no atoms (dual atoms)].

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1. Preliminaries

In this section, we list some basic concepts which are needed in the later sections.

For sets $X, Y$ and $Z, f = (f_1, f_2) : X \rightarrow Y \times Z$ is called a complex mapping if $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as $I$ and $L = (L, +, \cdot)$ denotes a lattice, where $"+"$ and $"\cdot"$ denote the sup and inf, respectively. For a general background of lattice theory, we refer to [4]. Moreover, we will denote by $R$ a ring having the zero "0", with respect to binary operations "$+"$ and "$\cdot"$.

Definition 1.1. [2,6] Let $X$ be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \rightarrow I \times I$ is called an intuitionistic fuzzy set (in short, IFS) in $X$ if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \rightarrow I$ and $\nu_A : X \rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to $A$, respectively. In particular, $0_\sim$ and $1_\sim$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in $X$ defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in $X$ as $\text{IFS}(X)$.

Definitions 1.2. [2] Let $X$ be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs on $X$. Then

1. $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
2. $A = B$ iff $A \subset B$ and $B \subset A$.
3. $A^c = (\nu_A, \mu_A)$.
4. $A \cap B = \left\{ \mu_A \land \mu_B, \nu_A \lor \nu_B \right\}$.
5. $A \cup B = \left\{ \mu_A \lor \mu_B, \nu_A \land \nu_B \right\}$.
6. $\left[ \right] A = \left( \mu_A, 1 - \mu_A \right)$, $< > A = (1 - \nu_A, \nu_A)$.

Definition 1.3. [6] Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in $X$, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

1. $\bigcap \ A_i = \left\{ \land \mu_{A_i}, \lor \nu_{A_i} \right\}$.
2. $\bigcup \ A_i = \left\{ \lor \mu_{A_i}, \land \nu_{A_i} \right\}$.
From Definitions 1.1, 1.2 and 1.3, it is clear that (IFS\(X\), \(\lor\), \(\land\)) is a complete lattice.

2. Lattice of intuitionistic fuzzy subrings

**Definition 2.1.** [10] Let \(A \in \text{IFS}(R)\). Then \(A\) is called an *intuitionistic fuzzy subring* (in short, IFSR) of \(R\) if it satisfies the following conditions: For any \(x, y \in R\),

(i) \(\mu_A(x + y) \geq \mu_A(x) \land \mu_A(y)\) and \(\nu_A(x + y) \leq \nu_A(x) \lor \nu_A(y)\).

(ii) \(\mu_A(-x) \geq \mu_A(x)\) and \(\nu_A(-x) \leq \nu_A(x)\).

(iii) \(\mu_A(xy) \geq \mu_A(x) \land \mu_A(y)\) and \(\nu_A(xy) \leq \nu_A(x) \lor \nu_A(y)\).

We will denote the set of all IFSRs of \(R\) as \(\text{IFS}(R)\). It can be easily verified from Proposition 2.6 in [10] that if \(A \in \text{IFS}(R)\), then

\[\mu_A(x) \leq \mu_A(0), \nu_A(x) \geq \nu_A(0)\] and \(A(x) = A(-x)\) for each \(x \in R\).

We shall call \(A(0)\) the *tip* of the intuitionistic fuzzy subring \(A\).

**Result 2.A.** [10, Proposition 4.2] Let \(A \in \text{IFS}(R)\). Then \(A \in \text{IFS}(R)\) if and only if for any \(x, y \in R\),

(i) \(\mu_A(x - y) \geq \mu_A(x) \land \mu_A(y)\) and \(\nu_A(x - y) \leq \nu_A(x) \lor \nu_A(y)\).

(ii) \(\mu_A(xy) \geq \mu_A(x) \land \mu_A(y)\) and \(\nu_A(xy) \leq \nu_A(x) \lor \nu_A(y)\).

**Proposition 2.2.** Let \(\{A_\alpha\}_{\alpha \in \Gamma} \subset \text{IFS}(R)\). Then \(\bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IFS}(R)\).
By the similar arguments, we have
\[ \mu_A(xy) \geq \mu_A(x) \land \mu_A(y) \quad \text{and} \quad \nu_A(xy) \leq \nu_A(x) \lor \nu_A(y). \]

Hence, by Result 2.A, \( A = \bigcap_{\alpha \in \Gamma} A_\alpha \in \text{IFSR}(R) \).

**Definition 2.3.** Let \( A \in \text{IFSR}(R) \). Then the intuitionistic fuzzy subring generated by \( A \) is the least intuitionistic fuzzy subring of \( R \) containing \( A \) and denoted by \( (A) \).

Here, we construct the lattice of intuitionistic fuzzy subrings of a ring and that of special types of intuitionistic fuzzy subrings such as intuitionistic fuzzy (left, right) ideals. The common feature of all these constructions is that the intersection of an arbitrary family of intuitionistic fuzzy subrings is always an intuitionistic fuzzy subring (See Proposition 2.2). Therefore, we consider the inf of a family of intuitionistic fuzzy subrings to be their intersection, whereas the union of two intuitionistic fuzzy subrings may not be an intuitionistic fuzzy subring. Hence, we shall always be taking the sup of a family of intuitionistic fuzzy subrings to be the intuitionistic fuzzy subring generated by the union of that family. The outcome of the above discussion can be described by the following propositions.

**Proposition 2.4.** \( \text{IFSR}(R) \) forms a complete lattice under the ordering of intuitionistic fuzzy set inclusion \( \subseteq \).

**Definition 2.5.** [10] Let \( A \in \text{IFSR}(R) \). Then \( A \) is called an:

1. intuitionistic fuzzy left ideal (in short, IFLI) of \( R \) if \( \mu_A(xy) \geq \mu_A(y) \) and \( \nu_A(xy) \leq \nu_A(x) \) for any \( x, y \in R \).
2. intuitionistic fuzzy right ideal (in short, IFRI) of \( R \) if \( \mu_A(xy) \geq \mu_A(x) \) and \( \nu_A(xy) \leq \nu_A(y) \) for any \( x, y \in R \).
3. intuitionistic fuzzy ideal (in short, IFI) of \( R \) if it is both an IFLI and an IFRI of \( R \).

We will denote the set of all IFIs [resp. IFLIs and IFRIs] of \( R \) as \( \text{IFI}(R) \) [resp. \( \text{IFLI}(R) \) and \( \text{IFRI}(R) \)]. In particular, \( \text{IFI}(\lambda_0, \mu_0)(R) \) denotes the set of all IFIs with the same tip \( \lambda_0, \mu_0 \). It is clear that \( \text{IFI}(R) = \text{IFLI}(R) \cap \text{IFRI}(R) \).

**Result 2.B.** [10, Proposition 4.5] Let \( A \in \text{IFS}(R) \). Then \( A \in \text{IFR}(R) \) \[ resp. \text{IFLI}(R) \text{ and } \text{IFRI}(R) \] if and only if for any \( x, y \in R \):

(i) \( \mu_A(x - y) \geq \mu_A(x) \land \mu_A(y) \) and \( \nu_A(x - y) \leq \nu_A(x) \lor \nu_A(y) \).
(ii) \( \mu_A(xy) \geq \mu_A(x) \lor \mu_A(y) \) and \( \nu_A(xy) \leq \nu_A(x) \land \nu_A(y) \) \[ resp. \mu_A(xy) \geq \mu_A(y), \nu_A(xy) \leq \nu_A(y) \text{ and } \mu_A(xy) \geq \mu_A(x), \nu_A(xy) \leq \nu_A(x) \].
The lattice of intuitionistic fuzzy ideals

The proof of the following result is similar to Proposition 2.2.

**Proposition 2.6.** Let \( \{A_\alpha\}_{\alpha \in \Gamma} \subseteq IFI(R) \) \( \text{resp. } IFLI(R) \) and \( IFRI(R) \). Then \( \bigcap_{\alpha \in \Gamma} A_\alpha \in IFI(R) \) \( \text{resp. } IFLI(R) \) and \( IFRI(R) \).

**Definition 2.7.** Let \( A \in IFS(R) \). Then the IFI \( \text{resp. } IFLI \) and \( IFRI \) generated by \( A \) is the least IFI \( \text{resp. } IFLI \) and \( IFRI \) of \( R \) containing \( A \) and denoted by \( (A) \).

The following is easily verified.

**Proposition 2.8.**
1. \( IFI(R) \) \( \text{resp. } IFLI(R) \) and \( IFRI(R) \) is a complete sublattice of \( IFS(R) \).
2. \( IFI(\lambda_0, \mu_0)(R) \) is a complete sublattice of \( IFS(R) \).

**Definition 2.9.** [9] Let \( X \) be a set and let \( A \in IFS(X) \). Then \( A \) is said to have the sup property if for each \( Y \in P(X) \), there exists \( y_0 \in Y \) such that \( A(y_0) = \left( \bigvee_{x \in Y} \mu_A(x), \bigwedge_{x \in Y} \nu_A(x) \right) \), where \( P(X) \) denotes the power set of \( X \).

**Definition 2.10.** Let \( A, B \in IFS(R) \). Then the sum \( A + B \) and the product \( A \circ B \) of \( A \) and \( B \) are defined as follows, respectively:
1. \( (A + B)(z) = \left( \bigvee_{z = x + y} \mu_A(x) \wedge \mu_B(y), \bigwedge_{z = x + y} \nu_A(x) \vee \nu_B(y) \right) \),
2. \( (A \circ B)(z) = \left( \begin{array}{ll} \left( \bigvee_{z = xy} \mu_A(x) \wedge \mu_B(y), \bigwedge_{z = xy} \nu_A(x) \vee \nu_B(y) \right) & \text{if } z = x \cdot y, \\ (0, 1) & \text{otherwise} \end{array} \right) \).

**Definition 2.11.** [9] Let \( X \) be a set, let \( A \in IFS(X) \) and let \( (\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \).
1. [6] The set \( A^{(\lambda, \mu)} = \{ x \in X : \mu_A(x) \geq \lambda \text{ and } \nu_A(x) \leq \mu \} \) is called a \( (\lambda, \mu) \)-level subset of \( A \).
2. The set \( A^{(\lambda, \mu)} = \{ x \in X : \mu_A(x) > \lambda \text{ and } \nu_A(x) < \mu \} \) is called a strong \( (\lambda, \mu) \)-level subset of \( A \).

**Lemma 2.12.** Let \( A, B \in IFR(R) \). If \( A \) and \( B \) have the sup property, then \( (A + B)^{(\lambda, \mu)} = A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \) for each \( (\lambda, \mu) \in I \times I \) with \( \lambda + \mu \leq 1 \).
Proof. Let \( z \in (A + B)^{(\lambda, \mu)} \). Then

\[
\mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \wedge \mu_B(y)] \geq \lambda \quad \text{and} \quad \nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \vee \nu_B(y)] \leq \mu. \tag{1}
\]

For each decomposition \( z = x + y \), we have either \( \mu_A(x) \leq \mu_B(y) \) and \( \nu_A(x) \geq \nu_B(y) \) or \( \mu_A(x) \geq \mu_B(y) \) and \( \nu_A(x) \leq \nu_B(y) \). This consideration leads as to define the following subsets of \( R \):

\[
X(z) = \left\{ x \in R : z = x + y \text{ for some } y \in R \text{ such that } \mu_A(x) \leq \mu_B(y) \text{ and } \nu_A(x) \geq \nu_B(y) \right\},
\]

\[
Y(z) = \left\{ y \in R : z = x + y \text{ for some } x \in R \text{ such that } \mu_A(x) \geq \mu_B(y) \text{ and } \nu_A(x) \leq \nu_B(y) \right\},
\]

\[
X^*(z) = \left\{ x \in R : z = x + y \text{ for some } y \in R \text{ such that } \mu_A(x) \geq \mu_B(y) \text{ and } \nu_A(x) \leq \nu_B(y) \right\}.
\]

Then clearly \( R = X(z) \cup X^*(z) \). Since \( A \) and \( B \) have the sup property, there exist \( x_0 \in X(z) \) and \( y_0 \in Y(z) \) such that

\[
\mu_A(x_0) = \bigvee_{x \in X(z)} \mu_A(x), \nu_A(x_0) = \bigwedge_{x \in X(z)} \nu_A(x)
\]

and

\[
\mu_B(y_0) = \bigvee_{y \in Y(z)} \mu_B(y), \nu_B(y_0) = \bigwedge_{y \in Y(z)} \nu_B(y). \tag{2}
\]

Since \( x_0 \in X(z) \), there exists \( y_0' \in R \) with \( z = x_0 + y_0' \) such that

\[
\mu_A(x_0) \leq \mu_B(y_0') \quad \text{and} \quad \nu_A(x_0) \geq \nu_B(y_0').
\]

Since \( y_0 \in Y(z) \), there exists \( x_0' \in R \) with \( z = x_0' + y_0 \) such that

\[
\mu_A(x_0') \geq \mu_B(y_0) \quad \text{and} \quad \nu_A(x_0') \leq \nu_B(y_0).
\]

But for \( A(x_0) \) and \( B(y_0) \), we have either \( \mu_A(x_0) \geq \mu_B(y_0) \) and \( \nu_A(x_0) \leq \nu_B(y_0) \) or \( \mu_A(x_0) \leq \mu_B(y_0) \) and \( \nu_A(x_0) \geq \nu_B(y_0) \).

Case (i) : Suppose \( \mu_A(x_0) \geq \mu_B(y_0) \) and \( \nu_A(x_0) \leq \nu_B(y_0) \). Then

\[
\bigvee_{z=x+y} [\mu_A(x) \wedge \mu_B(y)] = \bigvee_{x \in R} [\mu_A(x) \wedge \mu_B(z - x)] \quad \text{(Since } y = z - x)\]
Case (i), it follows that $x \leq y$ and $\lambda, \mu$. Thus, by (1),

$$x = \left( \bigvee_{x \in X(z)} \left[ \mu_A(x) \land \mu_B(z - x) \right] \right) \lor \left( \bigvee_{x \in X^*(z)} \left[ \mu_A(x) \land \mu_B(z - x) \right] \right)$$

(Since $R = X(z) \cup X^*(z)$)

$$= \left( \bigvee_{x \in X(z)} \left[ \mu_A(x) \land \mu_B(y) \right] \right) \lor \left( \bigvee_{y \in Y(z)} \left[ \mu_A(x) \land \mu_B(y) \right] \right)$$

$$= \left( \bigvee_{x \in X(z)} \mu_A(x) \right) \lor \left( \bigvee_{y \in Y(z)} \mu_B(y) \right)$$

$$= \mu_A(x_0) \lor \mu_B(y_0) \text{ (By (2))}$$

$$= \mu_A(x_0) \text{ (By the hypothesis)}$$

and

$$\bigwedge_{x = x+y} [\nu_A(x) \lor \nu_B(y)]$$

$$= \bigwedge_{x \in R} [\nu_A(x) \lor \nu_B(z - x)]$$

$$= \left( \bigwedge_{x \in X(z)} [\nu_A(x) \lor \nu_B(z - x)] \right) \land \left( \bigwedge_{x \in X^*(z)} [\nu_A(x) \lor \nu_B(z - x)] \right)$$

$$= \left( \bigwedge_{x \in X(z)} [\nu_A(x) \lor \nu_B(y)] \right) \land \left( \bigwedge_{y \in Y(z)} [\nu_A(x) \lor \nu_B(y)] \right)$$

$$= \left( \bigwedge_{x \in X(z)} \nu_A(x) \right) \land \left( \bigwedge_{y \in Y(z)} \nu_B(y) \right)$$

$$= \nu_A(x_0) \land \nu_B(y_0)$$

Thus, by (1),

$$\mu_A(x_0) = \mu_{A+B}(z) \geq \lambda \text{ and } \nu_A(x_0) = \nu_{A+B}(z) \leq \mu.$$

So $x_0 \in A^{(\lambda, \mu)}$. Since $\mu_B(y_0) \geq \mu_A(x_0)$ and $\nu_B(y_0) \leq \mu_A(x_0)$, $\mu_B(y_0) \geq \lambda$ and $\nu_B(y_0) \leq \mu$. Then $y_0 \in B^{(\lambda, \mu)}$. Thus $z = x_0 + y_0 \in A^{(\lambda, \mu)} + B^{(\lambda, \mu)}$.

Case (ii) : Suppose $\mu_A(x_0) \leq \mu_B(y_0)$ and $\nu_A(x_0) \geq \nu_B(y_0)$. Then as in Case (i), it follows that $x_0 \in A^{(\lambda, \mu)}$ and $y_0 \in B^{(\lambda, \mu)}$. Thus $z = x_0 + y_0 \in A^{(\lambda, \mu)} + B^{(\lambda, \mu)}$. So, in either case, $z \in A^{(\lambda, \mu)} + B^{(\lambda, \mu)}$. Hence $(A + B)^{(\lambda, \mu)} \subseteq A^{(\lambda, \mu)} + B^{(\lambda, \mu)}$. Now let $z \in A^{(\lambda, \mu)} + B^{(\lambda, \mu)}$. Then there exist $x_0 \in A^{(\lambda, \mu)}$ and $y_0 \in B^{(\lambda, \mu)}$ such that $z = x_0 + y_0$. Thus $\mu_A(x_0) \geq \lambda, \nu_A(x_0) \leq \mu$ and
\( \mu_B(y_0) \geq \lambda, \nu_B(y_0) \leq \mu \). So
\[
\mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \land \mu_B(y)] \geq \lambda \text{ and } \nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \lor \nu_B(y)] \leq \mu.
\]
Thus \( z \in (A + B)^{(\lambda, \mu)} \). Hence \( A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \subset (A + B)^{(\lambda, \mu)} \). Therefore \( (A + B)^{(\lambda, \mu)} = A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \). This completes the proof. \( \square \)

**Lemma 2.12.** Let \( A, B \in IFS(R) \) and let \((\lambda, \mu) \in [0, 1) \times [0, 1)\) with \(\lambda + \mu \leq 1\). Then \( (A + B)^{(\lambda, \mu)} = A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \).

**Proof.** Suppose \((A + B)^{(\lambda, \mu)} = \phi\). Then clearly \((A + B)^{(\lambda, \mu)} \subset A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \).

Suppose \((A + B)^{(\lambda, \mu)} \neq \phi\) and let \( z \in (A + B)^{(\lambda, \mu)} \). Then
\[
\mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \land \mu_B(y)] > \lambda
\]
and
\[
\nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \lor \nu_B(y)] < \mu.
\]
Thus there exist \( x_0, y_0 \in R \) with \( z = x_0 + y_0 \) such that
\[
\mu_A(x_0) \land \mu_B(y_0) > \lambda \text{ and } \nu_A(x_0) \lor \nu_B(y_0) < \mu.
\]
So \( \mu_A(x_0) > \lambda \), \( \nu_A(x_0) < \mu \) and \( \mu_B(y_0) > \lambda \), \( \nu_B(y_0) < \mu \). Then \( x_0 \in A^{(\lambda, \mu)} \) and \( y_0 \in B^{(\lambda, \mu)} \). Thus \( z = x_0 + y_0 \in A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \). So \( (A + B)^{(\lambda, \mu)} \subset A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \). Now for each \((\lambda, \mu) \in [0, 1) \times [0, 1)\) with \(\lambda + \mu \leq 1\), suppose
\[
\left( \bigvee_{x \in R} \mu_A(x) \right) \land \left( \bigvee_{y \in R} \mu_B(y) \right) \leq \lambda \text{ and } \left( \bigwedge_{x \in R} \nu_A(x) \right) \lor \left( \bigwedge_{y \in R} \nu_B(y) \right) \geq \mu.
\]
Then one of \( A^{(\lambda, \mu)} \) and \( B^{(\lambda, \mu)} \) is \( \phi \). Thus \( A^{(\lambda, \mu)} + B^{(\lambda, \mu)} = \phi \subset (A + B)^{(\lambda, \mu)} \).

Otherwise, \( A^{(\lambda, \mu)} \neq \phi \) and \( B^{(\lambda, \mu)} \neq \phi \). Then \( A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \neq \phi \). Let \( z \in A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \). Then that exist \( x_0 \in A^{(\lambda, \mu)} \) and \( y_0 \in B^{(\lambda, \mu)} \) such that \( z = x_0 + y_0 \) and
\[
\mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \land \mu_B(y)] \geq \mu_A(x_0) \land \mu_B(y_0) > \lambda
\]
and
\[
\nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \lor \nu_B(y)] \leq \nu_A(x_0) \lor \nu_B(y_0) < \mu.
\]
Thus \( z \in (A + B)^{(\lambda, \mu)} \). So \( A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \subset (A + B)^{(\lambda, \mu)} \). Hence \( (A + B)^{(\lambda, \mu)} = A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \). This completes the proof. \( \square \)
Lemma 2.13. Let $A \in IFS(R)$. Then $A \in IFI(R)$ \textbf{[resp. IFLI(R) and IRRI(R)]} if and only if $A^{(\lambda, \mu)} = \phi$ or $A^{(\lambda, \mu)} \in I(R)$ \textbf{[resp. LI(R) and RI(R)]} for each $(\lambda, \mu) \in [0,1) \times (0,1]$ with $x + y \leq 1$, where $I(R)$ \textbf{[resp. LI(R) and RI(R)]} denotes the set of all ideals \textbf{[resp. left ideals and right ideals]} of $R$.

Proof. We prove this lemma for left ideal, since other cases are similar. It is clear that $A = 0_*$ if and only if $A^{(\lambda, \mu)} = \phi$ for each $(\lambda, \mu) \in [0,1) \times (0,1]$ with $x + y \leq 1$. Now we assume that $A \neq 0_*$. 

$(\Rightarrow)$: Suppose $A \in IFLI(R)$ and let $(\lambda, \mu) \in [0,1) \times (0,1]$ with $x + y \leq 1$. Suppose $A \neq 0_*$. Let $x, y \in A^{(\lambda, \mu)}$ and let $z \in R$. Then

\[
\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y) \quad \text{(Since } A \in IFLI(R))
\]

and

\[
\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y) < \mu.
\]

Also

\[
\mu_A(zx) \geq \mu_A(x) \quad \text{(Since } A \in IFLI(R))
\]

and

\[
\nu_A(zx) \leq \nu_A(x) < \mu.
\]

Thus $x - y \in A^{(\lambda, \mu)}$ and $zx \in A^{(\lambda, \mu)}$. Hence $A^{(\lambda, \mu)} \subseteq LI(R)$.

$(\Leftarrow)$: Suppose the necessary condition holds. For any $x, y \in R$, let $A(x) = (\lambda, \mu)$ and let $A(y) = (s, t)$ such that $\lambda \leq s$ and $\mu \geq t$.

Case(i): Suppose $\lambda = 0$ and $\mu = 1$. Then

\[
\mu_A(x - y) \geq \lambda = \mu_A(x) \wedge \mu_A(y), \quad \nu_A(x - y) \leq \mu = \nu_A(x) \vee \nu_A(y)
\]

and

\[
\mu_A(zx) \geq \lambda = \mu_A(x) \quad \text{and} \quad \nu_A(zx) \leq \mu = \nu_A(x), \quad \text{for each } z \in R.
\]

Thus $A \in IFLI(R)$.

Case(ii): Suppose $\lambda \neq 0$ and $\mu \neq 1$. For each $\epsilon > 0$, let $\epsilon < \lambda$. Then we have

\[
\mu_A(y) > s - \epsilon \geq \lambda - \epsilon, \quad \nu_A(y) < t + \epsilon \leq \mu + \epsilon.
\]

and

\[
\mu_A(x) > \lambda - \epsilon, \quad \nu_A(x) < \mu + \epsilon.
\]

Thus $x, y \in A^{(\lambda - \epsilon, \mu + \epsilon)}$. By the hypothesis, $A^{(\lambda - \epsilon, \mu + \epsilon)} \subseteq I(R)$. So $x - y \in A^{(\lambda - \epsilon, \mu + \epsilon)}$ and $z x \in A^{(\lambda - \epsilon, \mu + \epsilon)}$ for each $z \in R$. Then

\[
\mu_A(x - y) > \lambda - \epsilon, \quad \nu_A(x - y) < \mu + \epsilon.
\]
and 
\[ \mu_A(zx) > \lambda - \varepsilon, \nu_A(zx) < \mu + \varepsilon \]
for each \( z \in \mathbb{R} \).

Since \( \varepsilon \) is arbitrary,
\[ \mu_A(x - y) \geq \lambda = \mu_A(x) \land \mu_A(y), \quad \nu_A(x - y) \leq \mu = \nu_A(x) \lor \nu_A(y) \]
and
\[ \mu_A(xz) \geq \lambda = \mu_A(x), \quad \nu_A(xz) \leq \mu = \nu_A(x). \]

Hence \( A \in \text{IFLI}(R) \). This completes the proof. \( \square \)

**Theorem 2.14.** \( \text{IFLI}_{\lambda_0,\mu_0}(R) \), \( \text{IFRI}_{\lambda_0,\mu_0}(R) \), and \( \text{IFI}_{\lambda_0,\mu_0}(R) \) are sublattices of \( \text{IFS}R(R) \) and for any \( A, B \in \text{IFLI}_{\lambda_0,\mu_0}(R) \) [resp. \( \text{IFRI}_{\lambda_0,\mu_0}(R) \) and \( \text{IFI}_{\lambda_0,\mu_0}(R) \)], \( A \lor B = A + B \).

**Proof.** It is easy to see that \( \text{IFLI}_{\lambda_0,\mu_0}(R) \), \( \text{IFRI}_{\lambda_0,\mu_0}(K) \), and \( \text{IFI}_{\lambda_0,\mu_0}(R) \) are sublattices of \( \text{IFS}R(R) \). We prove \( A \lor B = A + B \) for any \( A, B \in \text{IFLI}_{\lambda_0,\mu_0}(R) \) (For \( \text{IFRI}_{\lambda_0,\mu_0}(R) \) and \( \text{IFI}_{\lambda_0,\mu_0}(R) \), the proofs are similar).

Let \( z \in R \). Then
\[ \mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \land \mu_B(y)] \leq \mu_A(0) \land \mu_B(0) = \lambda_0 \]
and
\[ \nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \lor \nu_B(y)] \geq \nu_A(0) \lor \nu_B(0) = \mu_0. \]
Thus \( \bigvee_{z \in R} \mu_{A+B}(z) \leq \lambda_0 \) and \( \bigwedge_{z \in R} \nu_{A+B}(z) \geq \mu_0 \). On the other hand,
\[ \bigvee_{z \in R} \mu_{A+B}(z) \geq \mu_{A+B}(0) = \bigvee_{0=x+y} [\mu_A(x) \land \mu_B(y)] \geq \mu_A(0) \land \mu_B(0) = \lambda_0 \]
and
\[ \bigwedge_{z \in R} \nu_{A+B}(z) \leq \nu_{A+B}(0) = \bigwedge_{0=x+y} [\nu_A(x) \lor \nu_B(y)] \leq \nu_A(0) \lor \nu_B(0) = \mu_0. \]

So
\[ \left( \bigvee_{z \in R} \mu_{A+B}(z), \bigwedge_{z \in R} \nu_{A+B}(z) \right) = (A + B)(0) = (\lambda_0, \mu_0). \] (3)

For each \( (\lambda, \mu) \in [0,1] \times (0,1] \) with \( \lambda < \lambda_0 \) and \( \mu > \mu_0 \), \( (A + B)^{\lambda,\mu} \neq \phi \). By Lemma 2.12 , \( (A + B)^{\lambda,\mu} = A^{\lambda,\mu} + B^{\lambda,\mu} \). Since \( A, B \in \text{IFLI}(R) \), by Lemma 2.13, \( A^{\lambda,\mu}, B^{\lambda,\mu} \in \text{LI}(R) \). Thus \( (A + B)^{\lambda,\mu} \in \text{LI}(R) \). So, by Lemma 2.13, \( A + B \in \text{IFLI}(R) \). Moreover, \( (A + B) \in \text{IFLI}_{\lambda_0,\mu_0}(R) \). (4)

Let \( z \in R \). Then
\[ \mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \land \mu_B(y)] \geq \mu_A(z) \land \mu_B(0) = \mu_A(z) \]
and
\[ \nu_{A+B}(z) = \bigwedge_{x+y} [\nu_A(x) \lor \nu_B(y)] \leq \nu_A(z) \lor \nu_B(0) = \nu_A(z). \]

Thus \( A \subset A + B \). By the similar arguments, we have \( B \subset A + B \). So \( A \subset A + B \) and \( B \subset A + B \).

Now let \( C \in \text{IFLI}(R) \) such that \( A \subset C \) and \( B \subset C \) and let \( z \in R \). Then
\[
\mu_{A+B}(z) = \bigvee_{x+y} [\mu_A(x) \land \mu_B(y)] \leq \bigvee_{x+y} [\mu_C(x) \land \mu_C(y)]
\]
\[
\leq \bigvee_{x+y} \mu_C(z) \quad \text{(Since \( \mu_C(z) = \mu_C(x+y) \geq \mu_C(x) \land \mu_C(y) \))}
\]
\[ = \mu_C(z) \]
and
\[
\nu_{A+B}(z) = \bigwedge_{x+y} [\nu_A(x) \lor \nu_B(y)] \geq \bigwedge_{x+y} [\nu_C(x) \lor \nu_C(y)]
\]
\[
\leq \bigwedge_{x+y} \nu_C(z) = \nu_C(z).
\]

Thus \( A + B \subset C \).

Hence, by (3), (4), (5) and (6), \( A + B = A \lor B \). This completes the proof. □

**Remark 2.15.**

1. \( A \lor B = A + B \) is not true in \( \text{IFSR}(R) \), \( \text{IFLI}(R) \), \( \text{IFRI}(R) \) and \( \text{IFI}(R) \)(See Example 2.16).

2. As well-known, \( S + T \) is not subring in general, where \( S \) and \( T \) are subrings of \( R \). Hence \( A \lor B = A + B \) is not true in \( \text{IFSR}(\lambda_0,\mu_0)(R) \).

**Example 2.16.** We define two complex mappings \( A : R \to I \times I \) and \( B : R \to I \times I \) as follows, respectively: for each \( x \in R \),
\[ A(x) = (0.5, 0.4) \quad \text{and} \quad B(x) = (0.3, 0.6). \]
Then clearly \( A, B \in \text{IFSR}(R) \) [resp. \( \text{IFI}(R) \), \( \text{IFRI}(R) \) and \( \text{IFI}(R) \)]. Moreover, it is easy to see that \( (A + B)(0) = (0.3, 0.6) \) and \( (A \lor B)(0) = (0.5, 0.4) \).

**Example 2.17.** Let \( R = \{(a, b) : a, b \in \mathbb{Z}\} \), where \( \mathbb{Z} \) is the ring of integers. We define the additive operation and the multiplicative operation on \( R \) as follows, respectively: for any \((a, b), (c, d) \in R\),
\[ (a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (0, 0). \]
Then \( (R, +, \cdot) \) forms a ring with zero \( (0, 0) \). Now we define three complex mappings \( A, B, C : R \to I \times I \) as follows, respectively: for each \((x, y) \in R\),
\[ A(x, y) = \begin{cases} 
(\frac{1}{4}, \frac{1}{4}) & \text{if } y = 0, \\
(0, 1) & \text{if } y \neq 0,
\end{cases} \]
\[ B(x, y) = \begin{cases} 
(\frac{1}{4}, \frac{1}{4}) & \text{if } x = 0, \\
(0, 1) & \text{if } x \neq 0,
\end{cases} \]
\[ C(x, y) = \begin{cases} \left\{ \frac{1}{3}, \frac{3}{5} \right\} & \text{if } x = y, \\ (0, 1) & \text{if } x \neq y \end{cases} \]

Then it is easy to see that \( A, B, C \in \text{IFI}(R) \). Let \((x, y) \in R\). Then

\[
\mu_{A+B}(x, y) = \bigvee_{(x, y) = (x_1, y_1) + (x_2, y_2)} \left[ \mu_A(x_1, y_1) \land \mu_B(x_2, y_2) \right]
\leq [\mu_A(x, 0) \land \mu_B(0, y)] = \frac{1}{3}
\]

and

\[
\nu_{A+B}(x, y) = \bigwedge_{(x, y) = (x_1, y_1) + (x_2, y_2)} \left[ \nu_A(x_1, y_1) \lor \nu_B(x_2, y_2) \right]
\leq \nu_A(x, 0) \lor \nu_B(0, y) = \frac{3}{5}
\]

Thus

\[
\mu_{C \land (A+B)}(x, y) = \mu_C(x, y) \land \mu_{A+B}(x, y) = \mu_C(x, y)
\]

and

\[
\nu_{C \land (A+B)}(x, y) = \nu_C(x, y) \lor \nu_{A+B}(x, y) = \nu_C(x, y).
\]

So \( C \land (A + B) = C \). On the other hand,

\[
\mu_{C \land A}(x, y) = \mu_C(x, y) \land \mu_A(x, y) = \begin{cases} \frac{1}{3} & \text{if } (x, y) = (0, 0), \\ 0 & \text{if } (x, y) \neq (0, 0), \end{cases}
\]

and

\[
\nu_{C \land A}(x, y) = \nu_C(x, y) \lor \nu_A(x, y) = \begin{cases} \frac{3}{5} & \text{if } (x, y) = (0, 0), \\ 1 & \text{if } (x, y) \neq (0, 0). \end{cases}
\]

Also

\[
\mu_{C \land B}(x, y) = \mu_C(x, y) \land \mu_B(x, y) = \begin{cases} \frac{1}{3} & \text{if } (x, y) = (0, 0), \\ 0 & \text{if } (x, y) \neq (0, 0), \end{cases}
\]

and

\[
\nu_{C \land B}(x, y) = \nu_C(x, y) \lor \nu_B(x, y) = \begin{cases} \frac{3}{5} & \text{if } (x, y) = (0, 0), \\ 1 & \text{if } (x, y) \neq (0, 0). \end{cases}
\]

Thus

\[
\mu_{(C \land A) + (C \land B)}(x, y) = \begin{cases} \frac{1}{3} & \text{if } (x, y) = (0, 0), \\ 0 & \text{if } (x, y) \neq (0, 0), \end{cases}
\]

and

\[
\nu_{(C \land A) + (C \land B)}(x, y) = \begin{cases} \frac{3}{5} & \text{if } (x, y) = (0, 0), \\ 1 & \text{if } (x, y) \neq (0, 0). \end{cases}
\]

So \( C \land (A + B) \neq (C \land A) + (C \land B) \). Hence \( \text{IFI}(R) \) is not distributive. \qed
The following is the immediate result of Propositions 2.18 and 2.19 in [10], Proposition 3.7 in [9] and Proposition 2.3 in [14].

**Theorem 2.18.** Let \( A \in IFS(R) \). Then \( A \in IFI(R) \) if and only if \( A(\lambda,\mu) \) is an ideal for each \((\lambda,\mu) \in I \times I \) with \( \lambda \leq \mu_A(0) \) and \( \mu \geq \nu_A(0) \).

**Lemma 2.19.** Let \( A, B \in IFI(R) \). If \( A \) and \( B \) have the sup property, then the following holds:

1. \( A + B \) has the sup property.
2. \( A \cap B \) has the sup property.

**Proof.** (1) Let \( S \) be any subset of \( R \). Then

\[
\bigvee_{z \in S} \mu_{A+B}(z) = \bigvee_{z \in S} \left( \bigvee_{z = x + y} [\mu_A(x) \land \mu_B(y)] \right) = \bigvee_{z \in S, z = x + y} [\mu_A(x) \land \mu_B(y)]
\]

and

\[
\bigwedge_{z \in S} \nu_{A+B}(z) = \bigwedge_{z \in S} \left( \bigwedge_{z = x + y} [\nu_A(x) \lor \nu_B(y)] \right) = \bigwedge_{z \in S, z = x + y} [\nu_A(x) \lor \nu_B(y)].
\]

Let us define two subsets \( X(S) \) and \( Y(S) \) of \( R \) by

\[
X(S) = \left\{ x \in R : z \in S, z = x + y \text{ for some } y \in R \text{ such that } \mu_A(x) \leq \mu_B(y) \text{ and } \nu_A(x) \geq \nu_B(y) \right\},
\]

\[
Y(S) = \left\{ y \in R : z \in S, z = x + y \text{ for some } x \in R \text{ such that } \mu_A(x) \geq \mu_B(y) \text{ and } \nu_A(x) \leq \nu_B(y) \right\}.
\]

Since \( A \) and \( B \) have the sup property, there exist \( x' \in X(S) \) and \( y'' \in Y(S) \) such that

\[
\mu_A(x') = \bigvee_{x \in X(S)} \mu_A(x), \nu_A(x') = \bigwedge_{x \in X(S)} \nu_A(x)
\]

and

\[
\mu_B(y'') = \bigvee_{y \in Y(S)} \mu_B(y), \nu_B(y'') = \bigwedge_{y \in Y(S)} \nu_B(y).
\]

(7)

Since \( x' \in X(S) \), there exists \( z_1 \in S \) such that \( z_1 = x' + y' \) for some \( y' \in R \) satisfying \( \mu_A(x') \leq \mu_B(y') \) and \( \nu_A(x') \geq \nu_B(y') \). Also, since \( y'' \in Y(S) \), there exists \( z_2 \in S \) such that \( z_2 = x'' + y'' \) for some \( x'' \in R \) satisfying \( \mu_A(x'') \geq \mu_B(y'') \) and \( \nu_A(x'') \leq \nu_B(y'') \).

On the other hand, we have either \( \mu_A(x') \geq \mu_B(y'') \), \( \nu_A(x') \leq \nu_B(y'') \) or \( \mu_A(x') \leq \mu_B(y'') \), \( \nu_A(x') \geq \nu_B(y'') \).
Case (i) : Suppose $\mu_A(x') \geq \mu_B(y'')$ and $\nu_A(x') \leq \nu_B(y'')$. Then

$$\bigvee_{z \in S, z = x+y} [\mu_A(x) \land \mu_B(y)]$$

$$= \bigvee_{x \in X(S)} [\mu_A(x) \land \mu_B(y)] \lor \left( \bigvee_{y \in Y(S)} [\mu_A(x) \land \mu_B(y)] \right)$$

(As in Lemma 2.12)

$$= \left( \bigvee_{x \in X(S)} \mu_A(x) \right) \lor \left( \bigvee_{y \in Y(S)} \mu_B(y) \right)$$

$$= \mu_A(x') \lor \mu_B(y'') \quad \text{(By (7))}$$

$$= \mu_A(x') \quad \text{(By the hypothesis)}$$

and

$$\bigwedge_{z \in S, z = x+y} [\nu_A(x) \lor \nu_B(y)]$$

$$= \bigwedge_{x \in X(S)} [\nu_A(x) \lor \nu_B(y)] \land \left( \bigwedge_{y \in Y(S)} [\nu_A(x) \lor \nu_B(y)] \right)$$

$$= \left( \bigwedge_{x \in X(S)} \nu_A(x) \right) \land \left( \bigwedge_{y \in Y(S)} \nu_B(y) \right)$$

$$= \nu_A(x') \land \nu_B(y'')$$

$$= \nu_A(x').$$

Thus

$$\bigvee_{z \in S} \mu_{A+B}(z) = \mu_A(x') \quad \text{and} \quad \bigwedge_{z \in S} \nu_{A+B}(z) = \nu_A(x'). \quad \text{(8)}$$

Now we show that

$$\bigvee_{z \in S} \mu_{A+B}(z) = \mu_{A+B}(z_1) \quad \text{and} \quad \bigwedge_{z \in S} \nu_{A+B}(z) = \nu_{A+B}(z_1).$$

For decompositions $z_1 = x_i' + y_i'$, we have

$$\mu_{A+B}(z_1) = \bigvee_{z_1 = x_i' + y_i'} [\mu_A(x_i') \land \mu_B(y_i')] \quad \text{and} \quad \nu_{A+B}(z_1) = \bigwedge_{z_1 = x_i' + y_i'} [\nu_A(x_i') \lor \nu_B(y_i')].$$

Again, we construct subset $X(z_1)$ and $Y(z_1)$ of $R$ as follows :

$$X(z_1) = \{ x_i' \in R : z_1 = x_i' + y_i \text{ for some } y_i \in R \text{ such that } \mu_A(x_i') \leq \mu_B(y_i) \text{ and } \nu_A(x_i') \geq \nu_B(y_i) \},$$
\[ Y(\bar{z}_1) = \{y_i' \in R : \bar{z}_1 = x_i' + y_i' \text{ for some } x_i' \in R \text{ such that } \mu_A(x_i') \geq \mu_B(y_i') \text{ and } \nu_A(x_i') \leq \nu_B(y_i') \}. \]

Then
\[
\bigvee_{\bar{z}_1 = x_i' + y_i'} [\mu_A(x_i') \land \mu_B(y_i')] \\
= \left( \bigvee_{x_i' \in X(\bar{z}_1)} [\mu_A(x_i') \land \mu_B(y_i')] \right) \lor \left( \bigvee_{y_i' \in Y(\bar{z}_1)} [\mu_A(x_i') \land \mu_B(y_i')] \right)
\]

(As in Lemma 2.12)
\[
= \left( \bigvee_{x_i' \in X(\bar{z}_1)} \mu_A(x_i') \right) \lor \left( \bigvee_{y_i' \in Y(\bar{z}_1)} \mu_B(y_i') \right)
\]

and
\[
\bigwedge_{\bar{z}_1 = x_i' + y_i'} [\nu_A(x_i') \lor \nu_B(y_i')] \\
= \left( \bigwedge_{x_i' \in X(\bar{z}_1)} [\nu_A(x_i') \lor \nu_B(y_i')] \right) \land \left( \bigwedge_{y_i' \in Y(\bar{z}_1)} [\nu_A(x_i') \lor \nu_B(y_i')] \right)
\]
\[
= \left( \bigwedge_{x_i' \in X(\bar{z}_1)} \nu_A(x_i') \right) \land \left( \bigwedge_{y_i' \in Y(\bar{z}_1)} \nu_B(y_i') \right).
\]

Since \( X(\bar{z}_1) \subset X(S) \) and \( x_i' \in X(\bar{z}_1) \),
\[
\mu_A(x') \leq \bigvee_{x_i' \in X(\bar{z}_1)} \mu_A(x_i') \leq \bigvee_{x \in X(S)} \mu_A(x) = \mu_A(x')
\]
and
\[
\nu_A(x') \geq \bigwedge_{x_i' \in X(\bar{z}_1)} \nu_A(x_i') \geq \bigwedge_{x \in X(S)} \nu_A(x) = \nu_A(x').
\]

Thus
\[
\bigvee_{x_i' \in X(\bar{z}_1)} \mu_A(x_i') = \mu_A(x') \text{ and } \bigwedge_{x_i' \in X(\bar{z}_1)} \nu_A(x_i') = \nu_A(x').
\]

Also, since \( Y(\bar{z}_1) \subset Y(S) \) and \( y_i'' \in Y(\bar{z}_1) \), we have
\[
\bigvee_{y_i'' \in Y(\bar{z}_1)} \mu_B(y_i'') = \mu_B(y'') \text{ and } \bigwedge_{y_i'' \in Y(\bar{z}_1)} \nu_B(y_i'') = \nu_B(y'').
\]

By the hypothesis,
\[
\bigvee_{x_i' \in X(\bar{z}_1)} \mu_A(x_i') = \mu_A(x') \geq \mu_B(y'') = \bigvee_{y_i'' \in Y(\bar{z}_1)} \mu_B(y'').
\]
and
\[ \bigwedge_{x' \in X(z_1)} \nu_A(x') = \nu_A(x') \leq \nu_B(y'') = \bigwedge_{y'_i \in Y(z_1)} \nu_B(y'_i). \]
Thus
\[ \mu_{A+B}(z_1) = \left( \bigvee_{x'_i \in X(z_1)} \mu_A(x'_i) \right) \lor \left( \bigvee_{y'_i \in Y(z_1)} \mu_B(y'_i) \right) = \mu_A(x'). \]
and
\[ \nu_{A+B}(z_1) = \left( \bigwedge_{x'_i \in X(z_1)} \nu_A(x'_i) \right) \land \left( \bigwedge_{y'_i \in Y(z_1)} \nu_B(y'_i) \right) = \nu_A(x'). \] (9)

So, by (8) and (9),
\[ \bigvee_{z \in S} \mu_{A+B}(z) = \mu_{A+B}(z_1) \text{ and } \bigwedge_{z \in S} \nu_{A+B}(z) = \nu_{A+B}(z_1). \]

Case (ii) Suppose \( \mu_A(x') \leq \mu_B(y'') \) and \( \nu_A(x') \geq \nu_B(y'') \). By proceeding in a similar way as in Case (i), we can verify that
\[ \bigvee_{z \in S} \mu_{A+B}(z) = \mu_{A+B}(z_2) \text{ and } \bigwedge_{z \in S} \nu_{A+B}(z) = \nu_{A+B}(z_2) \]
for some \( z_2 \in S \). Hence, in all, \( A + B \) has the sup property.

(2) The proof is left as an exercise for the reader. This completes the proof.

\[ \square \]

**Theorem 2.20.** Let IFI\(_{(\lambda_0, \mu_0)}(R)\) be the set of all IFIs with the sup property and same tip \"(\lambda_0, \mu_0)\". Then IFI\(_{(\lambda_0, \mu_0)}(R)\) forms a sublattice of IFI\(_{(\lambda_0, \mu_0)}(R)\) and hence of IFI\(_{(R)}\).

**Proof.** Let \( A, B \in \text{IFI}_{(\lambda_0, \mu_0)}(R) \). We show that \( A \lor B = A + B \). Since \( A, B \in \text{IFI}(R) \), by Theorem 2.18, \( A^{(\lambda, \mu)} \) and \( B^{(\lambda, \mu)} \) are ideals for each \( (\lambda, \mu) \in I \times I \) with \( \lambda \leq \mu_A + \mu_B(0) = \lambda_0 \) and \( \mu \geq \nu_{A+B}(0) = \mu_0 \). Then \( A^{(\lambda, \mu)} + B^{(\lambda, \mu)} \) is an ideal of \( R \). Since \( A \) and \( B \) have the sup property, by Lemma 2.12, \( A^{(\lambda, \mu)} + B^{(\lambda, \mu)} = (A + B)^{(\lambda, \mu)} \). Thus \( (A + B)^{(\lambda, \mu)} \) is an ideal of \( R \). So, by Theorem 2.18, \( A + B \in \text{IFI}(R) \). Since \( A \) and \( B \) have the same tip \"(\lambda_0, \mu_0)\", we have
\[ \mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \land \mu_B(y)] \geq \mu_A(z) \land \mu_B(0) = \mu_A(z) \]
and
\[ \nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \lor \nu_B(y)] \leq \nu_A(z) \lor \nu_B(0) = \nu_A(z) \]
for each \( z \in S \). Then \( A \subset A + B \). By the similar arguments, we have \( B \subset A + B \).

Now let \( C \in \text{IFI}(R) \) contain \( A \) and \( B \) and let \( z \in S \) such that \( z = x + y \). Then
\[ \mu_C(z) = \mu_C(x + y) \geq \mu_C(x) \land \mu_C(y) \]
and
\[ \nu_C(z) = \nu_C(x + y) \leq \nu_C(x) \lor \nu_C(y) \].
The lattice of intuitionistic fuzzy ideals

and
\[ \nu_C(z) = \nu_C(x + y) \leq \nu_C(x) \lor \nu_C(y). \]

Thus
\[ \mu_{A+B}(z) = \bigwedge_{z=x+y} [\mu_A(x) \land \mu_B(y)] \]
\[ \leq \bigwedge_{z=x+y} [\mu_C(x) \land \mu_C(y)] \quad \text{(Since } A \subseteq C \text{ and } B \subseteq C \text{)} \]
\[ = \mu_C(z) \]
and
\[ \nu_{A+B}(z) = \bigwedge_{z=x+y} [\nu_A(x) \lor \nu_B(y)] \]
\[ \geq \bigwedge_{z=x+y} [\nu_C(x) \lor \nu_C(y)] \]
\[ = \nu_C(z). \]

So \( A + B \subseteq C \). Hence \( A + B \) is the least intuitionistic fuzzy ideal containing \( A \) and \( B \). Therefore \( A + B = A \lor B \). On the other hand, by Lemma 2.19(1), \( A \lor B \) has the sup property. Thus \( A \lor B \in \text{IFI}_{s}(\lambda_0, \mu_0)(R) \). Also, by Lemma 2.19(2), \( A \land B \) has the sup property. Thus \( A \land B \in \text{IFI}_{s}(\lambda_0, \mu_0)(R) \). So \( \text{IFI}_{s}(\lambda_0, \mu_0)(R) \) forms a sublattice of \( \text{IFI}_{s}(\lambda_0, \mu_0)(R) \) and hence of \( \text{IFI}(R) \). This completes the proof. \[
\Box
\]

The following lattice diagram is the interrelationship of different sublattices of the lattice \( \text{IFSR}(R) \):

Now we obtain an intuitionistic fuzzy analog of a well-known result that the set of ideals of a ring forms a modular lattice.
3. Intuitionistic fuzzy ideals and modularity

In the previous section, we discussed various sublattices of the lattice of intuitionistic fuzzy ideals of a ring. Hence, we obtain an intuitionistic fuzzy analog of a well known result that the set of ideals of a ring forms a modular lattice.

**Lemma 3.1.** Let \( A \in \text{IFSR}(R) \). If \( \mu_A(x) < \mu_A(y) \) and \( \nu_A(x) > \nu_A(y) \) for some \( x, y \in R \), then \( A(x + y) = A(x) \).

**Proof.** Since \( A \in \text{IFSR}(R) \),

\[
\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y) = \mu_A(x)
\]

and

\[
\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y) = \nu_A(x).
\]

Assume that \( \mu_A(x + y) > \mu_A(x) \) and \( \nu_A(x + y) < \nu_A(x) \). Then

\[
\mu_A(x) = \mu_A(y + x - y) \geq \mu_A(x + y) \wedge \mu_A(y) > \mu_A(x)
\]

and

\[
\nu_A(x) = \nu_A(y + x - y) \leq \nu_A(x + y) \vee \nu_A(y) < \nu_A(x).
\]

This contradicts the fact that \( A(x) = A(x) \). Hence \( A(x + y) = A(x) \). \( \square \)

**Theorem 3.2.** The sublattice \( \text{IFL}_{(\lambda_0, \mu_0)}(R) \) is modular.

**Proof.** Since the modular inequality is valid for every lattice, for any \( A, B, C \in \text{IFL}_{(\lambda_0, \mu_0)}(R) \) with \( B \subset A \), we have

\[
B \vee (A \wedge C) \subset A \wedge (B \vee C).
\]

Assume that \( A \wedge (B \vee C) \neq B \vee (A \wedge C) \). Then there exits \( z \in R \) such that

\[
\mu_{A \wedge (B \vee C)}(z) > \mu_{B \vee (A \wedge C)}(z) \quad \text{and} \quad \nu_{A \wedge (B \vee C)}(z) < \nu_{B \vee (A \wedge C)}(z).
\]

Thus, by Theorem 2.20,

\[
\mu_A(z) \wedge \mu_{B+C}(z) > \mu_{B+(A \wedge C)}(z) \quad \text{and} \quad \nu_A(z) \vee \nu_{B+C}(z) < \nu_{B+(A \wedge C)}(z).
\]

So

\[
\mu_A(z) > \mu_{B+(A \wedge C)}(z), \nu_A(z) < \nu_{B+(A \wedge C)}(z) \quad (10)
\]

and

\[
\mu_{B+C}(z) > \mu_{B+(A \wedge C)}(z), \nu_{B+C}(z) < \nu_{B+(A \wedge C)}(z).
\]

Then there exist \( x_0, y_0 \in R \) with \( z = x_0 + y_0 \) such that

\[
\mu_B(x_0) \wedge \mu_C(y_0) > \mu_{B+(A \wedge C)}(z)
\]

and

\[
\nu_B(x_0) \vee \nu_C(y_0) < \nu_{B+(A \wedge C)}(z)
\]
Thus
\[ \mu_B(x_0) > \mu_{B+(A \cap C)}(z), \nu_B(x_0) < \nu_{B+(A \cap C)}(z) \]
and
\[ \mu_C(y_0) > \mu_{B+(A \cap C)}(z), \nu_C(y_0) < \nu_{B+(A \cap C)}(z) \]
On the other hand,
\[ \mu_{B+(A \cap C)}(z) = \bigvee_{z=x+y} [\mu_B(x) \land \mu_{A \cap C}(y)] \]
\[ \geq \mu_B(x_0) \land \mu_{A \cap C}(y_0) = \mu_B(x_0) \land \mu_A(y_0) \land \mu_C(y_0) \]
and
\[ \nu_{B+(A \cap C)}(z) = \bigwedge_{z=x+y} [\nu_B(x) \lor \nu_{A \cap C}(y)] \]
\[ \leq \nu_B(x_0) \lor \nu_{A \cap C}(y_0) = \nu_B(x_0) \lor \nu_A(y_0) \lor \nu_C(y_0). \]
Then, by (10), (11) and (12),
\[ \mu_A(z), \mu_B(x_0), \mu_C(y_0) > \mu_B(x_0) \land \mu_A(y_0) \land \mu_C(y_0) \]
and
\[ \nu_A(z), \nu_B(x_0), \nu_C(y_0) < \nu_B(x_0) \lor \nu_A(y_0) \lor \nu_C(y_0). \]
Thus \( \mu_B(x_0) \land \mu_A(y_0) \land \mu_C(y_0) = \mu_A(y_0) \) and \( \nu_B(x_0) \lor \nu_A(y_0) \lor \nu_C(y_0) = \nu_A(y_0). \)
So \( \mu_A(-y_0) = \mu_A(y_0) < \mu_A(x_0 + y_0) = \mu_A(z) \) and \( \nu_A(-y_0) = \nu_A(y_0) > \nu_A(x_0 + y_0) = \nu_A(z). \) By Lemma 3.1,
\[ \mu_A(y_0) = \mu_A(x_0 + y_0 - y_0) = \mu_A(x_0) \]
and
\[ \nu_A(y_0) = \nu_A(x_0 + y_0 - y_0) = \nu_A(x_0). \]
Then \( \mu_B(x_0) > \mu_A(y_0) = \mu_A(x_0) \) and \( \nu_B(x_0) < \nu_A(y_0) = \nu_A(x_0). \) This contradicts the fact that \( B \subset A. \) Hence \( A \land (B \lor C) = B \lor (A \land C). \) Therefore \( \text{IFI}_{\lambda(0, \mu_0)}(R) \) is modular. This completes the proof. \( \square \)

**Remark 3.3.** As a special case, \( \text{IFI}_{(1,0)}(R) \) is a complete sublattice of \( \text{IFI}(R) \) and \( \text{IFI}_{(1,0)}(R) \) is a modular sublattice of \( \text{IFI}(R). \)

**Theorem 3.4.** (The generalization of Theorem 3.2) \( \text{IFI}_{\lambda(0, \mu_0)}(R), \text{IFI}_{\lambda(0, \mu_0)}(R) \) and \( \text{IFI}_{\lambda(0, \mu_0)}(R) \) are all modular.

**Proof.** The proofs are similar to Theorem 3.2. \( \square \)

**Proposition 3.5.** \( \text{IFI}(R) \) is bounded.
Proof. It is clear that $0 \sim \in \text{IFI}(R)$ and $1 \sim \in \text{IFI}(R)$. Moreover, $0 \sim A \subset 1 \sim$ for each $A \in \text{IFI}(R)$. Hence IFI($R$) is bounded. □

**Proposition 3.6.** (1) IFI($R$) is not complemented.
(2) IFI($R$) has no atoms.
(3) IFI($R$) has no dual atoms.

Proof. (1) We define a complex mapping $A = (\mu_A, \nu_A) : R \to I \times I$ as follows : for each $x \in R$, $A(x) = (\frac{1}{2}, \frac{1}{2})$. Then clearly $A, A^c \in \text{IFI}(R)$. But $A \cup A^c \neq 1 \sim$ and $A \cap A^c \neq 0 \sim$. Thus $A$ has no complement in IFI($R$). Hence IFI($R$) is not complemented.

(2) Suppose $A \in \text{IFI}(R)$ with $A \neq 0 \sim$. We define a complex mapping $B = (\mu_B, \nu_B) : R \to I \times I$ as follows : for each $x \in R$, $\mu_B(x) = \frac{1}{2} \mu_A(x)$ and $\nu_B(x) = 1 - \frac{1}{2} \nu_A(x)$. Then clearly $B \in \text{IFI}(R)$. Moreover $0 \sim \subseteq B \subseteq 1 \sim$. Hence IFI($R$) has no atoms.

(3) Suppose $A \in \text{IFI}(R)$ with $A \neq 1 \sim$. We define a complex mapping $B = (\mu_B, \nu_B) : R \to I \times I$ as follows : for each $x \in R$,

\[
\mu_B(x) = \frac{1}{2} + \frac{1}{2} \mu_A(x) \quad \text{and} \quad \nu_B(x) = \frac{1}{2} - \frac{1}{2} \mu_A(x).
\]

Then clearly $A \subseteq B \subseteq 1 \sim$. Now let $x, y \in R$. Then

\[
\mu_B(xy) = \frac{1}{2} + \frac{1}{2} \mu_A(xy)
\]

\[
\geq \frac{1}{2} + \frac{1}{2} (\mu_A(x) \lor \mu_A(y)) \quad \text{(Since $A \in \text{IFI}(R)$)}
\]

\[
= \left( \frac{1}{2} + \frac{1}{2} \mu_A(x) \right) \lor \left( \frac{1}{2} + \frac{1}{2} \mu_A(y) \right)
\]

\[
= \mu_B(x) \lor \mu_B(y)
\]

and

\[
\nu_B(xy) = \frac{1}{2} - \frac{1}{2} \mu_A(xy)
\]

\[
\leq \frac{1}{2} - \frac{1}{2} (\mu_A(x) \lor \mu_A(y)) = \left( \frac{1}{2} - \frac{1}{2} \mu_A(x) \right) \land \left( \frac{1}{2} - \frac{1}{2} \mu_A(y) \right)
\]

\[
= \nu_B(x) \land \nu_B(y).
\]

Also,

\[
\mu_B(x - y) = \frac{1}{2} + \frac{1}{2} \mu_A(x - y)
\]

\[
\geq \frac{1}{2} + \frac{1}{2} (\mu_A(x) \wedge \mu_A(y)) \quad \text{(Since $A \in \text{IFI}(R)$)}
\]

\[
= \left( \frac{1}{2} + \frac{1}{2} \mu_A(x) \right) \land \left( \frac{1}{2} + \frac{1}{2} \mu_A(y) \right) = \mu_B(x) \land \mu_B(y)
\]
and

\[ \nu_B(x - y) = \frac{1}{2} \left( \frac{1}{2} \mu_A(x - y) \right) \leq \frac{1}{2} - \frac{1}{2} \mu_A(x) \land \mu_A(y) \]

\[ = \frac{1}{2} \left( \frac{1}{2} \mu_A(x) \right) \lor \frac{1}{2} \left( \frac{1}{2} \mu_A(y) \right) = \nu_B(x) \lor \nu_B(y). \]

So \( B \in \text{IFI}(R) \). Hence IFI\((R)\) has no dual atoms. □

References


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