CONTACT CR-WARPED PRODUCT
SUBMANIFOLDS IN KENMOTSU SPACE FORMS

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Abstract. Recently, Chen studied warped products which are CR-submanifolds in Kaehler manifolds and established general sharp inequalities for CR-warped products in Kaehler manifolds. In the present paper, we obtain sharp estimates for the squared norm of the second fundamental form (an extrinsic invariant) in terms of the warping function for contact CR-warped products isometrically immersed in Kenmotsu space forms. The equality case is considered. Some applications are derived.

1. Introduction

Let \( \tilde{M} \) be a Hermitian manifold and denote by \( J \) the canonical almost complex structure on \( \tilde{M} \). According to the behavior of the tangent bundle \( TM \) with respect to the action of \( J \), we may distinguish two special classes of submanifolds \( M \) in \( \tilde{M} \).

a) complex submanifolds, i.e., \( J(T_pM) = T_pM, \forall p \in M \).

b) totally real submanifolds, i.e., \( J(T_pM) \subset T_p^\bot M, \forall p \in M \), where \( T_pM \) (resp. \( T_p^\bot M \)) is the tangent (resp. the normal) space of \( M \) at \( p \). Such submanifolds were defined and studied by Chen and Ogiue[10].

On the other hand, Yano and Ishihara (see [16]) considered a submanifold \( M \) whose tangent bundle \( TM \) splits into a complex subbundle \( D \) and a totally real subbundle \( D^\bot \). Later, such a submanifold was called a CR-submanifold ([3, 4]). Blair and Chen[3] proved that

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a $CR$-submanifold of a locally conformal Kaehler manifold is a Cauchy-Riemann manifold in the sense of Greenfield.

The first main result on $CR$-submanifolds was obtained by Chen [4]: any $CR$-submanifold of a Kaehler manifold is foliated by totally real submanifolds (i.e., the totally real subbundle is involutive).

As non-trivial examples of $CR$-submanifolds, we can mention the (real) hypersurfaces of Hermitian manifolds. Recently, Chen [7] introduced the notion of a $CR$-warped product submanifold in a Kaehler manifold. In a series of papers ([5, 7]), he investigated such submanifolds. In particular, he established a sharp relationship between the warping function $f$ of a warped product $CR$-submanifold $M_1 \times_f M_2$ of a Kaehler manifold $\tilde{M}$ and the squared norm of the second fundamental form $\|h\|^2$ (see [7]).

For other results on warped product submanifolds in complex space forms we refer to [13].

In the present paper, we study contact $CR$-warped product submanifolds in Kenmotsu space forms. We prove estimates of the squared norm of the second fundamental form in terms of the warping function. Equality cases are investigated. Obstructions to the existence of contact $CR$-warped product submanifolds in Kenmotsu space forms are derived.

2. Kenmotsu manifolds and their submanifolds

Tanno [15] has classified, into 3 classes, the connected almost contact Riemannian manifolds whose automorphisms groups have the maximum dimensions:

1. homogeneous normal contact Riemannian manifolds with constant $\phi$-holomorphic sectional curvature;
2. global Riemannian products of a line or circle and a Kaehlerian space form;
3. warped product spaces $L \times_f F$, where $L$ is a line and $F$ a Kaehlerian manifold.

Kenmotsu [12] studied the third class and characterized it by tensor equations. Later, such a manifold was called a Kenmotsu manifold.

A $(2m + 1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be a Kenmotsu manifold if it admits an endomorphism $\phi$ of its tangent
bundle $T\tilde{M}$, a vector field $\xi$ and a 1-form $\eta$, which satisfy:

\[
\begin{cases}
\phi^2 = -Id + \eta \otimes \xi, & \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), & \eta(X) = g(X, \xi), \\
(\nabla_X \phi) Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \\
\nabla_X \xi = X - \eta(X)\xi,
\end{cases}
\]

for any vector fields $X, Y$ on $\tilde{M}$, where $\nabla$ denotes the Riemannian connection with respect to $g$.

We denote by $\omega$ the fundamental 2-form of $\tilde{M}$, i.e.,

\[\omega(X, Y) = g(\phi X, Y), \quad \forall X, Y \in \Gamma(T\tilde{M}).\]

It was proved that the pairing $(\omega, \eta)$ defines a locally conformal cosymplectic structure, i.e.,

\[d\omega = 2\omega \wedge \eta, \quad d\eta = 0.\]

A Kenmotsu manifold with constant $\phi$-holomorphic sectional curvature $c$ is called a Kenmotsu space form and is denoted by $\tilde{M}(c)$. Then its curvature tensor $\tilde{R}$ is expressed by [12]

\[
4\tilde{R}(X, Y)Z = (c - 3\{g(Y, Z)X - g(X, Z)Y\} \\
+ (c + 1)[\{\eta(X)Y - \eta(Y)X\}\eta(Z)] \\
+ \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi \\
+ \omega(Y, Z)\phi X - \omega(X, Z)\phi Y - 2\omega(X, Y)\phi Z].
\]

Let $\tilde{M}$ be a Kenmotsu manifold and $M$ an $n$-dimensional submanifold tangent to $\xi$. For any vector field $X$ tangent to $M$, we put

\[\phi X = PX + FX,\]

where $PX$ (resp. $FX$) denotes the tangential (resp. normal) component of $\phi X$. Then $P$ is an endomorphism of tangent bundle $TM$ and $F$ is a normal bundle valued 1-form on $TM$.

The equation of Gauss is given by

\[\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) \\
+ g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),\]

for any vectors $X, Y, Z, W$ tangent to $M$.

We denote by $H$ the mean curvature vector, i.e.,

\[H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),\]
where \( \{e_1, ..., e_n\} \) is an orthonormal basis of the tangent space \( T_pM, \)
\( p \in M. \)

Also, we set
\[
h'_{ij} = g(h(e_i, e_j), e_r), \quad i, j = 1, ..., n; \ r = n + 1, ..., 2m + 1,
\]
and
\[
\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).
\]

By analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a Kenmotsu manifold were considered (see, for example, [16]).

A submanifold \( M \) tangent to \( \xi \) is said to be invariant (resp. anti-invariant) if \( \phi(T_pM) \subset T_pM, \forall p \in M \) (resp. \( \phi(T_pM) \subset T_p^\perp M, \forall p \in M \)).

A submanifold \( M \) tangent to \( \xi \) is called a contact CR-submanifold [16] if there exists a pair of orthogonal differentiable distributions \( D \) and \( D^\perp \) on \( M \), such that:

i) \( TM = D \oplus D^\perp \oplus \{\xi\} \), where \( \{\xi\} \) is the 1-dimensional distribution spanned by \( \xi \);

ii) \( D \) is invariant by \( \phi \), i.e., \( \phi(D_p) \subset D_p, \forall p \in M \);

iii) \( D^\perp \) is anti-invariant by \( \phi \), i.e., \( \phi(D^\perp_p) \subset T_p^\perp M, \forall p \in M \).

In particular, if \( D^\perp = \{0\} \) (resp. \( D = \{0\} \)), \( M \) is an invariant (resp. anti-invariant) submanifold.

3. Contact CR-warped product submanifolds

Let \((M_1, g_1)\) and \((M_2, g_2)\) be two Riemannian manifolds and \( f \) a positive differentiable function on \( M_1 \). The warped product of \( M_1 \) and \( M_2 \) is the Riemannian manifold
\[
M_1 \times_f M_2 = (M_1 \times M_2, g),
\]
where \( g = g_1 + f^2 g_2 \) (see, for instance, [8]).

It is well-known that the notion of warped products plays some important role in Differential Geometry as well as in Physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [6].

We recall the following general formulae on a warped product (see [8])
\[
\nabla_X Z = \nabla_Z X = \frac{1}{f}(X f) Z,
\]
for any vector fields \( X, Z \) tangent to \( M_1, M_2 \), respectively.
If $X$ and $Z$ are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by $X$ and $Z$ is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f} \{(\nabla_X X)f - X^2 f\}.$$ 

By reference to [11], a warped product submanifold $M_1 \times_f M_2$ of a Kenmotsu manifold $\widetilde{M}$, with $M_1$ a $(2\alpha + 1)$-dimensional invariant submanifold tangent to $\xi$ and $M_2$ a $\beta$-dimensional anti-invariant submanifold of $\widetilde{M}$, is said to be a contact CR-warped product submanifold.

We state the following estimate of the squared norm of the second fundamental form for contact CR-warped products in Kenmotsu manifolds.

**Theorem 3.1.** Let $\widetilde{M}$ be a $(2m + 1)$-dimensional Kenmotsu manifold and $M = M_1 \times_f M_2$ an $n$-dimensional contact CR-warped product submanifold, such that $M_1$ is a $(2\alpha + 1)$-dimensional invariant submanifold tangent to $\xi$ and $M_2$ a $\beta$-dimensional anti-invariant submanifold of $\widetilde{M}$. Then

(i) The squared norm of the second fundamental form of $M$ satisfies

$$\|h\|^2 \geq 2\beta[\|\nabla(\ln f)\|^2 - 1],$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

(ii) If the equality sign of (7) holds identically, then $M_1$ is a totally geodesic submanifold and $M_2$ is a totally umbilical submanifold of $\widetilde{M}$.

Moreover, $M$ is a minimal submanifold of $\widetilde{M}$.

**Proof.** Let $M = M_1 \times_f M_2$ be a warped product submanifold of a Kenmotsu manifold $\widetilde{M}$, such that $M_1$ is an invariant submanifold tangent to $\xi$ and $M_2$ an anti-invariant submanifold of $\widetilde{M}$.

For any unit vector fields $X$ tangent to $M_1$ and $Z, W$ tangent to $M_2$ respectively, we have:

$$g(h(\phi X, Z), \phi Z) = g(\nabla_Z \phi X, \phi Z)$$
$$= g(\phi \nabla_Z X, \phi Z)$$
$$= g(\nabla_Z X, Z)$$
$$= g(\nabla_X X, Z)$$
$$= (X \ln f)g(Z, W).$$

(8)
On the other hand, since the ambient manifold $\tilde{M}$ is a Kenmotsu manifold, it is easily seen that
\begin{equation}
\tag{9}
h(\xi, Z) = 0.
\end{equation}

Obviously, (5) implies $\xi \ln f = 1$. Therefore, by (8) and (9), the inequality (7) follows immediately.

Denote by $h''$ the second fundamental form of $M_2$ in $M$. Then, we get
\begin{equation}
\tag{10}
g(h''(Z, W), X) = g(\nabla_Z W, X) = -(X \ln f)g(Z, W),
\end{equation}
or equivalently
\begin{equation}
\tag{11}
h''(Z, W) = -g(Z, W)\nabla(\ln f).
\end{equation}

If the equality sign of (7) holds identically, then we obtain
\begin{equation}
\tag{11}
h(\mathcal{D}, \mathcal{D}) = 0, \quad h(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0, \quad h(\mathcal{D}, \mathcal{D}^\perp) \subset \phi \mathcal{D}^\perp.
\end{equation}

The first condition (11) implies that $M_1$ is totally geodesic in $M$. On the other hand, one has
\begin{equation*}
g(h(X, \phi Y), \phi Z) = g(\tilde{\nabla}_X \phi Y, \phi Z) = g(\nabla_X Y, Z) = 0.
\end{equation*}
Thus $M_1$ is totally geodesic in $\tilde{M}$.

The second condition (11) and (10) imply that $M_2$ is totally umbilical in $\tilde{M}$. Moreover, by (11), it follows that $M$ is a minimal submanifold of $\tilde{M}$.

In particular, if the ambient space is a Kenmotsu space form, one has the following.

**Corollary 3.2.** Let $\tilde{M}(c)$ be a $(2m + 1)$-dimensional Kenmotsu space form of constant $\phi$-sectional curvature $c$ and $M = M_1 \times_f M_2$ an $n$-dimensional non-trivial contact CR-warped product submanifold, satisfying
\begin{equation*}
\|h\|^2 = 2\beta[\|\nabla(\ln f)\|^2 - 1].
\end{equation*}
Then, we have
\begin{enumerate}
\item[(a)] $M_1$ is a totally geodesic invariant submanifold of $\tilde{M}(c)$. Hence $M_1$ is a Kenmotsu space form of constant $\phi$-sectional curvature $c$.
\item[(b)] $M_2$ is a totally umbilical anti-invariant submanifold of $\tilde{M}(c)$. Hence $M_2$ is a real space form of sectional curvature $\varepsilon > \frac{\sqrt{3}}{4}$.
\item[(c)] If $\beta > 1$, then the warping function $f$ satisfies $\|\nabla f\|^2 = (\varepsilon - \frac{\sqrt{3}}{4})f^2$.
\end{enumerate}
Proof. Statement (a) follows from Theorem 3.1.

Also, we know that $M_2$ is a totally umbilical submanifold of $\tilde{M}(c)$. Gauss equation implies that $M_2$ is a real space form of constant sectional curvature $\varepsilon \geq \frac{c-3}{4}$. Moreover, by (5), we see that $\varepsilon = \frac{c-3}{4}$ if and only if the warping function $f$ is constant.

Let $R''$ denote the Riemann curvature tensor of $M_2$. Then, we have (see, for instance, [8])

$$R(Z,W)V = R''(Z,W)V - \|\nabla (\ln f)\|^2 [g(W,V)Z - g(Z,V)W],$$

for any vectors $Z, W, V$ tangent to $M_2$.

By applying Gauss equation, we obtain statement (c).

4. Another inequality

In the present section, we will improve the inequality (7) for contact CR-warped product submanifolds in Kenmotsu space forms. Equality case is characterized.

THEOREM 4.1. Let $\tilde{M}$ be a $(2m+1)$-dimensional Kenmotsu space form and $M = M_1 \times_f M_2$ an $n$-dimensional contact CR-warped product submanifold, such that $M_1$ is a $(2\alpha+1)$-dimensional invariant submanifold tangent to $\xi$ and $M_2$ a $\beta$-dimensional anti-invariant submanifold of $\tilde{M}(c)$. Then

(i) The squared norm of the second fundamental form of $M$ satisfies

$$\|h\|^2 \geq 2\beta [\|\nabla (\ln f)\|^2 - \Delta (\ln f) - 1] + \alpha \beta (c + 1),$$

where $\Delta$ is the Laplacian operator on $M_1$.

(ii) The equality sign of (12) holds identically if and only if we have:

(a) $M_1$ is a totally geodesic invariant submanifold of $\tilde{M}(c)$. Hence $M_1$ is a Kenmotsu space form of constant $\phi$-sectional curvature $c$.

(b) $M_2$ is a totally umbilical anti-invariant submanifold of $\tilde{M}(c)$. Hence $M_2$ is a real space form of sectional curvature $\varepsilon \geq \frac{c-3}{4}$.

Proof. Let $M = M_1 \times_f M_2$ be a warped product submanifold of a $(2m+1)$-dimensional Kenmotsu space form $\tilde{M}(c)$, such that $M_1$ is an invariant submanifold tangent to $\xi$ and $M_2$ an anti-invariant submanifold of $\tilde{M}(c)$.

We denote by $\nu$ the normal subbundle orthogonal to $\phi(TM_2)$. Obviously, we have

$$T^\perp M = \phi(TM_2) \oplus \nu, \quad \phi\nu = \nu.$$
For any vector fields $X$ tangent to $M_1$ and orthogonal to $\xi$ and $Z$ tangent to $M_2$, equation (2) gives
\[
\tilde{R}(X, \phi X, Z, \phi Z) = \frac{c+1}{2} g(X, X)g(Z, Z).
\]
On the other hand, by Codazzi equation, we have
\[
\tilde{R}(X, \phi X, Z, \phi Z) = -g(\nabla_{\phi X}^\perp h(X, Z), \phi Z) + g(\nabla_{\phi}^\perp h(X, Z) - h(\phi X, Z), \phi Z).
\]
(13)

By using the equation (5) and structure equations of a Kenmotsu manifold, we get
\[
g(\nabla_{\phi X}^\perp h(X, Z), \phi Z) = Xg(h(X, Z), \phi Z) - g(h(X, Z), \nabla_{\phi} h(X, Z))
\]
\[
= X((X \ln f)g(Z, Z)) - (X \ln f)g(h(X, Z), \phi Z) - g(h(X, Z), \nabla_{\phi} h(X, Z))
\]
\[
= (X^2 \ln f)g(Z, Z) + (X \ln f)^2 g(Z, Z) - h_{\nu}(X, Z)^2.
\]
where we denote by $h_{\nu}(X, Z)$ the $\nu$-component of $h(X, Z)$.

Also, by (8) and (5), we obtain respectively
\[
g(h(\nabla_X \phi X, Z), \phi Z) = ((\nabla_X \phi X) \ln f)g(Z, Z),
\]
\[
g(h(\phi X, \nabla_Z X), \phi Z) = (X \ln f)g(h(\phi X, Z), \phi Z)
\]
\[
= (X \ln f)^2 g(Z, Z).
\]

Substituting the above relations in (13), we find
\[
\tilde{R}(X, \phi X, Z, \phi Z)
\]
\[
= 2||h_{\nu}(X, Z)||^2 - (X^2 \ln f)g(Z, Z) + ((\nabla_X \phi X) \ln f)g(Z, Z)
\]
\[
- ((\phi X)^2 \ln f)g(Z, Z) + ((\nabla_{\phi} X \phi X) \ln f)g(Z, Z).
\]
(14)

Then the equation (14) becomes
\[
2||h_{\nu}(X, Z)||^2 = \left[ \frac{c+1}{2} g(X, X) + (X^2 \ln f) - ((\nabla_X \phi X) \ln f)
\right.
\]
\[
+ ((\phi X)^2 \ln f) - ((\nabla_{\phi} X \phi X) \ln f)]g(Z, Z).
\]
(15)

Let $\{X_0 = \xi, X_1, \ldots, X_\alpha, X_{\alpha+1} = \phi X_1, \ldots, X_{2\alpha} = \phi X_\alpha, Z_1, \ldots, Z_\beta\}$ be a local orthonormal frame on $M$ such that $X_0, \ldots, X_{2\alpha}$ are tangent to...
Contact CR-Warped product submanifolds in Kenmotsu space forms

$M_1$ and $Z_1, \ldots, Z_\beta$ are tangent to $M_2$. Therefore

\begin{equation}
(16) \quad 2 \sum_{j=1}^{2\alpha} \sum_{t=1}^{\beta} \| h_\nu(X_j, Z_t) \|^2 = \alpha \beta (c + 1) - 2 \beta \Delta (\ln f).
\end{equation}

Combining (7) and (16), we obtain the inequality (12). The equality case can be solved similarly to the Corollary 3.2.

**Corollary 4.2.** Let $\tilde{M}(c)$ be a Kenmotsu space form with $c < -1$. Then there do not exist contact CR-warped product submanifolds $M_1 \times_f M_2$ in $\tilde{M}(c)$ such that $\ln f$ is a harmonic function on $M_1$.

**Proof.** Assume there exists a contact CR-warped product submanifold $M_1 \times_f M_2$ in a Kenmotsu space form $\tilde{M}(c)$ such that $\ln f$ is a harmonic function on $M_1$. Then (16) implies $c \geq -1$.

**Corollary 4.3.** Let $\tilde{M}(c)$ be a Kenmotsu space form with $c \leq -1$. Then there do not exist contact CR-warped product submanifolds $M_1 \times_f M_2$ in $\tilde{M}(c)$ such that $\ln f$ is a non-negative eigenfunction of the Laplacian on $M_1$ corresponding to an eigenvalue $\lambda > 0$.

**References**


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