ON THE STABILITY OF A BETA
TYPE FUNCTIONAL EQUATIONS

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Abstract. In this paper we investigate the generalized Hyers-Ulam-Rassias sta-
bility for a functional equation of the form \( f(\varphi(x, y)) = \phi(x, y)f(x, y) \), where \( x, y \) lie in the set \( S \). As a consequence we obtain stability in the sense of Hyers, Ulam,
Rassias, Gavruta, for some well-known equations such as the gamma, beta and
\( G \)-function type equations.

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1. Introduction

In 1940, The stability problem raised by S. M. Ulam [19] had solved by D. H.
Hyers [5], it has been generalized to the unbounded case by Th. M. Rassias
[15], his result is extended by P. Gavruta[3] and Ger[4].

The functional equation which we interested in this article is derived from
the gamma functional equation \( f(x + 1) = xf(x) \), which was proved by S.
M.Jung([8],[9],[10]) originally. This equation is generalized to the gamma types
functional equation \( f(x + p) = \varphi(x)f(x) \) and the beta types functional equation
\( f(x + p, y + p) = \varphi(x, y)f(x, y) \) by author([7],[11],[12],[13],[14]). Recently T. Trif
[18] researched the more generalized gamma types functional equation \( f(\varphi(x)) =
\phi(x)f(x) + \psi(x) \).

In this paper, we will be investigated the generalized Hyers-Ulam-Rassias
stability in the sense of Gavruta for the functional equation

\[
f(\varphi(x, y)) = \phi(x, y)f(x, y),
\]

(1.1)
where \( \varphi, \phi \) are the given functions, while \( f \) is the unknown function, and \( x, y \) lie in the set \( S \). And our obtained results also apply to the beta type functional equations, and the application to well-known results concerned with gamma and G type functions.

In section 2, we contribute the stability for the functional equations (1.1) in the sense of Gavruta.

In section 3, our results shown in the section 2 apply to the gamma, G types functional equations and some examples suitably restricted by the domain in one or two variables.

Throughout this paper, let \( B \) be a Banach space over the field \( K \). \( K \) will be either the field \( R \), or the field \( C \) of complex numbers. Each positive real number \( \delta \) is fixed, and \( n_0 \) is a fixed natural number. \( R \) and \( R^+ \) denote the set of real numbers and the set of all nonnegative real numbers, respectively. Given the nonempty set \( S \) and the function \( \varphi : S^2 \rightarrow S^2 \), we put \( \varphi_0(x, y) := (x, y) \) and \( \varphi_n(x, y) := \varphi(\varphi_{n-1}(x, y)) \) for all positive integers \( n \) and all points \( x, y \in S \). The functions \( \phi : S^2 \rightarrow K \setminus \{0\}, \psi : S^2 \rightarrow B \) and \( \varepsilon : S^2 \rightarrow R^+ \) are defined.

2. The generalization of Hyers-Ulam stability of the Equation (1.1) and (2.8)

Let the functions \( \varphi, \phi, \varepsilon \) be the given functions such that

\[
\omega(x, y) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x, y))}{\prod_{j=0}^{k} |\phi(\varphi_j(x, y))|} < \infty \quad \forall x, y \in S. \tag{2.1}
\]

**Theorem 1.** Let the functions \( \varphi, \phi, \varepsilon \) hold the condition (2.1). If a function \( f : S^2 \rightarrow B \) satisfies the inequality

\[
||f(\varphi(x, y)) - \phi(x, y)f(x, y)|| \leq \varepsilon(x, y) \quad \forall x, y \in S, \tag{2.2}
\]

then there exists a unique solution \( g : S^2 \rightarrow B \) of the equation (1.1) with

\[
||g(x, y) - f(x, y)|| \leq \omega(x, y) \quad \forall x, y \in S. \tag{2.3}
\]

**Proof.** For any \( x, y \in S \) and for every positive integer \( n \), let \( \omega_n : S^2 \rightarrow R^+ \) and \( g_n : S^2 \rightarrow B \) be the functions defined by

\[
\omega_n(x, y) := \sum_{k=0}^{n-1} \frac{\varepsilon(\varphi_k(x, y))}{\prod_{j=0}^{k} |\phi(\varphi_j(x, y))|}
\]


and

\[ g_n(x, y) = \frac{f(\varphi_n(x, y))}{\prod_{j=0}^{n-1} \phi(\varphi_j(x, y))} \]

for all \( x, y \in S \), respectively.

By (2.2), it follows that

\[ \left| \frac{f(\varphi(x, y))}{\phi(x, y)} - f(x, y) \right| \leq \frac{\varepsilon(x, y)}{|\phi(x, y)|} \quad \text{for all } x, y \in S. \]

Substituting \((x, y)\) by \(\varphi_n(x, y)\) in this inequality, and then dividing both sides of the obtained inequality by \(\prod_{j=0}^{n-1} |\phi(\varphi_j(x, y))|\), we get

\[ ||g_{n+1}(x, y) - g_n(x, y)|| = \frac{\varepsilon(\varphi_n(x, y))}{\prod_{j=0}^{n-1} |\phi(\varphi_j(x, y))|}. \quad (2.4) \]

By induction on \(n\) we prove that

\[ ||g_n(x, y) - f(x, y)|| \leq \omega_n(x, y). \quad (2.5) \]

for all \( x, y \in S \), and for all positive integers \(n\). For the case \(n = 1\), the inequality (2.5) is an immediate consequence of (2.2).

Assume that the inequality (2.5) holds true for some \(n\). Then we obtain for \(n + 1\) by (2.4) that

\[ ||g_{n+1}(x, y) - f(x, y)|| \leq ||g_{n+1}(x, y) - g_n(x, y)|| + ||g_n(x, y) - f(x, y)|| \]

\[ \leq \frac{\varepsilon(\varphi_n(x, y))}{\prod_{j=0}^{n-1} |\phi(\varphi_j(x, y))|} + \omega_n(x, y) \]

\[ = \omega_{n+1}(x, y). \]

We claim that \(\{g_n(x, y)\}\) is a Cauchy sequence. Indeed, by (2.4) and (2.1), we have for \(n > m\) that

\[ ||g_n(x, y) - g_m(x, y)|| \leq \sum_{k=m}^{n-1} ||g_{k+1}(x, y) - g_k(x, y)|| \]

\[ \leq \sum_{k=m}^{n-1} \frac{\varepsilon(\varphi_k(x, y))}{\prod_{j=0}^{k-1} |\phi(\varphi_j(x, y))|} \rightarrow 0 \]

as \(m \rightarrow \infty\).

Hence we can define a function \(g : S^2 \rightarrow B\) by

\[ g(x, y) := \lim_{n \rightarrow \infty} g_n(x, y). \quad (2.6) \]
From the definition of $g_n$, we have $g_n(\varphi(x, y)) = \phi(x, y)g_{n+1}(x, y)$, hence the function $g$ satisfies (1.1).

We show from (2.5) that $g$ satisfies the inequality (2.3) as follows:

$$||g(x, y) - f(x, y)|| = \lim_{n \to \infty} ||g_n(x, y) - f(x, y)||$$

$$\leq \lim_{n \to \infty} \omega_n(x, y)$$

$$= \omega(x, y)$$

for all $x, y \in S$.

If $h : S^2 \to B$ is another function which satisfies (1.1) and (2.3), then we have

$$||g(x, y) - h(x, y)||$$

$$= ||g(\varphi_n(x, y)) - h(\varphi_n(x, y))|| \cdot \prod_{j=0}^{n-1} \frac{1}{|\phi(\varphi_j(x, y))|}$$

$$\leq 2\omega_n(\varphi_n(x, y)) \cdot \prod_{j=0}^{n-1} \frac{1}{|\phi(\varphi_j(x, y))|}$$

$$= 2 \left( \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_{n+k}(x, y))}{\prod_{j=0}^{k} \phi(\varphi_{n+j}(x, y))} \right) \cdot \prod_{j=0}^{n-1} \frac{1}{|\phi(\varphi_j(x, y))|}$$

$$= 2 \sum_{k=n}^{\infty} \frac{\varepsilon(\varphi_k(x, y))}{\prod_{j=0}^{k} |\phi(\varphi_j(x, y))|}$$

for all $x, y \in S$ and all positive integers $n$. Which tends to zero as $n \to \infty$, since $\omega(x, y)$ is bounded. This implies the uniqueness of $g$. \hfill \Box

As a direct consequence of Theorem 1, the following results provide the Hyers-Ulam stability of equation (1.1), the Hyers-Ulam stability and stability in the sense of Gavruta for the equations (2.8).

Let the function $\varphi, \phi$ satisfy

$$\mu(x, y) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\phi(\varphi_j(x, y))|} < \infty \quad \forall x, y \in S. \quad (2.7)$$

**Corollary 1.** Let the function $\varphi, \phi$ hold the condition (2.7). If a function $f : S^2 \to B$ satisfies the inequality

$$||f(\varphi(x, y)) - \phi(x, y)f(x, y)|| \leq \delta \quad \forall x, y \in S,$$
then there exists a unique solution $g : S^2 \rightarrow B$ of the equation (1.1) with
\[ ||g(x, y) - f(x, y)|| \leq \delta \mu(x, y) \quad \forall x, y \in S. \]

Proof. Putting $\varepsilon(x, y) = \delta$ in Theorem 1.

We consider the special case of the equation (1.1) as follow: that is $\varphi(x, y) = (x + p, y + q), S = (0, \infty), \text{ and } B = R$. Then we can get the same results for the modified functional equations:
\[ f(x + p, y + q) = \phi(x, y)f(x, y), \quad (2.8) \]
\[ f(x + 1, y + 1) = \phi(x, y)f(x, y), \quad (2.9) \]
where each positive real number $p, q$ is fixed, $x, y \in (0, \infty)$.

The conditions for the functions $\phi$ and $\varepsilon$ in (2.1) and (2.7) change to
\[ \omega'(x, y) := \sum_{k=0}^{\infty} \frac{\varepsilon(x + kp, y + kq)}{\prod_{j=0}^{k} |\phi(x + jp, y + jq)|} < \infty \quad \text{and} \quad (2.10) \]
\[ \mu'(x, y) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\phi(x + jp, y + jq)|} < \infty \quad \text{for all } x, y > 0. \quad (2.11) \]

**Theorem 2.** Let the functions $\phi, \varepsilon$ hold the condition (2.10). If a function $f : (0, \infty)^2 \rightarrow R$ satisfies the inequality
\[ |f(x + p, y + q) - \phi(x, y)f(x, y)| \leq \varepsilon(x, y) \quad \forall x, y > n_0, \]
then there exists a unique solution $g : (0, \infty)^2 \rightarrow R$ of the equation (2.8) with
\[ |g(x, y) - f(x, y)| \leq \omega'(x, y) \quad \forall x, y > n_0. \]

Proof. Putting $S = (0, \infty), B = R, \varphi(x, y) = (x + p, y + q)$ in Theorem 1, then Theorem 1 implies the required result except for the domain of the function. For this we define the function $g_0 : (n_0, \infty)^2 \rightarrow R$ by
\[ g_0(x, y) := \lim_{n \to \infty} g_n(x, y), \]
in substituting $g$ defined in (2.6) to $g_0$. 
Now we extend the function \( g_0 \) to \((0, \infty)^2\). We define for each \( 0 < x, y \leq n_0 \)
\[
g(x, y) := \frac{g_0(x + kv, y + kv)}{\prod_{k=0}^{n-1} \phi(x + np, y + nq)},
\]
where \( k \) is the smallest natural number satisfying the inequalities \( x_i + kp_i > n_0 \)
for each \( i \).

Then \( g(x + p, y + q) = \phi(x, y)g(x, y) \) for all \( x, y > 0 \) and \( g(x, y) = g_0(x, y) \)
for all \( x, y > n_0 \). Also the inequality
\[
|g(x, y) - f(x, y)| < \omega'(x, y)
\]
holds for all \( x, y > n_0 \).

**Corollary 2.** Let the functions \( \phi \) hold the condition (2.11). If a function \( f : (0, \infty)^2 \rightarrow \mathbb{R} \) satisfies the inequality
\[
|f(x + p, y + q) - \phi(x, y)f(x, y)| \leq \delta, \quad \forall x, y > n_0,
\]
then there exists a unique solution \( g : (0, \infty)^2 \rightarrow \mathbb{R} \) of the equation (2.8) with
\[
|g(x, y) - f(x, y)| \leq \delta \mu' \quad \forall x, y > n_0.
\]

**Proof.** Putting \( \varepsilon(x, y) = \delta \) in Theorem 2.

**Corollary 3.** Let a function \( f : (0, \infty)^2 \rightarrow \mathbb{R} \) satisfies the inequality
\[
|f(x + 1, y + 1) - (x + y)f(x, y)| \leq \delta, \quad \forall x, y > n_0.
\]
Then there exists a unique solution \( g : (0, \infty)^2 \rightarrow \mathbb{R} \) of the equation (2.9) with
\[
|g(x, y) - f(x, y)| < \frac{\delta}{x + y} \left(1 + \frac{1}{2} + \frac{1}{2 \cdot 4} + \frac{1}{2 \cdot 4 \cdot 6} + \cdots\right) < \frac{5}{3} \frac{\delta}{x + y} \quad \forall x, y > n_0.
\]

**Proof.** Putting \( \phi(x, y) = x + y, p = q = 1 \) in Corollary 2.

3. Applications to the beta type, the gamma type functions, and the \( G \)-functional equation
(1) To the beta type functional equations

Let consider the special cases with $\phi(x, y) = \frac{xy}{(x+y)(x+y+1)}$ and $p = q = 1$ of the functional equation (2.8), then it implies the following functional equations

$$f(x+1, y+1) = \frac{(x+y)(x+y+1)}{xy}f(x, y),$$ (3.1)

$$f(x+1, y+1) = \frac{xy}{(x+y)(x+y+1)}f(x, y).$$ (3.1')

Since the beta function $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ satisfies the equation (3.1'), it is called the beta functional equation. The results of stability for equations (3.1) and (3.1') are given in [7] and [12]. The beta function $B(x, y)$ is closely related to the gamma function $\Gamma(x)$, that is, the relationship between them is

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y, x).$$

The condition (2.10) is replaced to

$$\omega_\beta(x, y) := \sum_{k=0}^{\infty} \varepsilon(x+k, y+k) \prod_{j=0}^{k} \frac{(x+j)(y+j)}{(x+j)+(y+j)((x+j)+(y+j)+1)} < \infty$$ (3.2)

for all $x, y > 0$.

**Corollary 4.** ([7], [12]). Let the function $\varepsilon$ hold the condition (3.2). If the function $f : (0, \infty) \times (0, \infty) \rightarrow R$ satisfies the inequality

$$|f(x+1, y+1)^{-1} - \frac{(x+y)(x+y+1)}{xy}f(x, y)^{-1}| \leq \varepsilon(x, y) \quad \forall x, y > n_0,$$

then there exists a unique solution $g : (0, \infty) \times (0, \infty) \rightarrow R$ of the equation (3.1) with

$$|g(x, y)^{-1} - f(x, y)^{-1}| \leq \omega_\beta(x, y) \quad \forall x, y > n_0.$$

**Remark 1.** The Corollary 4 with $\varepsilon(x) = \delta$ imply the Hyers-Ulam stability of equation (3.1), its direct proof can be found in the papers ([7], [12]).

By restricting to a single variable in the above shown results, we can apply to the gamma type functional equations, the $G$-functional equations, and also to the generalized forms of them, which is founded in the papers ([1], [7], [8], [9], [10], [11], [12], [13], [14]).
(1) To the gamma type functional equations

Restricting to a single variable of the functional equations (2.8) imply the following functional equations

\[ f(\varphi(x)) = \phi(x)f(x), \]
\[ f(\varphi(x)) = xf(x), \]  \hspace{1cm} (3.3)
\[ f(x + p) = \phi(x)f(x), \]  \hspace{1cm} (3.4)
\[ f(x + 1) = xf(x). \]  \hspace{1cm} (3.5)

The gamma function given by \( \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \) is a solution of the gamma functional equation (3.6).

For the stability of the equation (3.3) we restrict the condition (2.1) to a single variable, then it is represented by

\[ \omega_\gamma(x) := \sum_{k=0}^{\infty} \frac{\epsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|} < \infty \quad \forall x \in (0, \infty). \]  \hspace{1cm} (3.7)

**Corollary 5.** ([18]). Let the functions \( \varphi, \phi, \epsilon \) hold the condition (3.7). If a function \( f : S \to B \) satisfies the inequality

\[ ||f(\varphi(x)) - \phi(x)f(x)|| \leq \epsilon(x) \quad \forall x \in S, \]

then there exists a unique solution \( g : S \to B \) of the equation (3.3) with

\[ ||g(x) - f(x)|| \leq \omega_\gamma(x) \quad \forall x \in S. \]

The equation (3.4) can be considered the generalized form of Schröder functional equation \( f(\varphi(x)) = cf(x), c \neq 1 \): constant. In the case \( c > 1 \), Trif [18] proved the its stability, which is special case of the following Corollary.

**Corollary 6.** Let the function \( \varphi \) satisfies \( |\varphi_j(x)| > |x| > 1 \) for all \( j \). If a function \( f : (0, \infty) \to K \) satisfies the inequality

\[ |f(\varphi(x)) - xf(x)| \leq \delta \quad \forall x \in (0, \infty), \]

then there exists a unique solution \( g : (0, \infty) \to K \) of the equation (3.4) with

\[ |g(x) - f(x)| \leq \frac{\delta}{|x| - 1} \quad \forall x \in (0, \infty). \]
Proof. Putting $B = K, \phi(x) = x, \varepsilon(x) = \delta$ in Corollary 5.

Remark 2. Putting $\phi(x) = c > 1$ and $x = c$ in the Corollaries 5 and 6, we have the Hyers-Ulam stability of the Schröder functional equation $f(\varphi(x)) = cf(x)$, $c \neq 1$. It is founded in Trif [18].

Consider a single variable with $\phi(x) = x, p = 1$ in Theorem 2, then we have the stability of the gamma and generalized gamma functional equations (3.6) and (3.5). The condition (2.10) is represented by

$$\omega_\gamma p (x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x + kp)}{\prod_{j=0}^{k} |\phi(x + jp)|} < \infty$$

(3.8)

$$\omega_\gamma 1 (x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x + k)}{\prod_{j=0}^{k} |x + j|} < \infty.$$  

(3.9)

Corollary 7. ([11]). Let the functions $\phi, \varepsilon$ hold the condition (3.8). If a function $f : (0, \infty) \to R$ satisfies the inequality

$$|f(x + p) - \phi(x)f(x)| \leq \varepsilon(x) \quad \forall x > n_0,$$

then there exists a unique solution $g : (0, \infty) \to R$ of the equation (3.5) with

$$|g(x) - f(x)| \leq \omega_\gamma p (x) \quad \forall x > n_0.$$

Corollary 8. ([10], [11]). Let the function $\varepsilon$ hold the condition (3.9). If a function $f : (0, \infty) \to R$ satisfies the inequality

$$|f(x + 1) - xf(x)| \leq \varepsilon(x) \quad \forall x > n_0,$$

then there exists a unique solution $g : (0, \infty) \to R$ of the equation (3.6) with

$$|g(x) - f(x)| \leq \omega_\gamma 1 (x) \quad \forall x > n_0.$$

Remark 3. The Hyers-Ulam stability of equations (3.3), (3.5), (3.6) will be omitted, because it follows immediately from the Corollaries 5, 7, 8 with $\varepsilon(x) = \delta$, and also the generalized Hyers-Ulam stability of (3.4) follows immediately the Corollary 5.
(2) To the \(G\)-functional equation

The \(G\)-function introduced by E. W. Barnes [2]

\[
G(z) = (2\pi)^{\frac{z-1}{2}} e^{-\frac{\pi(z-1)}{2}} e^{-\gamma(z-1)^2} \prod_{k=1}^{\infty} \left(1 + \frac{z-1}{k} e^{1-\gamma(z-1)^2}ight)
\]

(3.10)
does satisfy the equation \(G(x + 1) = \Gamma(x)G(x)\) and \(\Gamma(1) = G(1) = 1\), where \(\gamma\) is the Euler-Mascheroni's constant defined by \(\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \approx 0.577215664\ldots\).

The properties and values of \(G\)-function are depend on those of the gamma function. Since the double gamma function is defined by the reciprocal of the \(G\)-function(see [2]), namely the double gamma function \(\Gamma_2\) is \(\Gamma_2(x) = 1/G(x)\), its functional equation is

\[
\Gamma_2(x + 1) = \Gamma_2(x)/\Gamma(x).
\]

Therefore the problem of stability for the \(G\)-function is equivalent to the stability for the reciprocal of the double gamma function.

Putting \(\phi(x) = \Gamma(x)\) and \(p = 1\) in equation (3.5), we obtain

\[
f(x + p) = \Gamma(x)f(x), \quad f(x + 1) = \Gamma(x)f(x).
\]

(3.11) (3.12)

The equation (3.12) may be called the \(G\)-functional equation since the \(G\)-function (3.10) is a solution of the equation (3.12).

The condition (3.5) is represented by

\[
\omega_{G_p}(x) := \sum_{k=0}^{\infty} \frac{\epsilon(x + kp)}{\prod_{j=0}^{k} |\Gamma(x + jp)|} < \infty.
\]

(3.13)

**Corollary 9.** Let the function \(\epsilon\) hold the condition (3.13). If a function \(f : (0, \infty) \to \mathbb{R}\) satisfies the inequality

\[
|f(x + p) - \Gamma(x)f(x)| \leq \epsilon(x) \quad \forall x > n_0,
\]

then there exists a unique solution \(g : (0, \infty) \to \mathbb{R}\) of the equation (3.11) with

\[
|g(x) - f(x)| \leq \omega_{G_p}(x) \quad \forall x > n_0.
\]
Remark 4. The Hyers-Ulam stability of equation (3.11) follows immediately from Corollary 9 with ε(x) = δ.

Remark 5. Stability for G-functional equation (the reciprocal of the double gamma functional equation). The Hyers-Ulam stability and the generalized Hyers-Ulam-Rassias stability of the equation (3.12) follows immediately the Corollary 9 and the Remark 4. If we compare the Hyers-Ulam stability for the gamma functional equation (3.6) and the G-functional equation (3.12), we can find that f is closer to g in the case \( \left( e^{\delta \Gamma(x)} \right) \) of the G-function than in the case \( \left( e^{\delta x} \right) \) of the gamma function in the interval \((0, 1)\) and \((\xi, \infty)\), where \( \xi \) is fixed point such that \( \Gamma(\xi) = \xi \neq 1 \).

4. Stability in the sense of Ger for the equation (1.1)

The following theorem is to give the stability in the sense of Ger for the equation (1.1).

Theorem 3. Let a function \( \varepsilon : S^2 \rightarrow (0, 1) \) satisfies
\[
\sum_{j=0}^{\infty} \varepsilon(\phi_j(x, y)) < \infty \quad \forall x, y \in S.
\]

If a function \( f : S^2 \rightarrow (0, \infty) \) be a function which satisfies the inequality
\[
\left| \frac{f(\phi(x, y))}{\phi(x, y) f(x, y)} - 1 \right| \leq \varepsilon(x, y) \quad \forall x, y \in S,
\]

then there exists a unique solution \( g : S^2 \rightarrow (0, \infty) \) of the equation (1.1) with
\[
\alpha(x, y) \leq g(x, y) \leq \beta(x, y),
\]
where \( \alpha(x, y) := \prod_{j=0}^{\infty} (1 - \varepsilon(\phi_j(x, y))) \) and \( \beta(x, y) := \prod_{j=0}^{\infty} (1 + \varepsilon(\phi_j(x, y))) \) for all \( x, y \in S \).

Proof. The condition (4.1) implies that \( \prod_{j=0}^{\infty} (1 \pm \varepsilon(\phi_j(x, y))) \) converge. Hence we can define the functions \( \alpha, \beta \) for all \( x, y \in S \) such that \(-\infty < \alpha := \prod_{j=0}^{\infty} (1 - \varepsilon(\phi_j(x, y))) < \prod_{j=0}^{\infty} (1 + \varepsilon(\phi_j(x, y))) := \beta < +\infty\), that is, these series are bounded.
For any \( x, y \in S \) and for every positive integer \( n \), we define
\[
g_n(x, y) = \frac{f(\varphi_n(x, y))}{\prod_{j=0}^{n-1} \varphi(\varphi_j(x, y))}.
\] (4.4)

For all positive integers \( m, n \) with \( n > m \), we have
\[
g_n(x, y) g_m(x, y) = \frac{f(\varphi_{m+1}(x, y))}{\varphi(\varphi_m(x, y)) f(\varphi_{m+1}(x, y))} \cdots \frac{f(\varphi_n(x, y))}{\varphi(\varphi_{n-1}(x, y)) f(\varphi_{n-1}(x, y))}.
\] (4.5)

It also follows from (4.2) that
\[
0 < 1 - \varepsilon(\varphi_j(x, y)) \leq \frac{f(\varphi_{j+1}(x, y))}{\varphi(\varphi_j(x, y)) f(\varphi_j(x, y))} \leq 1 + \varepsilon(\varphi_j(x, y))
\] (4.6)

for all \( x, y \in S \) and \( j = 0, 1, 2, \ldots \). From (4.5) and (4.6), we get
\[
\prod_{j=m}^{n-1} (1 - \varepsilon(\varphi_j(x, y))) \leq g_n(x, y) g_m(x, y) \leq \prod_{j=m}^{n-1} (1 + \varepsilon(\varphi_j(x, y)))
\]
or
\[
\sum_{j=m}^{n-1} \log(1 - \varepsilon(\varphi_j(x, y))) \leq \log g_n(x, y) - \log g_m(x, y) \leq \sum_{j=m}^{n-1} \log(1 + \varepsilon(\varphi_j(x, y))).
\]

Since \( \sum_{j=0}^{\infty} \log(1 - \varepsilon(\varphi_j(x, y))) = \log \alpha(x, y) \) and \( \sum_{j=0}^{\infty} \log(1 + \varepsilon(\varphi_j(x, y))) = \log \beta(x, y) \), it follows that \( \lim_{m \to \infty} \sum_{j=m}^{\infty} \log(1 - \varepsilon(\varphi_j(x, y))) = \lim_{m \to \infty} \sum_{j=m}^{\infty} \log(1 + \varepsilon(\varphi_j(x, y))) = 0 \) by boundedness of \( \alpha, \beta \).

Hence we note that \( \{\log g_n(x, y)\} \) is a Cauchy sequence for all \( X \in S \). It is reasonable to define a function \( g_0 : S^2 \to (0, \infty) \) by
\[
g_0(x, y) = e^{L(x, y)} = \lim_{n \to \infty} g_n(x, y) \quad \forall x, y \in S,
\] (4.7)

where \( L(x, y) := \lim_{n \to \infty} \log g_n(x, y) \).
We get easily that
\[ g_0(\varphi(x, y)) = \phi(x, y)g_0(x, y) \quad \forall x, y \in S. \quad (4.8) \]

Since
\[
\begin{align*}
\frac{g_n(x, y)}{f(x, y)} &= \frac{f(\varphi(x, y))}{\phi(x, y)f(x, y)} \cdot \frac{f(\varphi_2(x, y))}{\phi(\varphi(x, y)f(\varphi(x, y))} \\
&\quad \cdots \\
&\quad \frac{f(\varphi_n(x, y))}{\phi(\varphi_{n-1}(x, y)) f(\varphi_{n-1}(x, y))},
\end{align*}
\]
we get
\[
\prod_{j=0}^{n-1}(1 - \varepsilon(\varphi_j(x, y))) \leq \frac{g_n(x, y)}{f(x, y)} \leq \prod_{j=0}^{n-1}(1 + \varepsilon(\varphi_j(x, y))) \quad (4.9)
\]
for all \( x, y \in S \). This implies from (4.7), (4.9), and the definitions of \(\alpha\), \(\beta\) that
\[
\alpha(x, y) \leq \frac{g_0(x, y)}{f(x, y)} \leq \beta(x, y) \quad (4.10)
\]
for all \( x, y \in S \).

Assume \( h : S^2 \to (0, \infty) \) is a solution of equation (4.8) which satisfies the inequality (4.10). By (4.8), we have
\[
\begin{align*}
\frac{g_0(x, y)}{h(x, y)} &= \frac{g_0(\varphi_n(x, y))}{h(\varphi_n(x, y))} \frac{g_0(\varphi_n(x, y))}{f(\varphi_n(x, y))} \frac{f(\varphi_n(x, y))}{h(\varphi_n(x, y))} \\
&\quad \cdots \\
&\quad \frac{f(\varphi_n(x, y))}{h(\varphi_n(x, y))},
\end{align*}
\]
for any \( x, y \in S \) and for any natural number \( n \).

Hence we have
\[
\frac{\alpha(\varphi_n(x, y))}{\beta(\varphi_n(x, y))} \leq \frac{g_0(x, y)}{h(x, y)} \leq \frac{\beta(\varphi_n(x, y))}{\alpha(\varphi_n(x, y))}
\]
for any natural number \( n \). By the boundedness of the series \(\varepsilon\),
\[
\alpha(\varphi_n(x, y)) = \prod_{j=n}^{\infty}(1 - \varepsilon(\varphi_j(x, y))) \to 1
\]
as \( n \to \infty \). Similarly \( \beta(\varphi_n(x, y)) \to 1 \) as \( n \to \infty \).

Hence, it is obvious that \( h(x, y) \equiv g_0(x, y) \). \( \square \)

From the proof of Theorem 3, we can be known that the inequality (4.1) is a condition for the convergence of \(\alpha\) and \(\beta\). Hence the following corollary is natural.
Corollary 10. Let a function $f$ satisfies the inequality (4.2), in which $\varepsilon$ satisfies that

$$
\alpha(x, y) := \prod_{j=0}^{\infty} (1 - \varepsilon(\varphi_j(x, y))) \quad \text{and} \quad \beta(x, y) := \prod_{j=0}^{\infty} (1 + \varepsilon(\varphi_j(x, y)))
$$

are bounded for all $x, y \in S$. Then there exists a unique solution $g : S^2 \to (0, \infty)$ of the equation (1.1) with (4.3) for all $x, y \in S^2$.

Restrict Theorem 3 with $S = (0, \infty), \varphi(x, y) = (x + p, y + q)$ into two variable. Then we have the stability in the sense of Ger for the reciprocal of beta functional equation and the generalized functional equations of it as follow. Results in the weak condition for the function $\varepsilon$ can be found in papers([7],[11]).

Corollary 11. Let a function $f : (0, \infty) \times (0, \infty) \to \mathbb{R}^+$ be a function that satisfies the inequality

$$
\left| \frac{f(x+p, y+q)}{\phi(x, y)} \frac{f(x, y)}{f(x, y+1)} - 1 \right| \leq \varepsilon(x, y),
$$

where $\varepsilon : (0, \infty) \times (0, \infty) \to (0, 1)$ is a function such that

$$
\sum_{j=0}^{\infty} \varepsilon(x + jp, y + jq) < \infty \quad \forall x, y \in (0, \infty).
$$

Then there exists a unique solution $g : (0, \infty) \times (0, \infty) \to \mathbb{R}^+$ of the equation (2.8) with

$$
\alpha_{\varepsilon}(x, y) \leq g(x, y) \leq \beta_{\varepsilon}(x, y),
$$

where $\alpha_{\varepsilon}(x, y) := \prod_{j=0}^{\infty} (1 - \varepsilon(x + jp, x + jq))$ and $\beta_{\varepsilon}(x, y) := \prod_{j=0}^{\infty} (1 + \varepsilon(x + jq, x + jq))$.

Corollary 12. Let a function $f : (0, \infty) \times (0, \infty) \to \mathbb{R}^+$ be a function which satisfies the inequality

$$
\left| \frac{xy}{(x+y)(x+y+1)} \frac{f(x, y)}{f(x+1, y+1)} - 1 \right| \leq \varepsilon(x, y),
$$

where $\varepsilon : (0, \infty) \times (0, \infty) \to (0, 1)$ is a function such that

$$
\sum_{j=0}^{\infty} \varepsilon(x + j, y + j) < \infty \quad \forall x, y \in (0, \infty).
$$
Then there exists a unique solution \( g : (0, \infty) \times (0, \infty) \to R_+ \) of the functional equation (2.9) with
\[
\alpha_{\varepsilon_1}(x, y) \leq \frac{f(x, y)}{g(x, y)} \leq \beta_{\varepsilon_1}(x, y),
\]
where \( \alpha_{\varepsilon_1}(x, y) := \prod_{j=0}^{\infty} (1 - \varepsilon(x + j, x + j)) \) and \( \beta_{\varepsilon_1}(x, y) := \prod_{j=0}^{\infty} (1 + \varepsilon(x + j, x + j)) \).

Proof. Apply Corollary 10 with \( p = 1, q = 1 \), substitute \( f \) to \( f^{-1} \) and \( \phi(x, y) = \frac{1}{x+y(x+y+1)} \).

Restrict n-variables to a single variable with \( \varphi(x) = x + p \) in Theorem 3.

Corollary 13. Let a function \( f : (0, \infty) \to R_+ \) be a function that satisfies the inequality
\[
\left| \frac{f(x+1)}{\varphi(x) f(x)} - 1 \right| \leq \varepsilon(x),
\]
where \( \varepsilon : (0, \infty) \to (0, 1) \) is a function such that
\[
\sum_{j=0}^{\infty} \varepsilon(x + jp) < \infty \quad \forall x \in (0, \infty).
\]

Then there exists a unique solution \( g : (0, \infty) \to R_+ \) of the equation (3.5) with
\[
\alpha_{\gamma_p}(x) \leq \frac{g(x)}{f(x)} \leq \beta_{\gamma_p}(x)
\]
where \( \alpha_{\gamma_p}(x) := \prod_{j=0}^{\infty} (1 - \varepsilon(x + j)) \) and \( \beta_{\gamma_p}(x) := \prod_{j=0}^{\infty} (1 + \varepsilon(x + j)) \).

Corollary 14. Let \( \theta > 0 \) be given. If a mapping \( f : (0, \infty) \to R_+ \) satisfies the inequality
\[
\left| \frac{f(x+1)}{xf(x)} - 1 \right| \leq \frac{\delta}{x^{1+\theta}},
\]
then there exists a unique solution \( g : (0, \infty) \to (0, \infty) \) of the gamma functional equation (3.6) such that for any \( x > \max\{n_0, \delta^{-\frac{1}{\theta}}\} \)
\[
\alpha_\gamma(x) \leq \frac{g(x)}{f(x)} \leq \beta_\gamma(x),
\]
where \( \alpha_\gamma(x) := \prod_{j=0}^{\infty} (1 - \frac{\delta}{(x+j)^{1+\theta}}) \) and \( \beta_\gamma(x) := \prod_{j=0}^{\infty} (1 + \frac{\delta}{(x+j)^{1+\theta}}) \).
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