SPACES OF CONFORMAL VECTOR FIELDS
ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We study Riemannian or pseudo-Riemannian mani-
folds which carry the space of closed conformal vector fields of at
least 2-dimension. Subject to the condition that at each point the
set of closed conformal vector fields spans a non-degenerate sub-
space of the tangent space at the point, we prove a global and a
local classification theorems for such manifolds.

1. Introduction

Conformal mappings and conformal vector fields are important in
general relativity, as is well known since the early 1920's [6, 18]. In
1925, Brinkmann studied conformal mappings between Riemannian or
pseudo-Riemannian Einstein spaces [1]. Later conformal vector fields,
or infinitesimal conformal mappings on Einstein spaces were reduced
to the case of gradient vector fields, leading to a very fruitful theory
of conformal gradient vector fields in general. Brinkmann's work has
attracted renewed interest, especially in the context of general relativity
[2, 3, 4, 5, 9, 13, 16], and the following local theorems have been shown:

Proposition 1.1. [3] Let \((M^4, g)\) be a 4-dimensional Ricci-flat Lorentz
manifold. If \(M^4\) admits a nonhomothetic conformal vector field, then
\(M^4\) is a plane gravitational wave.

Proposition 1.2. [4] Let \((M^4, g)\) be a 4-dimensional Einstein but
not Ricci flat Lorentz manifold. If \(M^4\) admits a nonisometric conformal
vector field, then \(M^4\) has constant sectional curvature.

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mal vector field.
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D00026).
Proposition 1.3. (Kerckhove [9]) Let \((M^n, g)\) be an \(n\)-dimensional Einstein but not Ricci flat pseudo-Riemannian manifold with \(\text{Ric} = (n - 1)kg, k \neq 0\), which carries a conformal vector field. Here we denote by \(\text{Ric}\) the Ricci tensor of \((M^n, g)\). If each subspace \(\Delta(p)\) spanned by the set of conformal gradient vector fields at \(p \in M^n\) is a non-degenerate subspace of \(T_pM\) whose dimension \(m\) is independent of the choice of the point \(p\), then \((M^n, g)\) is locally isometric to a warped product \(B^m(k) \times_f F\). The base \(B\) is an \(m\)-dimensional space of constant sectional curvature \(k\); the fibre \((F, g_F)\) is an Einstein manifold with \(\text{Ric}_F = (n - m - 1)\alpha g_F\) for some constant \(\alpha\).

For an arbitrary pseudo-Riemannian manifold \((M^n, g)\) we denote by \(A(M^n, g)\) and \(\varphi\) the space of functions \(f\) on \(M^n\) whose hessian tensor \(H_f\) satisfies \(H_f = fg\) and the symmetric bilinear form on the space \(A(M^n, g)\) defined by \(\varphi(f, h) = \langle \nabla f, \nabla h \rangle - fh\), respectively. Then for an arbitrary complete connected pseudo-Riemannian manifold \((M^n, g)\) with \(A(M^n, g) \neq \{0\}\), in [7] Kerbrat shows the following global theorems:

Proposition 1.4. (Kerbrat [7]) If \(\dim A(M^n, g) = m > 0\), and the bilinear form \(\varphi\) is positive definite, then \((M^n, g)\) is isometric to a warped product \(H^m \times_f F\), where the base \(H^m\) is the hyperbolic space with constant curvature \(-1\), and the fibre \(F\) is a complete pseudo-Riemannian manifold satisfying \(A(F, g_F) = \{0\}\).

Proposition 1.5. (Kerbrat [7]) If the metric \(g\) is indefinite and \(\varphi(f, f) < 0\) for some \(f \in A(M^n, g)\), then \((M^n, g)\) is isometric to a space form or to a covering of a space form.

The case in which \(\varphi\) is positive semi-definite and degenerate was not treated by Kerbrat (See [9], p.825).

In this paper, we study pseudo-Riemannian manifolds which carry the space of closed conformal vector fields of at least 2-dimension. In Section 3 we improve the global theorems of Kerbrat (Proposition 1.4 and Proposition 1.5) and the local theorem of Kerckhove (Proposition 1.3) under the condition that each subspace \(\Delta(p)\) is nondegenerate, which is a necessary condition for \((M^n, g)\) to admit a warped product structure in the sense of Kerbrat or of Kerckhove (Theorem 3.1 and Theorem 3.2). Furthermore, we give a necessary and sufficient condition on the fiber space \(F\) for any closed conformal vector fields on the warped product space \(M^n = B^m(k) \times_f F\) to be lifted from the base space (Theorem 3.4).
2. Preliminaries and closed conformal vector fields on space forms

We consider an \( n \)-dimensional connected pseudo-Riemannian manifold \((M^n, g)\) carrying a closed conformal vector fields \(V\). Hence there is a smooth function \(\phi\) on \(M^n\) such that

\[
(2.1) \quad \nabla_X V = \phi X
\]

for all vector fields \(X\). Here \(\nabla\) denotes the Levi-Civita connection on \(M^n\). Then for every point \(p \in M^n\) one can find a neighborhood \(U\) and a function \(f\) such that \(V = \nabla f\), where \(\nabla f\) denotes the gradient of \(f\). It follows that the Hessian \(\nabla^2 f\) satisfies

\[
(2.2) \quad \nabla^2 f = \phi g.
\]

Therefore, \(\Delta f = \text{div} V = n\phi\).

From equation (2.1) we immediately obtain the following Ricci identity for the Riemannian curvature tensor:

\[
(2.3) \quad R(X, Y)V = X(\phi)Y - Y(\phi)X,
\]

and by contraction we get

\[
(2.4) \quad \text{Ric}(X, V) = (1 - n)X(\phi).
\]

We denote by \(CC(M^n, g)\) the vector space of closed conformal vector fields. First of all, we state some useful lemmas for later use.

**Lemma 2.1.** Let \(V\) be a non-trivial closed conformal vector field.

1. If \(\gamma : [0, \ell] \to M^n\) is a geodesic with \(V(\gamma(0)) = a\gamma'(0)\) for some \(a \in \mathbb{R}\), then we have

\[
(2.5) \quad V(\gamma(t)) = (a + \int_0^t \phi(\gamma(s))ds)\gamma'(t).
\]

2. If \(V(p) = 0\), then \(\text{div} V(p) = n\phi(p) \neq 0\), in particular, all zeros of \(V\) are isolated.

**Proof.** See Propositions 2.1 and 2.3 in [15]. \(\Box\)
**Lemma 2.2.** Let $(M^n, g)$ be an $n$-dimensional connected pseudo-Riemannian manifold. Then the following hold:

1. $\dim \text{CC}(M^n, g) \leq n + 1$.
2. If $\dim \text{CC}(M^n, g) \geq 2$, there exists a constant $k \in \mathbb{R}$ such that for all $V \in \text{CC}(M^n, g)$
   \[ \nabla \phi = -kV, \]
   where $n\phi$ is the divergence of $V$.

**Proof.** See Proposition 2.3 in [15] and Proposition 4 in [7]. □

In [15], W. Kühnel and H. B. Rademacher observed that if the dimension of the space of closed conformal vector fields is maximal, that is, $\dim \text{CC}(M^n, g) = n + 1$, then the manifold has constant sectional curvature [15, Remark 2.4].

Now we prove the following:

**Proposition 2.3.** Let $(M^n, g)$ be an $n$-dimensional pseudo-Riemannian manifold. If $\dim \text{CC}(M^n, g) \geq \max\{2, n-1\}$, then $M^n$ has constant sectional curvature.

**Proof.** Since $\dim \text{CC}(M^n, g) \geq 2$, Lemma 2.2 together with (2.3) shows that there exists a constant $k \in \mathbb{R}$ which satisfies

\[ R(X, Y)V = k\{< V, Y > X - < V, X > Y\}, \quad X, Y \in TM \]

for all $V \in \text{CC}(M^n, g)$. Choose $V_1, \cdots, V_{n-1}$ in $\text{CC}(M^n, g)$ in a way that they are linearly independent. Let $U$ be the set of all points $p$ at which $V_1(p), \cdots, V_{n-1}(p)$ are linearly independent. Then (2.1) and Lemma 2.1 show that $U$ is an open dense subset of $M^n$.

For each fixed $p \in U$, choose $V_n(p)$ so that $\{V_1(p), \cdots, V_{n-1}(p), V_n(p)\}$ forms a basis for $T_pM$. It suffices to show that (2.6) holds for $V_n$ instead of $V$ on $U$. Then the open dense set $U$ has constant sectional curvature $k$. By continuity, $M^n$ has constant sectional curvature $k$.

For $V = V_i, \quad 1 \leq i \leq n - 1$, we have from (2.6)

\[ < R(X, Y)V_i, V > = -< R(X, Y)V, V_i > = k\{< V_i, Y > X - < V_i, X > Y\}, V >. \]

For $V = V_n$, (2.7) is trivial, and hence (2.6) holds for $V = V_n$ due to nondegeneracy of the metric. This completes the proof. □
In [10], using the work of Kühnel, W. ([12]), the authors characterize the Riemannian space forms in terms of the dimension of the space of conformal gradient vector fields.

The model spaces $B^n_k$ of constant sectional curvature $k = \epsilon a^2$ with $\epsilon = \pm 1$, $a > 0$ and index $\nu$ are the hyperquadrics in pseudo-Euclidean space:

$$S^n_\nu(a^2) = \{x \in R^{n+1}_\nu| < x, x > = 1/a^2\},$$

$$H^n_\nu(-a^2) = \{x \in R^{n+1}_\nu+1| < x, x > = -1/a^2\}.$$

For a fixed vector $T$ in $R^{n+1}_\nu$ or $R^{n+1}_\nu+1$, let $\sigma_T$ be the height function in the direction of $T$ defined by $\sigma_T(x) = <T, x>$. Then one can easily show that on $B^n_k$,

$$\nabla \sigma_T(x) = T - k\sigma_T(x)x,$$

$$(2.8)$$

$$\nabla X \nabla \sigma_T = -k\sigma_T X$$

for all vector fields $X$ of $B^n_k$ ([9]). (2.9) implies that for any constant vector $T$ in $R^{n+1}_\nu$ or $R^{n+1}_\nu+1$, $\nabla \sigma_T$ is a closed conformal vector field on the hyperquadric $B^n_k(k = \epsilon a^2)$. Furthermore, by counting dimensions (Lemma 2.2) we see that $\nabla \sigma_T$ represents every element of $CC(B^n_k)$.

For the flat space form $R^n_\nu$ with index $\nu$, the vector field $V$ defined by $V(x) = bx + c, b \in R, c \in R^n_\nu$ is a closed conformal vector field. Obviously, by counting dimensions, we have

$$CC(R^n_\nu, g) = \{bx + c|b \in R, c \in R^n_\nu\}.$$

For the space of conformal vector fields of pseudo-Riemannian space forms, the authors et al. gave a complete description about it ([11]).

Now we introduce a function space $A_k(M^n, g)(k \neq 0)$ and a symmetric bilinear form $\Phi_k$ on the space as follows:

$$A_k(M^n, g) = \{f \in C^\infty(M)| \nabla X \nabla f = -kfX, \ X \in TM\},$$

$$(2.10)$$

$$\Phi_k(f, h) = <\nabla f, \nabla h > +kh, \ f, h \in A_k(M^n, g).$$

In [7], Kerbrat defined a function space $A(M^n, g)$ by

$$A(M^n, g) = \{f \in C^\infty(M)| \nabla X \nabla f = fX, \ X \in TM\}.$$
and a symmetric bilinear map $\varphi$ on the space by

$$\varphi(f, h) = \langle \nabla f, \nabla h \rangle - fh, \quad f, h \in A(M^n, g).$$

By the scale change $g \rightarrow -kg$, we see that

$$A_k(M^n, g) = A(M^n, -kg), \quad \Phi_k(f, f) = -k\varphi(f, f).$$

For the non-flat space form $B^n(k), k = \epsilon a^2$, (2.8) shows that

$$\Phi_k(\sigma_T, \sigma_S) = \langle \nabla \sigma_T, \nabla \sigma_S \rangle + k\sigma_T \sigma_S = \langle T, S \rangle.$$

This implies that the symmetric bilinear form $\Phi_k$ is just the usual scalar product on the ambient pseudo-Euclidean space.

3. Closed conformal vector fields

In this section we consider the vector space $CC(M^n, g)$ of closed conformal vector fields on a pseudo-Riemannian manifold $(M^n, g)$ with indefinite metric $g$. For $p \in M^n$, let $\Delta(p)$ be the span of the set of closed conformal vector fields at $p$, that is,

$$\Delta(p) = \{ V(p) \in T_p(M) | V \in CC(M^n, g) \}.$$

Suppose that $CC(M^n, g)$ is of dimension $m \geq 2$. Then (2.1) and Lemma 2.2 imply that there exists a constant $k \in \mathbb{R}$ such that for all $V$ in $CC(M^n, g)$ with $\phi = (1/n)\text{div}V$

$$\nabla \phi = -kV,$$

so that we have

$$\nabla_X \nabla \phi = -k\phi X, \quad X \in TM.$$

Hence, if $k$ is nonzero, then the space $CC(M^n, g)$ may be identified with the space $A_k(M^n, g)$ and we have $\Delta(p) = \{ \nabla f(p) \neq f \in A_k(M^n, g) \}$.

First of all, we establish a global classification theorem.
Theorem 3.1. Let \((M^n, g)\) be an \(n\)-dimensional connected and complete pseudo-Riemannian manifold with indefinite metric \(g\). Suppose that there exists \(k = \epsilon a^2\) with \(\epsilon = \pm 1, a > 0\) such that \((M^n, g)\) satisfies

(a) \(\dim A_k(M^n, g) = m \geq 1\),

(b) each subspace \(\Delta(p)\) is nondegenerate.

Then one of the following holds:

1. \((M^n, g)\) is isometric to \(S^n_\epsilon(a^2), H^n_\epsilon(-a^2)\) or a covering space of \(S^{n-1}_\epsilon(a^2)\) or \(H^{n+1}_\epsilon(-a^2)\).
2. \((M^n, g)\) is isometric to a warped product space \(S^m_\epsilon(a^2) \times \sigma T F^{n-m}\) (\(\epsilon = 1\)) or \(H^m(-a^2) \times \sigma T F^{n-m}(\epsilon = -1)\), where the fiber \((F, g_F)\) is an \((n-m)\)-dimensional connected and complete pseudo-Riemannian manifold with \(A_k(F, g_F) = \{0\}\), and \(T\) is a vector in \(R^{m+1}_\epsilon(\epsilon = 1)\) or \(R^{m+1}_1(\epsilon = -1)\) with \(<T, T> = k\).

Proof. First, we show that if \((M^n, g)\) is not isometric to a space form in case (1), then the bilinear form \(\Phi_k\) is definite on \(A_k(M^n, g)\). Suppose that \(\Phi_k(f, f)\) is trivial for some nontrivial function \(f \in A_k(M^n, g)\). If there exists \(h \in A_k(M^n, g)\) such that \(\Phi_k(f, h) \neq 0\), then it is obvious that

\[\Phi_k(tf + h, tf + h) = 2t\Phi_k(f, h) + \Phi_k(h, h)\]

for all \(t \in R\). This implies that there exists a function \(l \in A_k(M^n, g)\) such that \(e\Phi_k(l, l) > 0\). Hence, by (2.12) \((M^n, -kg)\) carries a function \(l \in A(M^n, -kg)\) which satisfies \(\varphi(l, l) < 0\). By Proposition 1.5, we see that \((M^n, -kg)\) is isometric to a space form listed in (1) with curvature \(-1\), so that \((M^n, g)\) is a space form in (1), which is a contradiction. This shows that for the function \(f \in A_k(M^n, g)\) we have \(\Phi_k(f, h) = 0\) for all \(h \in A_k(M^n, g)\). For any point \(m \notin f^{-1}(0)\), (2.11) with the condition \(\Phi_k(f, f) = 0\) shows that \(\nabla f(m)\) is not zero. Since the metric \(g\) is indefinite, we can always choose a null vector \(v\) in \(T_m M\) such that \(<v, \nabla f(m)> \neq 0\). Let \(\gamma\) be the null geodesic with initial velocity vector \(v\). Then (2.10) implies that

\[(f \circ \gamma)'(t) = -k(f \circ \gamma)(t) < \gamma'(t), \gamma'(t) > = 0,\]

so that we have

\[f(\gamma(t)) = f(m) + <v, \nabla f(m) > > t.\]

Hence we see that \(f^{-1}(0)\) is not empty. Fix a point \(p \in f^{-1}(0)\). Then \(\nabla f(p)\) is a nonzero vector (Lemma 2.1) with the property that for all \(h \in A_k(M^n, g)\)

\[<\nabla f(p), \nabla h(p) > = <\nabla f(p), \nabla h(p) > + kf(p)h(p) = \Phi_k(f, h) = 0,\]
which means that $\Delta(p)$ is degenerate. This contradiction shows the 
definiteness of $\Phi_k$.

If $\epsilon \Phi_k(f, f) > 0$ for some function $f \in A_k(M^n, g)$, then as above, 
Proposition 1.5 with (2.12) shows that $(M^n, g)$ is isometric to a space 
form in (1). Hence we may assume that $\epsilon \Phi_k$ is negative definite on 
$A_k(M^n, g)$, that is, $\varphi$ is positive definite on $A(M^n, -kg)$. Then Proposition 1.4 shows that $(M^n, -kg)$ is isometric to a warped product space $H^m(-1) \times F$ with metric $g_{H^m(-1)} + f^2\tilde{g}_F$, where $f$ is given by $\sqrt{\rho}$ ([7]) and $(F, \tilde{g}_F)$ is a connected complete pseudo-Riemannian manifold with 
$A(F, \tilde{g}_F) = \{0\}$. Hence $(M^n, g)$ is isometric to $B^m(k) \times F (F, g_F)$, where 
the base $B^m(k)$ is a space form $H^m(-a^2)(\epsilon = -1)$ or $S^m_m(a^2)(\epsilon = 1)$ and 
the metric $g_F$ is given by $\frac{1}{k}\tilde{g}_F$. It is straightforward to show that $f = \sqrt{\rho}$ belongs to $A(H^m(-1))$ with $\varphi(f, f) = -1$. Therefore $f \in A_k(B^m(k))$ 
with $\Phi_k(f, f) = k$ due to (2.12). Hence $f$ is a height function $\sigma_T$ for some 
vector $T$ in $R^{m+1}_1$ or $R^{m+1}_m$ with $<T, T> = k$. Since $A(F, \tilde{g}_F) = \{0\}$, 
we also have $A_k(F, g_F) = \{0\}$. \hfill \Box

In the case (2) of Theorem 3.1, if the base space form $B^m(k)$ is neither 
$S^m_m(a^2)$ nor $H^m(-a^2)$, then for any constant vector $T$ in the ambient 
pseudo-Euclidean space the function $\sigma_T$ vanishes on a hypersurface in 
$B^m(k)$ preventing the warped product construction from extending over all 
of $B^m(k)$. By contrast, if $B^m(k)$ is either $S^m_m(a^2)$ or $H^m(-a^2)$ and 
$T$ satisfies $<T, T> = k$, then the function $\sigma_T$ is nowhere zero since $T$ 
is nowhere tangent to $B^m(k)$.

Now we prove a local classification theorem(cf. [8, 9]), which is a 
generalization of Kerckhove’s results(Proposition 1.3).

**Theorem 3.2.** Let $(M^n, g)$ be an $n$-dimensional connected pseudo-
Riemannian manifold. Suppose that there exists a nonzero constant 
k $\in R$ such that

(a) $\dim A_k(M^n, g) = m \geq 1$,

(b) each subspace $\Delta(p)$ is nondegenerate.

Then, for a fixed $p_0 \in M^n$ the following hold:

(1) If $\dim \Delta(p_0) < m$, then $(M^n, g)$ is locally isometric to a space 
$B^n(k)$ of constant sectional curvature $k$.

(2) If $\dim \Delta(p_0) = m$, then $(M^n, g)$ is locally isometric to a warped 
product space $B^m(k) \times F$, where the base $B^m(k)$ is a space of constant 
sectional curvature $k$ and the fiber $(F^{n-m}, g_F)$ is a pseudo-Riemannian 
manifold. Furthermore, $F$ satisfies the following :

(i) In case $<T, T> \neq 0$, we have $A_\alpha(F, g_F) = \{0\}$, where $\alpha = <T, T>$.
(ii) In case $<T,T>=0$, $F$ carries no nontrivial homothetic gradient vector fields.

In either case, we have $A_k(M^n,g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), <S,T>=0\}$, where $\tilde{\sigma}_S$ denotes the lifting of $\sigma_S$.

**Proof.** (1) If $\dim \Delta(p_0) < m$, there exists a nontrivial function $f \in A_k(M^n,g)$ which satisfies $\nabla f(p_0) = 0$. Lemma 2.1 shows that $f(p_0) \neq 0$. Since $\Phi_k(f,f) = kf(p_0)^2$, by the scale change $g \to -kg$ we have $\varphi(f,f) = -f(p_0)^2 < 0$. Hence (1) follows from Proposition 1.5 with the scale change.

(2) If $\dim \Delta(p_0) = m$, then $\dim \Delta(p) = m$ in a neighborhood of $p_0$ in $M^n$. Hence, as in the proof of Proposition 1.3 [9], it can be shown that there exists a neighborhood $U$ of $p_0$ which is isometric to a warped product space $B^m(k) \times_f F$ for some positive function $f$ on $B^m(k)$. The base space is the integral submanifold of $\Delta$ through $p_0$ and has constant sectional curvature $k$. Note that the fibre $p \times F$ is totally umbilic. Hence the second fundamental form $h$ of the fibre satisfies $h(V, W) = <V, W> H$. For any $\sigma \in A_k(M^n,g)$, we have the following:

$$<V, W><H, \nabla \sigma> = <h(V, W), \nabla \sigma>$$
$$= V <W, \nabla \sigma> - <W, \nabla_V \nabla \sigma>$$
$$= -k \sigma <V, W> .$$

Thus we obtain $<H, \nabla \sigma> = -k \sigma$. Since the mean curvature vector field $H$ is given by $-\nabla f/f$, we have

$$<\nabla f, \nabla \sigma> + kf \sigma = 0$$

for all $\sigma \in A_k(M^n,g)$. By taking the covariant derivative of (3.3) with respect to any vector field $X$ on $B^m(k)$, we find

$$<\nabla_X \nabla f, \nabla \sigma> + <kf X, \nabla \sigma> = 0.$$  

Since $\nabla \sigma(\sigma \in A_k(M^n,g))$ spans the tangent spaces of $B^m(k)$, (3.4) shows that $f$ belongs to $A_k(B^m(k))$. Hence $f$ is a height function $\sigma_T$ for some vector $T$ in $R^{m+1}$. It is easy to show that if $S$ is a vector in $R^{m+1}$ with $<S,T>=0$, then the lifting $\tilde{\sigma}_S$ of the height function $\sigma_S$ belongs to $A_k(M^n,g)$. By counting dimensions, we see that $A_k(M^n,g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), <S,T>=0\}$.

Suppose that the constant vector $T$ satisfies $<T,T>= \alpha \neq 0$ and $h$ belongs to $A_\alpha(F, g_F)$. Then we have

$$\nabla(\sigma_T h) = h\nabla \sigma_T + \frac{1}{\sigma_T} \nabla^* h,$$
where $\nabla^* h$ denotes the gradient vector of $h$ on $F$. Hence the condition $\Phi_k(\sigma_T, \sigma_T) = \langle T, T \rangle = \alpha$ on $B^m(k)$ shows that the function $\sigma_T h$ lies in the function space $A_k(M^n, g)$. Since the leaves $B^m(k) \times q$, $q \in F$ are the integral submanifolds of the distribution $\Delta$, we see that $\nabla_{\sigma_T h}$ must be tangent to the leaves. This shows that $A_{\alpha}(F, g_{F}) = \{0\}$.

Finally, we suppose that $T$ is a null vector in $R^{m+1}$ and $\nabla^* h$ is a homothetic gradient vector field on $F$ with $\nabla^* \nabla^* h = cV$, $c \in R$, $V \in TF$, where we denote by $\nabla^*$ the Levi-Civita connection on $F$. Then it is not difficult to show that for a null vector $\bar{T}$ in $R^{m+1}$ with $\langle T, \bar{T} \rangle = -1$, the function $l$ defined by $l = h\nabla_{\sigma_T} + \frac{1}{\sigma_T} \nabla^* h + c\nabla_{\sigma_T} \bar{T}$ must be tangent to the leaves. This completes the proof of our theorem. \(\square\)

Note that if $(F, g_{F})$ has constant sectional curvature $\alpha = \langle T, T \rangle$, then so does $M^n = B^m(k) \times_{\sigma} F$. Hence we see that not all closed conformal vector fields on the warped product need to be lifted from the base. Thus it is worthwhile to find a condition on the fibre $F$ which guarantee that any closed conformal vector fields on the warped product space $M^n = B^m(k) \times_{\sigma} F$ to be lifted from the base space.

To find the condition, we state a useful lemma which can be easily shown. Recall that $\Sigma_V g$ denotes the Lie derivative of $g$ with respect to $V$.

**Lemma 3.3.** [20] Let $(M^n, g)$ be a totally umbilic submanifold of a pseudo-Riemannian space $(\bar{M}, \bar{g})$. If $V$ is a conformal vector field on $\bar{M}$ with $\Sigma_V g = 2\sigma g$, then the tangential part $V^T$ of $V$ on $M^n$ is a conformal vector field on $M^n$ with

$$\Sigma_{V^T} g = 2\{\sigma + \bar{g}(V, H)\} g,$$

where $H$ denotes the mean curvature vector field of $M^n$ in $\bar{M}$.

In [11], the authors et al. proved a converse of Lemma 3.3 for hypersurfaces of a pseudo-Riemannian space form.

Now we prove that the necessary condition on $F$ in Theorem 3.2 is sufficient for any closed conformal vector fields on the warped product space $M^n = B^m(k) \times_{\sigma} F$ to be lifted from the base space as follows.

**Theorem 3.4.** Let $(M^n, g)$ be a warped product space $B^m(k) \times_{\sigma} F$, where $T$ is a vector in the ambient pseudo-Euclidean space $R^{m+1}$ of $B^m(k)$. Suppose that the fibre $(F, g_{F})$ satisfies the following:
(1) In case \( < T, T > \neq 0 \), we have \( A_{\alpha}(F, g_F) = \{0\} \), where \( \alpha = < T, T > \).

(2) In case \( < T, T > = 0 \), \( F \) carries no nontrivial homothetic gradient vector fields.

Then \((M^n, g)\) satisfies the following:

\[
A_{k}(M^n, g) = \{ \tilde{\sigma}_S | \sigma_S \in A_{k}(B^m(k)) \}, < S, T > = 0 \}
\]

In particular, each subspace \( \Delta(p) \) is nondegenerate and of dimension \( m \).

**Proof.** First note that for a vector \( S \) in \( R^{m+1} \) the lifting \( \tilde{\sigma}_S \) of a height function \( \sigma_S \) belongs to \( A_{k}(M^n, g) \) if and only if \( < S, T > = 0 \) [9, p.824].

For such a nontrivial vector \( S \), (2.8) and \( \Phi_{k}(\sigma_S, \sigma_T) = < S, T > = 0 \) imply that \( \nabla \sigma_S(p) \neq 0 \) for any \( p \in B^m(k) \). This means that for each \( p \in B^m(k) \), \( \{ \nabla \sigma_S(p) | < S, T > = 0 \} \) spans \( T_pB^m(k) \).

For any \( f \in A_{k}(M^n, g) \) let \( f_p \) denote the restriction of \( f \) to the fibre \( p \times F \) and \( \nabla f_p \) the gradient of \( f_p \) on \( p \times F \). Then \( \nabla f_p \) is the vertical part of \( \nabla f \). Since each fibre \( p \times F \) is a totally umbilic submanifold of \( M^n \) with mean curvature vector field \( -\frac{1}{\sigma_T(p)} \nabla \sigma_T(p) \) and \( L_{\nabla f}g = -2kfg \) on \( M^n \), we obtain from Lemma 3.3

\[
\nabla \sigma_T(p) = \frac{-2}{\sigma_T(p)} \{ < \nabla f, \nabla \sigma_T(p) > + kf_p \sigma_T(p) \} \}
\]

Since \( \nabla f_p \) is closed, (3.5) implies for all \( V \in TF \)

\[
\nabla_{\nabla f_p}g|_{p \times F} = \frac{-2}{\sigma_T(p)} \{ < \nabla f, \nabla \sigma_T(p) > + kf_p \sigma_T(p) \} \}
\]

where \( \nabla^* \) denotes the Levi-Civita connection on \( F \).

Suppose that \( < T, T > = 0 \). Then \( \sigma_T \) belongs to \( A_{k}(M^n, g) \). Hence (3.6) shows that

\[
\nabla^*_{\nabla f_p} = \frac{-1}{\sigma_T(p)} \Phi_{k}(\sigma_T, f)V, \quad V \in TF.
\]

Since \( g|_{p \times F} = \sigma_T(p)^2g_F \), it follows from (3.7) that on \( F \)

\[
\nabla^*_{\nabla f_p} = \sigma_T(p)\Phi_{k}(\sigma_T, f)V, \quad V \in TF.
\]

Hence the hypothesis shows that for each \( p \in B^m(k) \), \( f_p \) is constant. This implies that \( f \) is a function on the base \( B^m(k) \). Thus \( f \) is a height function \( \sigma_S \) for some vector \( S \) in \( R^{m+1} \) with \( < S, T > = 0 \).
Now, we suppose that $\langle T, T \rangle \neq 0$. Then the subspace $W = \{ \tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), \langle S, T \rangle = 0 \}$ of $A_k(M^n, g)$ is nondegenerate with respect to $\Phi_k$ because it is nothing but the orthogonal complement of $T$ in the ambient pseudo-Euclidean space. Hence it suffices to show that the orthogonal complement of $W$ with respect to $\Phi_k$ is trivial. For $f \in A_k(M^n, g)$ and a fixed point $p \in B^m(k)$, by differentiating both sides of (3.6) with respect to an arbitrary vector on $F$, we see that on $p \times F$

\begin{equation}
\langle \nabla f, \nabla \sigma_T \rangle + k\sigma_T(p)f_p = \frac{\langle T, T \rangle}{\sigma_T(p)} f_p + c
\end{equation}

for some constant $c$. Suppose that $f$ lies in the orthogonal complement of $W$. Since $\nabla \sigma_S(p)$ with $\langle S, T \rangle = 0$ generates the tangent space of $B^m(k)$ at $p$, we can choose $S_1, \ldots, S_m$ in the orthogonal complement of $T$ in $R^{m+1}$ such that $\langle \nabla \sigma_{S_i}(p), \nabla \sigma_{S_j}(p) \rangle = \epsilon_i \delta_{ij}$. Using $\Phi_k(f, \sigma_{S_i}) = \Phi_k(\sigma_T, \sigma_{S_i}) = 0$, on $p \times F$ we obtain

\begin{align*}
\langle \nabla f, \nabla \sigma_T \rangle &= k^2\sigma_T(p)f_p \sum \epsilon_i \sigma_{S_i}(p)^2, \\
\langle T, T \rangle &= k^2\sigma_T(p)^2 \sum \epsilon_i \sigma_{S_i}(p)^2.
\end{align*}

Hence (3.8) shows that the constant $c$ vanishes. Since $g|_{p \times F} = \sigma_T(p)^2g_F$, it follows from (3.6) and (3.8) that $f_p$ belongs to $A_{\alpha}(F)$, where $\alpha = \langle T, T \rangle$. Thus the hypothesis on $A_{\alpha}(F)$ shows that every element $f$ in the orthogonal complement of $W$ is trivial. Thus our theorem is proved. □

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References

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