A HIGHER ORDER MONOTONE ITERATIVE SCHEME FOR NONLINEAR NEUMANN BOUNDARY VALUE PROBLEMS

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Abstract. The generalized quasilinearization technique has been employed to obtain a sequence of approximate solutions converging monotonically and rapidly to a solution of the nonlinear Neumann boundary value problem.

1. Introduction

The method of generalized quasilinearization introduced by Lakshmikantham ([4, 5]) has been successfully employed to obtain a sequence of approximate solutions converging monotonically to a solution of the nonlinear problem, see, for example, [1-3, 6-10]. In this paper, we continue the study of nonlinear Neumann problems addressed in [1] and improve the convergence of a sequence of approximate solutions converging monotonically to a solution of the nonlinear Neumann boundary value problem. In fact, we establish the convergence of order $k(k \geq 2)$.

2. Some basic results

We know that the linear Neumann boundary value problem
\[-u''(t) = \lambda u(t), \quad t \in J = [0, \pi] \]
\[u'(0) = 0, \quad u'(\pi) = 0,\]
has a nontrivial solution if and only if $\lambda = m^2$, $m = 1, 2, ...$ and thus, for $\lambda \neq m^2$ and $\xi(t) \in C[0, \pi]$, the corresponding nonhomogeneous problem
\[-u''(t) - \lambda u(t) = \xi(t), \quad t \in J\]

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has a unique solution
\[ u(t) = \int_0^\pi G_\lambda(t, s) \xi(s) ds, \]
where \( G_\lambda \) is the Green’s function of the associated homogeneous problem and is given by
\[
G_\lambda = \begin{cases} 
\frac{1}{\sqrt{\lambda} \sin \sqrt{\lambda} \pi} \left\{ \cos[\sqrt{\lambda}(\pi - s)] \cos[\sqrt{\lambda}t], & \text{if } 0 \leq t \leq s \leq \pi \\
\cos[\sqrt{\lambda}s] \cos[\sqrt{\lambda}(\pi - t)], & \text{if } 0 \leq s \leq t \leq \pi 
\end{cases}
\]
for \( \lambda > 0 \),
\[
G_\lambda = \frac{1}{\sqrt{-\lambda} \sinh \sqrt{-\lambda} \pi} \begin{cases} 
\cosh[\sqrt{-\lambda}(\pi - s)] \cosh[\sqrt{-\lambda}t], & \text{if } 0 \leq t \leq s \leq \pi \\
\cosh[\sqrt{-\lambda}s] \cosh[\sqrt{-\lambda}(\pi - t)], & \text{if } 0 \leq s \leq t \leq \pi 
\end{cases}
\]
for \( \lambda < 0 \). We observe that \( G_\lambda \geq 0 \) for \( \lambda < 0 \). Now, we consider the nonlinear Neumann problem
\[
(1) \quad -u''(t) = f(t, u(t)), \quad t \in J \\
u'(0) = 0, \quad u'(\pi) = 0,
\]
where \( f : J \times R \to R \) is continuous. The problem (1) is equivalent to the integral equation
\[
(2) \quad u(t) = u(0) - \int_0^t (t - s)f(s, u(s)) ds
\]
with
\[
(3) \quad \int_0^t f(s, u(s)) ds = 0.
\]
We shall say that \( \alpha(t) \in C^2[J] \) is a lower solution of (1) if
\[
-\alpha''(t) \leq f(t, \alpha(t)), \quad t \in J \\
\alpha'(0) \geq 0, \quad \alpha'(-\pi) \leq 0,
\]
and \( \beta \in C^2[J] \) is an analogue upper solution of (1) if
\[
-\beta''(t) \geq f(t, \beta(t)), \quad t \in J \\
\beta'(0) \leq 0, \quad \beta'(\pi) \geq 0.
\]
The following theorem plays a crucial role in the forthcoming analysis.
and for its proof, see reference [11].

**Theorem 1.** Let $\alpha, \beta \in C^2[J, R]$ be lower and upper solutions of (1) respectively such that $\alpha(t) \leq \beta(t)$ on $J$. Then there exists a solution $u(t)$ of (1) such that $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in J$.

### 3. Higher order monotone iterative scheme

**Theorem 2.** Assume that

$\text{(B}_1\text{)}$ $\alpha, \beta \in C^2[J, R]$ such that $\alpha(t) \leq \beta(t)$ on $J$ are lower and upper solutions of (1) respectively.

$\text{(B}_2\text{)}$ $\frac{\partial f}{\partial u}(t, u), \ i = 1, 2, 3, ..., k$ exist and are continuous on $\Omega = \{(t, u) \in J \times R \}$ such that $\frac{\partial f}{\partial u}(t, u) < 0$, $\frac{\partial^{k+} f(t, u) + \phi(t, u)}{\partial u^k(t, u) \geq 0}$ for some function $\phi \in C^{0,k}[J \times R, R]$ such that $\frac{\partial^{k} \phi}{\partial u^k}(t, \xi) \leq 0$.

Then there exists a monotone nondecreasing sequence $\{\mu_n\}$ of solutions which converges uniformly to a solution of (1) with the order of convergence $k$ ($k \geq 2$).

**Proof.** Set

$$\phi(t, u) = F(t, u) - f(t, u), \quad t \in J.$$  

Using (B2) and generalized mean value theorem, we have

$$f(t, u) \geq \sum_{i=0}^{k-1} \frac{\partial^{i} F}{\partial u^{i}}(t, v) \frac{(u - v)^i}{(i)!} - \phi(t, u),$$

where $\alpha(t) \leq v(t) \leq u(t) \leq \beta(t)$. Now, we define

$$K(t, u, v) = \sum_{i=0}^{k-1} \frac{\partial^{i} F}{\partial u^{i}}(t, v) \frac{(u - v)^i}{(i)!} - \phi(t, u),$$

$$= \sum_{i=0}^{k-1} \frac{\partial^{i} f}{\partial u^{i}}(t, v) \frac{(u - v)^i}{(i)!} - \frac{\partial^{k} \phi}{\partial u^k}(t, \xi) \frac{(u - v)^k}{(k)!},$$

where $\alpha \leq v \leq u \leq \beta$ on $J$.

Observe that

$$K(t, u, v) \leq f(t, u), \quad K(t, u, u) = f(t, u).$$

Now, set $\mu_o = \alpha$ and consider the problem

$$-u''(t) = K(t, u(t), \mu_o(t)), \quad t \in J$$

$$u'(0) = 0, \quad u'(\pi) = 0.$$
Using \((B_1)\) and \((4)\), we get
\[
-\mu''(t) \leq f(t, \mu_o(t)) = K(t, \mu_u(t), \mu_o(t)), \quad t \in J \\
\mu'_o(0) \geq 0, \quad \mu'_o(\pi) \leq 0,
\]
and
\[
-\beta''(t) \geq f(t, \beta(t)) \geq K(t, \beta, \mu_o), \quad t \in J \\
\beta'(0) \leq 0, \quad \beta'(\pi) \geq 0,
\]
which imply that \(\mu_o\) and \(\beta\) are lower and upper solution of \((5)\) respectively. Hence, by Theorem 1, there exists a solution \(\mu_1\) of \((5)\) such that \(\mu_o \leq \mu_1 \leq \beta\) on \(J\). Next, we consider the problem
\[
-\mu''(t) = K(t, \mu(t), \mu_1(t)), \quad t \in J \\
\mu'(0) = 0, \quad \mu'(\pi) = 0.
\]
Employing the earlier arguments, it can be shown that there exists a solution \(\mu_2\) of \((6)\) such that \(\mu_1 \leq \mu_2 \leq \beta\) on \(J\), where \(\mu_1\) and \(\beta\) are lower and upper solution of \((6)\) respectively. Continuing this process successively, we obtain a monotone sequence \(\{\mu_n\}\) of solutions satisfying
\[
\mu_o \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_{n-1} \leq \mu_n \leq \beta,
\]
on \(J\), where the element \(\mu_n\) of the sequence is the solution of the problem
\[
-\mu''(t) = K(t, \mu(t), \mu_{n-1}(t)), \quad t \in J \\
\mu'(0) = 0, \quad \mu'(\pi) = 0.
\]
Since the sequence \(\{\mu_n\}\) is monotone, it follows that it has a pointwise limit \(\mu\). To show that \(\mu\) is in fact a solution of \((1)\), we observe that \(\mu_n\) is the solution of the following Neumann problem
\[
-\mu''(t) = f_n(t), \quad t \in J \\
\mu'(0) = 0, \quad \mu'(\pi) = 0,
\]
where \(f_n(t) = K(t, \mu_n(t), \mu_{n-1}(t))\). Since \(f_n(t)\) is continuous on \(\Omega\) and \(\alpha \leq \mu_n \leq \beta\) on \(\Omega\) for \(n = 1, 2, \ldots\), it follows that the sequence \(\{f_n(t)\}\) is bounded in \(C[J, R]\). This together with the the monotonicity of \(\{\mu_n\}\) implies that the sequence \(\{\mu_n\}\) uniformly converges to \(\mu\). Letting \(n \to \infty\), and using the uniform convergence of \(\{\mu_n\}\), we find that \(\mu\) satisfies the integral equation \((2)\) and \((3)\) and hence \(\mu\) is a solution of \((1)\).

To show that the convergence of the sequence is of order \(k\) \((k \geq 2)\), we set \(e_n = \mu - \mu_n, \quad a_n = \mu_{n+1} - \mu_n, \quad n = 1, 2, 3, \ldots\). Clearly, \(a_n \geq 0,\)
\( e_n \geq 0, \ e_n - a_n = e_{n+1}, \ a_n \leq e_n \) and \( a_n^k \leq e_n^k \). Using the mean value theorem repeatedly, we have
\[
-e''_n(t) = \mu''_n(t) - \mu''(t)
\]
\[
= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i(t, \mu_{n-1})} \left( \frac{e_{n-1}^i - a_{n-1}^i}{i!} \right)
+ \frac{\partial^k f}{\partial u^k(t, \zeta(t))} \frac{e_{n-1}^k}{k!} + \frac{\partial^k \phi}{\partial u^k(t, \zeta(t))} \frac{a_{n-1}^k}{k!}
\]
\[
\leq \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i(t, \mu_{n-1})} \frac{1}{(i)!} \sum_{j=0}^{i-1} e_{n-1}^j a_{n-1}^j e_n
+ \left( \frac{\partial^k f}{\partial u^k(t, \zeta(t))} + \frac{\partial^k \phi}{\partial u^k(t, \zeta(t))} \right) \frac{e_{n-1}^k}{k!}
\]
\[
\leq q_n(t) e_n + N e_n^k,
\]
where
\[
q_n(t) = \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i(t, \mu_{n-1})} \frac{1}{(i)!} \sum_{j=0}^{i-1} e_{n-1}^j a_{n-1}^j,
\]
and \( N > 0 \) provides bound for \( \frac{\partial^k F}{\partial u^k(t, \zeta(t))} \) on \( \Omega \). As \( \lim_{n \to \infty} q_n(t) = f_u(t, \mu) < 0 \), we can choose \( \lambda < 0 \) and \( n_o \in N \) such that for \( n \geq n_o \), \( q_n(t) < \lambda \), we have
\[
-e''_n(t) - \lambda(t) e_n(t) \leq (q_n(t) - \lambda)e_n(t) + N e_n^k \leq N e_n^k,
\]
whose solution is
\[
e_n(t) = \int_0^\pi G_\lambda(t, s) N e_{n-1}^k ds, \quad t \in J.
\]
Taking maximum over \([0, \pi]\), we obtain
\[
||e_n|| \leq C ||e_{n-1}||^k,
\]
where \( C \) provides a bound on \( N \int_0^\pi G_\lambda(t, s) ds \).
This completes the proof. \( \square \)

References

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