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MORE ON CUTTING A POLYGON INTO TRIANGLES OF EQUAL AREAS

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Abstract. In 2000 a general conjecture was proposed: *a special polygon cannot be cut into an odd number of triangles of equal areas*. It has been proved that the conjecture holds for polygons with at most six sides. In this paper we prove the existence of special n-polygon for any integer $n > 6$ and discuss the conjecture for special polygons with seven sides.

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1. Introduction

In [1] Stein proposed a generalized conjecture about cutting a polygon into triangles of equal areas. Let $\mathcal{P}$ be a simply connected polygon in the xy-plane with an oriented boundary. Orient the edges on the boundary of $\mathcal{P}$ to be consistent with the given orientation. Two edges of $\mathcal{P}$ are said to be equivalent if they are parallel. If the sum of the vectors in each equivalence class is the 0-vector, we call $\mathcal{P}$ a *special polygon*. Stein proposed the conjecture that a special polygon cannot be cut into an odd number of triangles of equal areas, and proved the conjecture for polygons with at most six sides. We call a dissection of a polygon into m triangles of equal areas an m-*equidissection*. An equidissection by an odd number of triangles is called an *odd equidissection*.

It is obvious that there are no special polygons with three or five sides, and some special polygons with four or six sides have been given in [1]. In this paper we prove the existence of special polygons with more than six sides. We

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conjecture that special polygons with seven sides have no odd equidissection and prove the conjecture for some typical cases.

First we summarize all necessary notations and lemmas from [1] for our discussion.

A function $\varphi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ is a 2-adic valuation on the reals if

(1) $\varphi(xy) = \varphi(x) + \varphi(y)$,
(2) $\varphi(x + y) \geq min\{\varphi(x), \varphi(y)\}$,
(3) $\varphi(2) = 1, \varphi(0) = \infty$.

The following are some properties of the 2-adic valuation (see [1]-[2]):

Property 1: $\varphi(1) = 0$, $\varphi(\frac{1}{x}) = -\varphi(x)$.
Property 2: for any odd integer n, $\varphi(n)= 0$.
Property 3: if $\varphi(x) < \varphi(y)$ then $\varphi(x + y) = \varphi(x)$.
Property 4: $\varphi(-x) = \varphi(x)$, and $\varphi(\frac{x}{y}) = \varphi(x) - \varphi(y)$.

For convenience we denote $\varphi(x)$ by x'.

By means of a 2-adic valuation the points in the xy-plane can be divided into three types:

- If $x' > 0$ and $y' > 0$, (x, y) is of type 0 and is denoted by p_0.
- If $x' \leq 0$ and $y' \geq x'$, (x, y) is of type 1 and is denoted by p_1.
- If $y' \leq 0$ and $y' < x'$, (x, y) is of type 2 and is denoted by p_2.

A line segment whose ends are a p_0 and a p_1 is called *complete.* A triangle whose vertices are labelled p_0, p_1, and p_2 is called *complete.* A polygon whose border contains an odd number of complete edges is called *complete.*

Lemma 1 ([1]-[4]). *If a complete polygon of area A is cut into m triangles of equal areas, then $m' \geq (2A)' = 1 + A'$. If $A' \geq -1$, then m is even.*

Lemma 2 ([5]). *If an m-equidissection of a polygon of area A contains a complete triangle, then $m' \geq (2A)' = 1 + A'$. If $A' \geq -1$, then m is even.*

Consider a simplicial dissection of a polygon in which each vertex is labelled p_0, p_1, or p_2. The adjective "simplicial" means that two overlapping (closed) triangles intersect either in a common vertex or in two vertices and the entire edge that joins them. Let $n_+(p_0p_1p_2)$ be the number of complete triangles in the dissection for which the order p_0 to p_1 to p_2 is counterclockwise. Let $n_-(p_0p_1p_2)$ be the number for which that order is clockwise. Similarly, let $n_+(p_0p_1)$ be the number of sections on the polygon labelled p_0p_1 for which the order p_0 to p_1 gives a counterclockwise orientation to the boundary. Let $n_-(p_0p_1)$ be the number for which that orientation is clockwise.

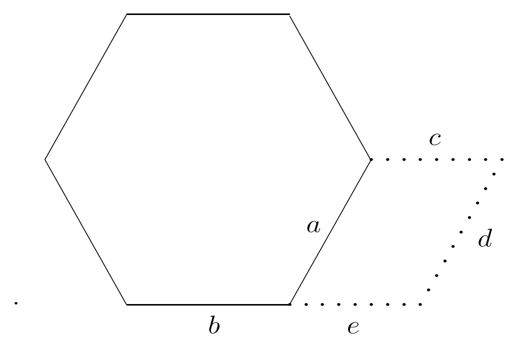


Figure 1

Lemma 3 ([5]). *For a simplicial dissection of a polygon in which each vertex is labelled p_0, p_1, or p_2,*

$$n_+(p_0p_1p_2) - n_-(p_0p_1p_2) = n_+(p_0p_1) - n_-(p_0p_1).$$

2. The main results

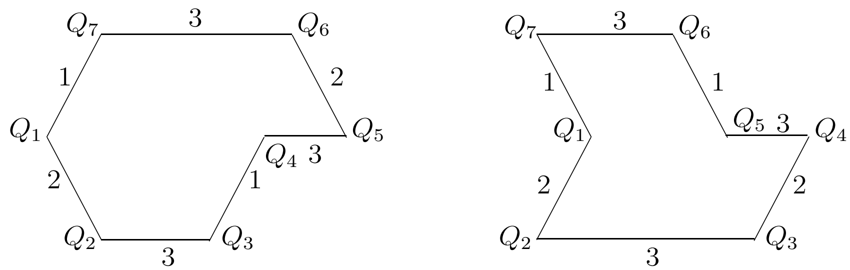
Theorem 1. *For any integer $n > 6$, there exists a special polygon with n sides.*

Proof. When n is even, a regular n-gon is a special polygon, and the conclusion is obvious.

When n is odd, suppose $n = 2k + 1$ ($k \geq 3$), and take a regular $2k$-gon, attach a parallelogram $acde$ to a side, say side a of the regular $2k$-gon, as shown in Figure 1, where side e is on the extended line of b which is adjacent to a. Replacing a by consecutive c, d, e we obtain a special $2k+1$-gon since the vector sum of c and e is zero. □

Now we consider different types of special polygons with seven sides. Seven sides of a special 7-gon can only be divided into three equivalence classes, two of which consist of two vectors (oriented sides), the other consists of three vectors. We use the same number to express the vectors in the same class. Then each special 7-gon can be expressed by an ordered index sequence $n_1n_2n_3n_4n_5n_6n_7$ where $n_i \in \{1, 2, 3\}$, and each index sequence consists of three 3's, two 1's and two 2's. The index sequence of a special 7-gon satisfies the following conditions:

(1) The components (numbers) in an index sequence are ordered circularly and hence a index sequence may start from any one of its components.



Type (I) Type (II)

Figure 2

(2) 1 and 2 in any index sequence are symmetric. Two index sequences are considered to be the same if one of them can be obtained from the other by interchanging 1 and 2 in its components.
(3) In any index sequence two consecutive numbers are always distinct.
(4) 1, 2, 3 must appear consecutively (not necessarily in a definite order) in some part of a index sequence.

It's easy to check that the following table shows all possible index sequences which satisfy conditions (1)-(4) :

	1321323
	1323123
1231323	1323213
1232313	1323132
	3213132
	3213231

But each index sequences in the right column can be changed into 1231323 or 1232313 by conditions (1) and (2).

Therefore, we obtain the following important lemma:

Lemma 4. *There exist only two types of special 7-gons, type (I) with index sequence 1231323 and type (II) with index sequence 1232313.*

In each type there are different configurations of special polygons. We conjecture that all possible configurations can be transformed into the two configurations shown in Figure 2.

Now we consider the special polygon of type(I) or type (II) in Figure 2.

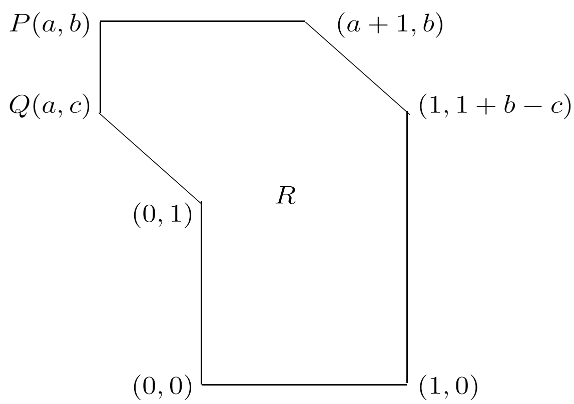


Figure 3

Assume that such a polygon, situated in the xy-plane, is tiled by m triangles of equal areas. Label its vertices by Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7. Each triplet of three consecutive vertices, Q_i, Q_{i+1}, Q_{i+2}, where the indices are viewed modulo 7, determines a triangle of area A_i. Let $Q_j,\ Q_{j+1},\ Q_{j+2}$ be a triplet for which A_j' is a minimum.

Theorem 2. *A special 7-gon of type (I) has no odd equidissection for* $j = 3, 4$.

Proof. When $j = 4$, let T be the affine transformation such that $T(Q_4) = (0,\ 1)$, $T(Q_5) = (0,\ 0)$, and $T(Q_6) = (1,\ 0)$. Let $\mathcal{R}$ be the image of the given polygon under T. Other vertices are labelled (a, b), (a, c), $(a+1, b)$, $(1, 1+b-c)$. Let $P = (a, b)$, Q $=(a, c)$. Figure 3 displays the labelling of $\mathcal{R}$.

The area of the triangle with vertices (0, 1), (0, 0), and (1, 0) is $\frac{1}{2}$. The area of the triangle with vertices (0, 0), (0, 1), and (a, c) is $\frac{-a}{2}$. The area of the triangle with vertices (a, b), $(a+1, b)$, and $(1, 1+b-c)$ is $\frac{c-1}{2}$. The area of the triangle with vertices (a, c), (a, b), and $(a+1, b)$ is $\frac{b-c}{2}$. Since T preserves ratios of areas, it follows that $(\frac{-a}{2})'$and $(\frac{c-1}{2})'$ and $(\frac{b-c}{2})'$ are at least $(\frac{1}{2})'$. Thus

$$a' \geq 0,\ b' \geq 0,\ c' \geq 0 \qquad (*)$$

Obviously the area of $\mathcal{R}$ is $b - ab + ac$.

The following are nine cases according to the types of P and Q:

(1) P is a p_0, and Q is a p_0, i.e., $a' > 0$, $b' > 0$, $c' > 0$.

When $b' \leq a'$, introduce the linear mapping $(x, y) \to (x, \frac{y}{b})$ and let $\mathcal{S}$

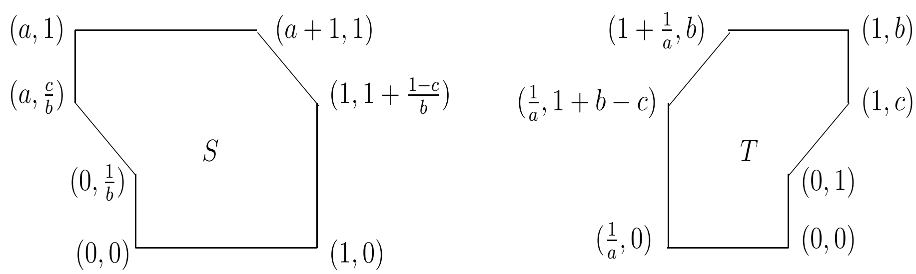


Figure 4

be the image of $\mathcal{R}$ under this mapping. Figure 4 (a) shows $\mathcal{S}$, which is complete.

Since the determinant of the mapping is $\frac{1}{b}$, the area of $\mathcal{S}$ is

$$\frac{b-ab+ac}{b}=1-a+\frac{ac}{b}.$$

Hence (area of $S)'=0>-1$. Thus m is even by Lemma 1 and the theorem is proved in this case.

When $b'>a'$, introduce the linear mapping $(x,y)\to(\frac{x}{a},y)$ and let $\mathcal{T}$ be the images of $\mathcal{R}$ under this mapping. Figure 4 (b) shows $\mathcal{T}$, which is complete. This mapping has determinant $\frac{1}{a}$, the area of $\mathcal{T}$ is

$$\frac{b-ab+ac}{a}=\frac{b}{a}-b+c.$$

Hence (area of $\mathcal{T})'>\prime$, m is even by Lemma 1 and the theorem is proved in this case.

(2) P is a p_0, and Q is a p_1, i.e., $a'>0$, $b'>0$; $a'\le 0$, $a'\le c'$.
This case leads to a contradiction.

(3) P is a p_0, and Q is a p_2, i.e., $a'>0$, $b'>0$; $c'\le 0$, $c'<a'$.
In addition, $c'\ge 0$ by $(*)$, so $c'=0$. Similar to case (1), we reach the conclusion.

(4) P is a p_1, and Q is a p_0, i.e., $a'\le b'$, $a'\le 0$; $a'>0$, $c'>0$.
This case leads to a contradiction.

(5) P is a p_1, and Q is a p_1, i.e., $a'\le b'$, $a'\le 0$; $a'\le 0$, $a'\le c'$.
In addition, $a'\ge 0$, by $(*)$, so $a'=0$. In this case $\mathcal{R}$ is complete, and since (area of $R)'=(b-ab+ac)'\ge 0$, m is even by Lemma 1 and the theorem is proved in this case.

(6) P is a p_1, and Q is a p_2, i.e., $a'\le b'$, $a'\le 0$; $c'\le 0$, $c'<a'$.

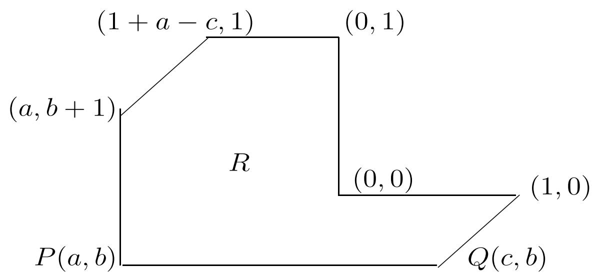


Figure 5

In addition, $c' \geq 0$ by $(*)$, so $c' = 0$, thus $a' > 0$, then the case leads to a contradiction.

(7) P is a p_2, and Q is a p_0, i.e., $b' \leq 0$, $b' < a'$; $a' > 0$, $c' > 0$.
In addition, $b' \geq 0$, by $(*)$, so $b' = 0$. In this case $\mathcal{R}$ is complete, and since (area of $R)' = (b - ab + ac)' = 0 > -1$, m is even by Lemma 1 and the theorem is proved in this case.

(8) P is a p_2, and Q is a p_1, i.e., $b' \leq 0$, $b' < a'$; $a' \leq 0$, $a' \leq c'$.
In addition, $b' \geq 0$ by $(*)$, so $b' = 0$, thus $a' > 0$, then the case leads to a contradiction.

(9) P is a p_2, and Q is a p_2, i.e., $b' \leq 0$, $b' < a'$; $c' \leq 0$, $c' < a'$.
In addition, $b' \geq 0$, $c' \geq 0$ by $(*)$, so $b' = 0$, $c' = 0$, thus $a' > 0$, $b' = 0$, $c' = 0$.
In this case $\mathcal{R}$ is complete, and since (area of $R)' = (b - ab + ac)' = 0 > -1$, m is even by Lemma 1 and the theorem is proved in this case.

The proof is complete for $j = 4$.

When $j = 3$, let T be the affine transformation such that

$$T(Q_3) = (0, 1),\ T(Q_4) = (0, 0),\ \text{and } T(Q_5) = (1, 0).$$

Let $\mathcal{R}$ be the image of the given polygon under T. Other vertices are labelled (a, b), (c, b), $(a, b+1)$, $(1+a-c, 1)$. Let $P = (a, b)$, Q $=(c, b)$. Figure 5 displays the labelling of $\mathcal{R}$.

The area of the triangle with vertices (0, 1), (0, 0) and (1, 0) is $\frac{1}{2}$.

The area of the triangle with vertices (0, 0), (1, 0), and (c, b) is $\frac{-b}{2}$.

The area of the triangle with vertices (a, b), $(a, b + 1)$, and $(1 + a - c, 1)$ is $\frac{1-c}{2}$.

The area of the triangle with vertices $(a, b + 1)$, (a, b), and (c, b) is $\frac{c-a}{2}$.

Since T preserves ratios of areas, it follows that $(\frac{-b}{2})'$, $(\frac{1-c}{2})'$ and $(\frac{c-a}{2})'$ are at least $(\frac{1}{2})'$. Thus

$$a' \geq 0,\ b' \geq 0,\ c' \geq 0 \tag{$*$}$$

Clearly the area of $\mathcal{R}$ is $ab - cb - a$.

As in the case $j = 4$, similarly we have nine cases to consider according to the types of P and Q, and here we need only to give the proofs for the following two cases:

(1) P is a p_2, and Q is a p_1.
i.e., $b' \leq 0$, $b' < a'$; $c' \leq 0$, $c' \leq b'$.
In addition, $b' \geq 0$, $c' \geq 0$ by $(*)$, so $b' = 0$, $c' = 0$, thus $a' > 0$.
The only case when the labelling is not complete occurs when $(a, b+1)$ is a p_0, and $(1 + a - c, 1)$ is a p_1, by using Lemma 2 and Lemma 3, $m' \geq (2(ab - cb - a))' \geq 2' = 1$, then m is even.

(2) P is a p_2, and Q is a p_2.
i.e., $b' \leq 0$, $b' < a'$; $b' \leq 0$, $b' < c'$.
In addition, $b' \geq 0$ by $(*)$, so $b' = 0$, thus $a' > 0$, $b' = 0$, $c' > 0$.
Similar to case (1), the only case when the labelling is not complete occurs when $(a, b + 1)$ is a p_0, and $(1 + a - c, 1)$ is a p_1, by using Lemma 2 and Lemma 3, $m' \geq (2(ab - cb - a))' \geq 2' = 1$, then m is even.

The proof is complete for $j = 3$.

□

In a similar way we obtain the following result.

Theorem 3. *A special 7-gon of type (II) has no odd equidissection for $j = 3, 4$.*

References

1. S. Stein, *A generalized conjecture about cutting a polygon into triangles of equal areas*, Discete Comput. Geom. **24** (2000), 141–145.
2. P. Monsky, *A conjecture of Stein on plane dissections*, Math. Z. **205** (1990), 583–592.
3. S. Stein, *Equidissections of centrally symmetic octagons*, Aequationes Math. **37** (1989), 113–118.
4. S. Stein, *Cutting a polyomino into triangles of equal areas*, Amer. Math. Monthly **106** (1999), 255–257.
5. S. Stein and S. Szabo, *Algebra and tiling, The Mathematical Association of America*, 1994
6. Rongxia Hao, Junliang Cai and Yanpei Liu, *Counting g-essential maps on surface with small genera*, Korean J. Comput. & Appl. Math. **9**(1-2) (2002), 451 –.

7. Qi Duan, Tzer-Shyong Chen, K. Djidjeli, W. G. Price and E. H. Twizell, *Convex control and approximation properties of interpolating curves*, Korean J. Comput. & Appl. Math. May (2000), 397 – .
8. Zhaoxiang Li and Yanpei Liu, *Chromatic sum of nonseperable simple maps on the plane*, J. Appl. Math. & Computing **12**(1-2) (2003), 120 – .

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