HEMIVARIATIONAL INEQUALITIES

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Abstract. The auxiliary principle is used to suggest and analyze some iterative methods for solving hemivariational inequalities under mild conditions. The results obtained in this paper can be considered as a novel application of the auxiliary principle technique. Since hemivariational inequalities include variational inequalities and nonlinear optimization problems as special cases, our results continue to hold for these problems.

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1. Introduction

Variational inequalities theory introduced in 1964 has emerged as a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure and applied sciences in a unified and general framework. The ideas and techniques of the variational inequalities are being applied in a variety of diverse areas and prove to be productive and innovative. Variational inequalities have been extended and generalized in several direction using novel and new techniques, see [2-23]. There are significant developments of variational inequalities related with multivalued, nonmonotone, nonconvex optimization and structural analysis. An important and useful generalization of variational inequalities is a class of variational inequalities, which is known as hemivariational inequalities. The hemivariational inequalities were introduced and investigated by Panagiotopoulos [19] by using the concept of the generalized directional derivatives of nonconvex and nondifferentiable functions. This class has important applications in structural analysis and nonconvex optimization. In particular, it has been shown [2] that if a nonsmooth and nonconvex
superpotential of a structure is quasidifferentiable then these problems can be studied via hemivariational inequalities. The solution of the hemivariational inequalities gives the position of the state equilibrium of the structure. It is worth mentioning that hemivariational inequalities can be viewed as a special case of mildly nonlinear variational inequalities, considered and introduced by Noor [11]. However, numerical techniques considered for solving mildly nonlinear variational inequalities can not be extended for hemivariational inequalities due to the presence of nonlinear and nondifferentiable terms. For the applications and formulation of the hemivariational inequalities, see [2, 10, 19, 20] and the references therein.

Variational inequalities and related optimization problems have witnessed an explosive growth in theoretical advances, algorithmic developments and applications across almost all disciplines of engineering, pure and applied sciences. As a result of interaction between different branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze various iterative algorithms for solving hemivariational inequalities and equilibrium problems. Analysis of these problems requires a blend of techniques and ideas from convex analysis, functional analysis, numerical analysis and nonsmooth analysis. There are several methods for solving variational inequalities and equilibrium problems. Due to the nature of the hemivariational inequalities, projection and resolvent methods can not be applied for solving hemivariational inequalities. In recent years, the auxiliary principle technique is being used to suggest and analyze some iterative methods for solving variational inequalities and equilibrium problems. This technique is basically due to Lions and Stampacchia [7] and was used by Noor [11] to obtain the existence results for the mildly (strongly) nonlinear variational inequalities. However, Glowinski, Lions and Tremolieres [5] used this technique to study the existence problem for mixed variational inequalities. The main idea involving this technique is to first consider an auxiliary problem and then to show that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. Noor [14-18] has used this approach to suggest and analyze some iterative methods for solving various classes of variational inequalities. In this paper, we show that this technique can be used to suggest some iterative schemes for hemivariational inequalities. We prove that the convergence of these methods require either pseudomonotonicity or partially relaxed strongly monotonicity. These are weaker conditions than monotonicity. As a special case, we obtain iterative schemes for solving variational inequalities. The comparison of these methods with other methods is a subject of future research.

2. Preliminaries
Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let $K$ be a nonempty closed convex set in $H$. Let $f : H \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. Let $\Omega$ be an open bounded and regular subset of $\mathbb{R}^n$. First of all, we recall the following concepts and results from nonsmooth analysis (see Clarke [1]).

**Definition 2.1.** Let $f$ be locally Lipschitz continuous at a given point $x \in H$ and $v$ be any other vector in $H$. The Clarke’s generalized directional derivative of $f$ at $x$ in the direction $v$, denoted by $f^0(x; v)$, is defined as

$$f^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$ 

The generalized gradient of $f$ at $x$, denoted $\partial f(x)$, is defined to be subdifferential of the function $f^0(x; v)$ at $0$. That is,

$$\partial f(x) = \{ w \in H : \langle w, v \rangle \leq f^0(x; v), \quad \forall v \in H \}.$$ 

**Lemma 2.1.** Let $f$ be a locally Lipschitz continuous at a given point $x \in H$ with a constant $L$. Then

(i) $\partial f(x)$ is a non-empty compact subset of $H$ and $\| \xi \| \leq L$ for each $\xi \in \partial f(x)$.

(ii) For every $v \in H$, $f^0(x; v) = \max\{ \langle \xi, c \rangle : \xi \in \partial f(x) \}$.

(iii) The function $v \mapsto f^0(x; v)$ is finite, positively homogeneous, subadditive, convex and continuous.

(iv) $f^0(x; -v) = (-f)^0(x; v)$.

(v) $f^0(x; v)$ is upper semicontinuous as a function of $(x; v)$.

(vi) $\forall x \in H$, there exists a constant $\alpha > 0$ such that

$$|f^0(x; v)| \leq \alpha \| v \|, \quad \forall v \in H.$$ 

If $f$ is convex on $K$ and locally Lipschitz continuous at $x \in K$, then $\partial f(x)$ coincides with the subdifferential $f'(x)$ of $f$ at $x$ in the sense of convex analysis, and $f^0(x; v)$ coincides with the directional derivative $f'(x; v)$ for each $v \in H$, that is, $f^0(x; v) = \langle f'(x; v) \rangle \quad \forall v \in H$.

For a given nonlinear operator $T : H \rightarrow H$, consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u \rangle + \int_{\Omega} f^0(x, u; v - u) d\Omega \geq 0, \quad \forall v \in K. \quad (1)$$

Here $f^0(x, u; v - u) := f^0(x, u(x); v(x) - u(x))$ denotes the generalized directional derivative of the function $f(x, \cdot)$ at $u(x)$ in the direction $v(x) - u(x)$. Problem of type (1) is called the hemivariational inequality introduced and studied by
Panagiotopoulos [19] in order to formulate variational principles associated with energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in engineering and industrial sciences, see [2, 10, 19, 20] and the references therein.

If \( f = 0 \), then problem (1) is equivalent to finding \( u \in K \) such that
\[
(Tu, v - y) \geq 0, \quad \forall v \in K.
\] (2)

Problem (2) is called the classical variational inequality, which is due to Stampacchia [22]. If \( f(.) \) is a smooth and convex function, then problem (1) is equivalent to finding \( u \in K \) such that
\[
(Tu, v - u) + (f'(u), v - u) \geq 0, \quad \forall v \in K,
\] (3)

which is known as the mildly nonlinear variational inequality, introduced and studied by Noor [11] in 1975. Here \( f'(u) \) is the differential of a convex function \( f \) at \( u \). For the applications, formulations and numerical methods of hemivariational inequalities and variational inequalities, see [2-23] and the references therein.

**Definition 2.2.** An operator \( T : H \rightarrow H \) is said to be:

(a) **monotone** if
\[
(Tu - Tv, v - u) \geq 0, \quad \forall u, v \in H.
\]

(b) **pseudomonotone** with respect to \( \int_{\Omega} f^0(x, u; v - u) d\Omega \) if
\[
(Tu, v - u) + \int_{\Omega} f^0(x, u; v - u) d\Omega \geq 0
\]
\[
\Rightarrow (Tv, v - u) + \int_{\Omega} f^0(x, u; v - u) d\Omega \geq 0, \quad \forall u, v \in H.
\]

(c) **partially relaxed strongly monotone** if there exists a constant \( \gamma > 0 \) such that
\[
(Tu - Tv, z - v) \geq -\gamma \| u - z \|^2, \quad \forall u, v, z \in H.
\]

(d) **hemicontinuous** if \( \forall t \in [0, 1] \) implies that \( (T(u + t(v - u)), v) \) is continuous, \( \forall u, v \in H \).

Note that for \( z = u \), partially relaxed strongly monotonicity reduces to monotonicity. This shows that partially relaxed strongly monotonicity implies monotonicity, but the converse is not true.
Definition 2.3. The function $\int_\Omega f^0(x, u; v - u) d\Omega$ is said to be partially relaxed strongly monotone if there exists a constant $\alpha > 0$ such that

$$\int_\Omega f^0(x, u; v - u) d\Omega + \int_\Omega f^0(x, z; u - v) d\Omega \leq \alpha \|z - v\|^2, \quad \forall u, v, z \in H.$$ 

Note that for $z = v$, partially relaxed strongly monotonicity reduces to monotonicity, that is,

$$\int_\Omega f^0(x, u; v - u) d\Omega + \int_\Omega f^0(x, v; u - v) d\Omega \leq 0, \quad \forall u, v \in H.$$ 

Lemma 2.2. Let the function $F(.,.)$ be hemicontinuous, pseudomonotone with respect to the function $\int_\Omega f^0(x, u; v - u) d\Omega$ and convex in the second argument. Then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tv, v - u \rangle + \int_\Omega f^0(x, u; v - u) d\Omega \geq 0, \quad \forall v \in K.$$

Proof. Let $u \in K$ be a solution of (1). Then

$$\langle Tu, v - u \rangle + \int_\Omega f^0(x, u; v - u) d\Omega \geq 0, \quad \forall v \in K,$$

which implies that

$$\langle Tv, v - u \rangle \leq \int_\Omega f^0(x, u; v - u) d\Omega, \quad \forall v \in K,$$

since $T$ is pseudomonotone with respect to $\int_\Omega f^0(x, u; v - u) d\Omega$.

$\forall u, v \in K, \quad t \in [0,1], \quad v_t = u + t(v - u) \in K$ since $K$ is a convex set. Taking $v = v_t$ in (4), we have

$$\langle Tv_t, v - u \rangle \leq \int_\Omega f^0(x, u; u - v) d\Omega.$$

Since $T$ is hemicontinuous, taking limit as $t \to 0$, we have

$$\langle Tu, v - u \rangle + \int_\Omega f^0(x, u; v - u) d\Omega \geq 0, \quad \forall v \in K,$$

the required result (1). \qed

Remark 2.1. From Lemma 2.2, we see that problems (1) and (4) are equivalent. Problem (4) is called the dual hemivariational inequality. One can easily show
that the solution set of problem (4) is a closed convex set. Lemma 2.1 can be viewed as a natural generalization of Minty’s result.

**Definition 2.4.** A function \( f : K \rightarrow H \) is said to be **strongly convex** if there exists a constant \( \beta > 0 \) such that
\[
f(u + t(v - u)) \leq (1 - t)f(u) + tf(v) - t(1 - t)\beta\|v - u\|^2, \quad \forall u, v \in K, \quad t \in [0, 1].
\]
If the strongly convex function is differentiable, then
\[
f(v) - f(u) \geq \langle f'(u), v - u \rangle + \beta\|v - u\|^2, \quad \forall u, v \in K,
\]
and conversely.

### 3. Iterative schemes

In this section, we suggest and analyze some iterative methods for hemivariational inequality problems (1) using the auxiliary principle technique of Glowinski, Lions and Tremolieres [5] as developed by Noor [14-18].

For a given \( u \in K \), consider the auxiliary problem of finding a unique \( w \in K \) such that
\[
\rho \langle Tw, v - w \rangle + (E'(w) - E'(u), v - w) + \rho \int_{\Omega} f^0(x, u; v - w) d\Omega \geq 0, \quad \forall v \in K,
\]  
(5)
where \( \rho > 0 \) is a constant and \( E'(u) \) is the differential of a strongly convex function \( E(u) \) at \( u \in K \). Since \( E(u) \) is a strongly convex function, problem (5) has an unique solution. We note that if \( w = u \), then clearly \( w \) is solution of the hemivariational inequality (1). This observation enables us to suggest and analyze the following iterative method for solving (1).

**Algorithm 3.1.** For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative scheme
\[
\rho \langle Tu_{n+1}, v - u_{n+1} \rangle + (E'(u_{n+1}) - E'(u_n), v - u_{n+1}) + \rho \int_{\Omega} f^0(x, u_n; v - u_{n+1}) d\Omega \geq 0, \quad \forall v \in K,
\]  
(6)
where \( \rho > 0 \) is a constant.

Algorithm 3.1 is called the **proximal method** for solving hemivariational inequality problem (1). In passing we remark that the proximal point method was suggested by Martinet [8] in the context of convex programming problems as regularization technique.

If \( f(x, u) = 0 \), then Algorithm 3.1 collapses to:
Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$
\rho \langle Tu_{n+1}, v - u_{n+1} \rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \geq 0, \quad \forall v \in K,
$$

for solving the classical variational inequalities (2).

In brief, for suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational-like inequalities and related problems.

Theorem 3.1. Let $T$ be pseudomonotone with respect to $\int_\Omega f^0(x, u; v - u) d\Omega$. Let $E$ be differentiable strongly convex function with module $\beta > 0$.

If $\int_\Omega f^0(x, u; v - u) d\Omega$ is partially relaxed strongly monotone with constant $\alpha > 0$ and $0 < \rho < \beta/\alpha$, then the approximate solution $u_{n+1}$ obtained from Algorithm 3.1 converges to the exact solution $u \in K$ satisfying (1).

Proof. Let $u \in K$ be a solution of (1). Then

$$
\langle Tu, v - u \rangle + \int_\Omega f^0(x, u; v - u) d\Omega \geq 0, \quad \forall v \in K,
$$

implies that

$$
\langle Tv, v - u \rangle + \int_\Omega f^0(x, u; v - u) d\Omega \geq 0, \quad \forall v \in K,
$$

(7)

since $T$ is pseudomonotone with respect to $\int_\Omega f^0(x, u; v - u) d\Omega$.

Taking $v = u$ in (6) and $v = u_{n+1}$ in (7), we have

$$
\rho \langle Tu_{n+1}, u - u_{n+1} \rangle + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \\
\geq -\rho \int_\Omega f^0(x, u_n; u - u_{n+1}) d\Omega
$$

(8)

and

$$
\langle Tu_{n+1}, u - u_{n+1} \rangle + \int_\Omega f^0(x, u; u_{n+1} - u) d\Omega \geq 0.
$$

(9)

We now consider the function

$$
B(u, w) = E(u) - E(w) - \langle E'(w), u - w \rangle \\
\geq \beta \|u - w\|^2, \quad \text{using strongly convexity of } E.
$$

(10)
Now combining (8), (9) and (10), we have
\[ B(u, u_n) - B(u, u_{n+1}) = E(u_{n+1}) - E(u_n) - \langle E'(u_{n+1}), u_{n+1} - u_n \rangle \]
\[ + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \]
\[ \geq \beta \| u_{n+1} - u_n \|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \]
\[ \geq \beta \| u_{n+1} - u_n \|^2 - \rho \langle Tu_{n+1}, u - u_{n+1} \rangle \]
\[ - \rho \int_\Omega f^0(x, u_n; u - u_{n+1}) d\Omega \]
\[ \geq \beta \| u_{n+1} - u_n \|^2 - \rho \left\{ \int_\Omega f^0(x, ; u_{n+1} - u_n) d\Omega \right\} \]
\[ + \int_\Omega f^0(x, u_n; u - u_{n+1}) d\Omega \]
\[ \geq (\beta - \rho \alpha) \| u_{n+1} - u_n \|^2, \]
where we have used the fact that \( \int_\Omega f^0(x, ; ..) d\Omega \) is partially relaxed strongly monotone with a constant \( \alpha > 0 \).

If \( u_{n+1} = u_n \), then clearly \( u_n \) is a solution of the hemivariational inequality problem (1). Otherwise, for \( 0 < \rho < \frac{\alpha}{\beta} \), it follows that \( B(u, u_n) - B(u, u_{n+1}) \) is nonnegative, and we must have
\[ \lim_{n \to \infty} \| u_{n+1} - u_n \| = 0. \]

Now using the technique of Zhu and Marcotte [23], it can be shown that the entire sequence \( \{ u_n \} \) converges to the cluster point \( u \) satisfying the hemivariational inequality problem (1).

It is well-known that to implement to the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving (1) using the auxiliary principle technique.

For a given \( u \in K \), find a unique \( w \in K \) such that
\[ \rho \langle Tu, v - w \rangle + \langle E'(w) - E'(u), v - w \rangle \]
\[ + \rho \int_\Omega f^0(x, u; v - w) d\Omega \geq 0, \quad \forall v \in K, \] (11)
where \( E'(u) \) is the differential of a strongly convex function \( E(u) \) at \( u \in K \). Problem (11) has a unique solution, since \( E \) is strongly convex function. Note that problems (5) and (11) are quite different problems. It is clear that for \( w = u \), \( w \) is a solution of (1). This fact allows us to suggest and analyze another iterative method for solving hemivariational inequality problem (1).
Algorithm 3.3. For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme

$$
\rho \langle Tu_n, v - u_n \rangle + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1} \rangle \\
+ \rho \int_{\Omega} (x, u_n; v - u_{n+1}) d\Omega \geq 0, \quad \forall v \in K,
$$

for solving the hemivariational inequalities (1).

For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving equilibrium problems and variational inequalities.

We now consider the convergence analysis of Algorithm 3.3 using essentially the technique of Theorem 3.1. For the sake of completeness and to convey an idea of the technique, we sketch the main points.

Theorem 3.2. Let $T$ and $\int_{\Omega} f^0(x, u; v - u) d\Omega$ be partially relaxed strongly monotone with constants $\gamma > 0$ and $\alpha > 0$ respectively. If $E$ is strongly convex function with modulus $\beta > 0$ and $0 < \rho < \beta / (\alpha + \gamma)$, then the approximate solution $u_{n+1}$ obtained from Algorithm 3.3 converges to a solution of (1).

Proof. Let $u \in K$ be solution of (1). Setting $v = u_{n+1}$ in (1) and $v = u$ in (12), we have

$$
\langle Tu, u_{n+1} - u \rangle + \int_{\Omega} f^0(x, u; u_{n+1} - u) d\Omega \geq 0
$$

and

$$
\rho \langle Tu_n, u - u_{n+1} \rangle + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \\
+ \rho \int_{\Omega} f^0(x, u_n; u - u_{n+1}) d\Omega \geq 0.
$$

As in Theorem 3.1 and from (13) and (14), we have

$$
B(u, u_n) - B(u, u_{n+1}) = E(u_{n+1}) - E(u_n) - \langle E'(u_{n+1}), u_{n+1} - u_n \rangle \\
+ \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \\
\geq \beta \|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1} \rangle \\
\geq \beta \|u_{n+1} - u_n\|^2 - \rho \langle Tu_n, u - u_{n+1} \rangle \\
- \rho \int_{\Omega} f^0(x, u_n; u - u_{n+1}) d\Omega
$$
\[ \geq \beta \|u_{n+1} - u_n\|^2 - \rho \left\{ \langle Tu - Tu_n, u_{n+1} - u \rangle \right\} \\
- \rho \left\{ \int_{\Omega} f^0(x, u_{n+1} - u) d\Omega \right\} \\
\quad + \left\{ \int_{\Omega} f^0(x, u_n; u - u_{n+1}) d\Omega \right\} \\
\geq \beta \|u_{n+1} - u_n\|^2 - \rho (\alpha + \gamma) \|u_{n+1} - u_n\|^2, \]

where we have used the fact that \( T \) and \( \int_{\Omega} f^0(x, \cdot; \cdot) d\Omega \) are partially relaxed strongly monotone with constants \( \alpha > 0 \) and \( \gamma > 0 \) respectively.

If \( u_{n+1} = u_n \), then clearly \( u_n \) is a solution of the hemivariational inequality problem (1). Otherwise, for \( 0 < \rho < \frac{\beta}{\alpha + \gamma} \), it follows that \( B(u, u_n) - B(u, u_{n+1}) \) is nonnegative, and we must have

\[ \lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \]

Now using the technique of Zhu and Marcotte [23], it can be shown that the entire sequence \( \{u_n\} \) converges to the cluster point \( u \) satisfying the hemivariational inequality problem (1).

\[ \square \]

4. Nonconvex hemiequilibrium problems

We would like to point out that the techniques and ideas of Section 3 can be extended for solving nonconvex hemiequilibrium problems, which are defined over the nonconvex sets. It is shown that the nonconvex hemiequilibrium problems include the hemivariational inequalities and hemiequilibrium problems as special cases. It is known that the nonconvex (\( g \)-convex) are not convex sets and include the convex sets as a special case. For this purpose, we recall the following concepts.

**Definition 4.1.** Let \( K \) be any set in \( H \). The set \( K \) is said to be \( g \)-convex if there exists a function \( g : K \to K \) such that

\[ g(u) + t(g(v) - g(u)) \in K, \quad \forall u, v \in K, t \in [0, 1]. \]

Note that every convex set is \( g \)-convex, but the converse is not true.

*From now onward, we assume that \( K \) is a \( g \)-convex set, unless otherwise specified.*
Definition 4.2. The function \( f : K \rightarrow H \) is said to be \( g \)-convex if
\[
f(g(u) + t(g(v) - g(u))) \leq (1 - t)f(g(u)) + tf(g(v)), \quad \forall u, v \in K, t \in [0, 1].
\]

Clearly every convex function is \( g \)-convex, but the converse is not true, see Noor [14].

Definition 4.3. A function \( f \) is said to be strongly \( g \)-convex on the \( g \)-convex set \( K \) with modulus \( \mu > 0 \) if \( \forall u, v \in K, t \in [0, 1], \)
\[
f(g(u) + t(g(v) - g(u))) \leq (1 - t)f(g(u)) + tf(g(v)) - t(1 - t)\mu\|g(v) - g(u)\|^2.
\]

Using the convex analysis techniques, one can easily show that the differentiable \( g \)-convex function \( f \) is strongly \( g \)-convex function if and only if
\[
f(g(v)) - f(g(u)) \geq \langle f'(g(u)), g(v) - g(u) \rangle + \mu\|g(v) - g(u)\|^2
\]
or
\[
\langle f'(g(u)) - f'(g(v)), g(u) - g(v) \rangle \geq 2\mu\|g(v) - g(u)\|^2,
\]
that is, \( f'(g(u)) \) is a strongly monotone operator.

It is well-known [14] that the \( g \)-convex functions are not convex function, but they have some nice properties which the convex functions have. Note that for \( g = I \), the \( g \)-convex functions are convex functions and definition 2.3 is a well known result in convex analysis.

For a given nonlinear continuous bifunction \( F(., .) : K \times K \rightarrow K \), consider the problem of finding \( u \in K \) such that
\[
F(g(u), g(v)) + \int_{\Omega} f^0(x, g(u); g(v) - g(u))d\Omega \geq 0, \quad \forall v \in K,
\]
(15)
which is called is called the nonconvex hemiequilibrium problem. For \( g = I \), where \( I \) is the identity operator, the \( g \)-convex set \( K \) becomes the convex set \( K \) and consequently, problem (15) is equivalent to finding \( u \in K \) such that
\[
F(u, v) + \int_{\Omega} f^0(x, u; u - v)d\Omega \geq 0, \quad \forall v \in K,
\]
(16)
which is called the hemiequilibrium problem introduced and studied by Noor [18].

We note that for \( F(g(u), g(v)) = \langle Tg(u), g(v) - g(v) \rangle \), where \( T : H \rightarrow H \) is a nonlinear continuous operator, problem (15) is equivalent to finding \( u \in K \) such that
\[
\langle Tg(u), g(v) - g(u) \rangle + \int_{\Omega} f^0(x, g(u); g(v) - g(u))d\Omega \geq 0, \quad \forall v \in K.
\]
(17)
Inequality (17) is known as the *nonconvex hemivariational inequality* and appears to be a new one.

If \( f^0(x, u; v - u) = 0 \), then problem (4) is equivalent to finding \( u \in K \) such that

\[
\langle T(g(u)), g(v) - g(u) \rangle \geq 0, \quad \forall v \in K,
\]

which is known as the nonconvex variational inequality introduced by Noor [14,16]. It is worth mentioning that nonconvex variational inequalities (18) are quite different from the so-called general variational inequalities introduced and studied by Noor. For the applications and numerical methods of general variational inequalities; see Noor [14, 16] and the references therein.

If \( g = I \), the identity operator, then the \( g \)-convex set \( K \) becomes the convex set \( K \), and consequently the nonconvex variational inequalities (17) are equivalent to finding \( u \in K \) such that

\[
(Tu, v - u) + \int_\Omega f^0(x, u; v - u) d\Omega \geq 0, \quad \forall \ v \in K,
\]

which is exactly the hemivariational inequality (1) considered in Section 3.

It is clear that problems (16) - (19) are special cases of the nonconvex hemiequilibrium problems (15). In brief, for a suitable and appropriate choice of the operators \( T \), \( g \), and the space \( H \), one can obtain a wide class of equilibrium, variational inequalities and complementarity problems. This clearly shows that problem (15) is quite general and unifying one. Furthermore, problem (15) has important applications in various branches of pure and applied sciences.

Using essentially the technique and ideas developed in Section 3, we can suggest and analyze the iterative methods for solving the nonconvex hemiequilibrium problems (15) such as:

**Algorithm 4.1.** For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative schemes

\[
\rho F(g(u_n), g(v)) + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \rangle \\
\geq -\rho \int_\Omega f^0(x, g(u_n); g(v) - g(u_{n+1})) d\Omega, \quad \forall v \in K,
\]

where \( \rho > 0 \) is a constant.
Algorithm 4.2. For a given $u_0 \in H$, compute $u_{n+1}$ by the iterative scheme

$$
\rho F(g(u_{n+1}), g(v)) + \langle E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1}) \rangle \\
\geq -\rho \int_{\Omega} f^0(x, g(u_{n+1}); g(v) - g(u_{n+1})) d\Omega, \quad \forall v \in K,
$$

which is known as the proximal point algorithm for solving nonconvex hemiequilibrium problem (15).

Remark 4.1. In this paper, we have shown that the auxiliary principle technique can be used for solving hemivariational inequalities and nonconvex hemiequilibrium problems. We note that this technique is independent of the projection or resolvent of the operator. We have also studied the convergence analysis of these new methods under mild conditions.

References

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