AN ESTIMATION FOR THE EIGENVALUES OF THE TRANSVERSAL DIRAC OPERATOR

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ABSTRACT. On a foliated Riemannian manifold \((M, g_M, \mathcal{F})\) with a transverse spin foliation \(\mathcal{F}\), we estimate a lower bound for the square of the eigenvalues of the transversal Dirac operator \(D_{tr}\).

1. Introduction

In 1963, A. Lichnerowicz([12]) proved that on a Riemannian spin manifold the square of the Dirac operator \(D\) is given by

\[ D^2 = \Delta + \frac{\sigma}{4}, \]

where \(\Delta\) is the positive spinor Laplacian and \(\sigma\) the scalar curvature. In 1980, T. Friedrich([3]) gave a lower bound for the square of the eigenvalues of the Dirac operator in the above equation, as follows:

\[ \lambda^2 \geq \frac{n}{4(n-1)} \sigma_0, \]

where \(\sigma_0 = \min_{\mathcal{F}} \sigma\). He also proved, in the limiting case, that the manifold is an Einstein.

In 1988, J. Brüning and F. W. Kamber ([2]) defined the transversal Dirac operator \(D_{tr}\) on \(M\) and proved the following equation:

\[ D_{tr}^2 = \nabla^*_{tr} \nabla_{tr} + R_{\nabla} + K_{\nabla}, \]

where \(R_{\nabla}\) is an endomorphism containing the curvature data and \(K_{\nabla}\) a function containing the mean curvature of the leaves.

This paper is a survey on the transversal Dirac operator \(D_{tr}\) and its eigenvalues on the foliated Riemannian manifold \(M\), which is based on the works [6,7,8] of the researcher.

2. Basic Laplacian

Let \((M, g_M, \mathcal{F})\) be a \((p+q)\)-dimensional Riemannian manifold with a foliation \(\mathcal{F}\) of codimension \(q\) and a bundle-like metric \(g_M\) with respect to \(\mathcal{F}\).

Let \(\Omega^r_B(\mathcal{F})\) be the space of all basic \(r\)-forms, i.e.,

\[ \Omega^r_B(\mathcal{F}) = \{ \phi \in \Omega^r(M) \mid i(X)\phi = 0, \theta(X)\phi = 0, \text{ for } X \in \Gamma L \}. \]

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The foliation $\mathcal{F}$ is said to be isoparametric if $\kappa \in \Omega^q_B(\mathcal{F})$, where $\kappa$ is a mean curvature form of $\mathcal{F}$ (see [10, 14] for details). We already know that $\kappa$ is closed, i.e., $d\kappa = 0$ if $\mathcal{F}$ is isoparametric ([14]). Since the exterior derivative preserves the basic forms (that is, $\theta(X)d\phi = 0$ and $i(X)d\phi = 0$ for $\phi \in \Omega^r_B(\mathcal{F})$), the restriction $d_B = d|_{\Omega^n_B(\mathcal{F})}$ is well defined. The basic Laplacian acting on $\Omega^n_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_Bd_B,$$

where $\delta_B$ is the adjoint operator of $d_B$. The differential operators $d_B$ and $\delta_B$ are locally given by the following lemma. (See [1, 8])

**Lemma 2.1.** Let $\mathcal{F}$ be a Riemannian foliation. Then the operators $d_B$ and $\delta_B$ on $\Omega^n_B(\mathcal{F})$ are given by

$$d_B = \sum a \theta_a \wedge \nabla_{E_a},$$

$$\delta_B = -\sum a i(E_a)\nabla_{E_a} + i(\kappa_B),$$

where $\{E_a\}$ is a local orthonormal basic frame in $Q$ and $\{\theta_a\}$ its $g_Q$-dual 1-form.

Lemma 2.1 can be proved by using the generalized Green’s theorem ([15]). Also, Lemma 2.1 is proved in [1] by different method.

The basic cohomology $H^*_B(\mathcal{F}) = H(\Omega^*_B,d_B)$ plays the role of the De Rham cohomology of the leaf space $M/\mathcal{F}$ of the foliation.

**Theorem 2.2.** ([9]) Let $\mathcal{F}$ be a transversally oriented Riemannian foliation on a compact orientable manifold $(M,g_M)$. Assume $g_M$ to be bundle-like with $\kappa \in \Omega^1_B(\mathcal{F})$. Then there is a decomposition into mutually orthogonal subspaces

$$\Omega^*_B(\mathcal{F}) \equiv imd_B \oplus im\delta_B \oplus \mathcal{H}^*_B(\mathcal{F})$$

with finite dimensional $\mathcal{H}^*_B(\mathcal{F})$, where $\mathcal{H}^*_B(\mathcal{F}) = Ker\Delta_B$ is a set of the harmonic basic $r$-forms.

Also, the following vanishing theorem about the basic cohomology is well known.

**Theorem 2.3.** ([15]) Let $\mathcal{F}$ be a Riemannian foliation of codimension $q \geq 2$ on a closed oriented Riemannian manifold $(M,g_M)$ with bundle-like metric $g_M$. Then the following holds:

(i) if $\rho^\nabla > 0$, then $H^*_B(\mathcal{F}) = 0$;

(ii) if $\mathcal{R}^\nabla > 0$, then $H^*_B(\mathcal{F}) = 0$ for $0 < r < q$.

If $\mathcal{F}$ is the foliation by points of $M$, the basic Laplacian is the ordinary Laplacian. Hence Theorem 2.2 and Theorem 2.3 are generalizations of the De-Rham Hodge Theory on the ordinary manifold.

3. THE TRANSVERSE DIRAC OPERATOR

Let $(M,g_M,\mathcal{F})$ be a Riemannian manifold with a transversally oriented Riemannian foliation $\mathcal{F}$ of codimension $q$ and a bundle-like metric $g_M$ with respect to $\mathcal{F}$. Let $SO(q) \to P \to M$ be the principal bundle of (oriented) transverse orthonormal framings. Then a transverse spin structure is a principal $Spin(q)$-bundle $\tilde{P}$ together with two sheeted covering $\xi: \tilde{P} \to P$ such that $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in \tilde{P}$, $g \in Spin(q)$, where $\xi_0: Spin(q) \to SO(q)$ is a covering. In this case, the
foliation $\mathcal{F}$ is called a transverse spin foliation. We then define the vector bundle $S$ associated with $\tilde{P}$ by
\begin{equation}
S = \tilde{P} \times_{Spin(q)} S_q,
\end{equation}
where $S_q$ is the irreducible spinor space associated to $Q$. The Hermitian metric on $S$ is induced from $g_Q$, and the Riemannian connection $\nabla$ on $P$ can be lifted to one on $\tilde{P}$, in particular, to one on $S$, which will be denoted by the same letter. $S$ is called the foliated spinor bundle. It is well known that the curvature transform $R^S$ is given as
\begin{equation}
R^S_{XY} \Phi = \frac{1}{4} \sum_{a,b} g_Q(R^\nabla_{XY} E_a, E_b)E_a \cdot E_b \Phi \quad \text{for } X, Y \in TM.
\end{equation}
On the foliated spinor bundle $S$, we have
\begin{equation}
R^\nabla = \frac{1}{4} \sigma^\nabla,
\end{equation}
\begin{equation}
\sum_a E_a \cdot R^S_{XE_a} \Phi = -\frac{1}{2} \rho^\nabla (X) \cdot \Phi
\end{equation}
for $X \in \Gamma Q([8])$. The transverse Dirac operator $D_{tr}$ on $S(\mathcal{F})$ is locally defined by
\begin{equation}
D_{tr} \Phi = \sum_a E_a \cdot \nabla_{E_a} \Phi - \frac{1}{2} \kappa \cdot \Phi,
\end{equation}
where $\{E_a\}$ is an orthonormal basis of $Q$, the normal bundle of the foliation $\mathcal{F}$ and $\kappa$ is a mean curvature form of $\mathcal{F}$ (See [2,4,8]). From (3.6) we know ([4,8]) that on an isoparametric transverse spin foliation with $\delta \kappa = 0$, the transverse Dirac operator $D_{tr}$ satisfies
\begin{equation}
D_{tr} = \nabla_{tr}^* \nabla_{tr} + \frac{1}{4} \sigma^\nabla + \frac{1}{4} |\kappa|^2.
\end{equation}
If $\mathcal{F}$ is a minimal foliation, we know that
\begin{equation}
D_{tr}^2 = d_B \delta_B + \delta_B d_B = \Delta_B.
\end{equation}

4. THE ESTIMATES

Let us introduce a new connection $\tilde{\nabla}$ on $S$ as
\begin{equation}
\tilde{\nabla}_X \Phi = \nabla_X \Phi + f \pi(X) \cdot \Phi \quad \text{for } X \in TM,
\end{equation}
where $f$ is a real valued basic function on $M$ and $\pi : TM \to Q$. Trivially, this connection $\tilde{\nabla}$ is a metric connection. Using this connection, we have
\begin{equation}
\tilde{\nabla}_{tr} \tilde{\nabla}_{tr} \Phi = \nabla_{tr}^* \nabla_{tr} \Phi - 2 f D_{tr} \Phi - \text{grad}_{\nabla} (f) \cdot \Phi + q f^2 \Phi,
\end{equation}
where $\text{grad}_{\nabla} (f) = \sum_a E_a (f) E_a$ is a transversal gradient of $f$. From (3.6) and (4.2), we have
\begin{equation}
\tilde{\nabla}_{tr}^* \tilde{\nabla}_{tr} \Phi = D_{tr}^2 \Phi - 2 f D_{tr} \Phi - \text{grad}_{\nabla} (f) \cdot \Phi + (q f^2 - \frac{1}{4} (\sigma^\nabla + |\kappa|^2)) \Phi.
\end{equation}
Let $D_{tr} \Phi = \lambda \Phi (\Phi \neq 0)$. Note that for all $X \in \Gamma Q$ and $\Phi \in \Gamma S$,
\begin{equation}
< X \cdot \Phi, \Phi > = < \Phi, X \cdot \Phi > = -< X \cdot \Phi, \Phi >.
\end{equation}
Hence (4.4) implies that $\langle \text{grad}_\nabla (f) \cdot \Phi, \Phi \rangle$ is a pure imaginary. Hence we have

$$\| f \nabla_{\text{tr}} \Phi \|^2 = \int_M (\lambda^2 - 2f\lambda + qf^2 - \frac{1}{4}(\sigma^\nabla + |\kappa|^2))|\Phi|^2,$$

(4.5)

$$\langle \text{grad}_\nabla (f) \cdot \Phi, \Phi \rangle = 0.$$

(4.6)

If we put $f = \frac{1}{q}$, then from (4.5), we have

$$\| f \nabla_{\text{tr}} \Phi \|^2 = \int_M (\frac{q-1}{q}\lambda^2 - \frac{1}{4} K_\sigma)|\Phi|^2,$$

where $K_\sigma = \sigma^\nabla + |\kappa|^2$. From (3.12), we have the following theorem (See [8]).

**Theorem 4.1.** Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with an isoparametric transverse spin foliation $\mathcal{F}$ of codimension $q > 1$ and bundle-like metric $g_M$ with respect to $\mathcal{F}$. Assume that the mean curvature $\kappa$ of $\mathcal{F}$ satisfies $\delta \kappa = 0$ and $\sigma^\nabla + |\kappa|^2 \geq 0$. Then the eigenvalue $\lambda$ of the transverse Dirac operator $D_{\text{tr}}$ satisfies

$$\lambda^2 \geq \frac{1}{4} q - 1 \min\{\sigma^\nabla + |\kappa|^2\}.$$

In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian with constant scalar curvature.

**Remark.** If $\mathcal{F}$ is a point foliation, then the transversal Dirac operator is just a Dirac operator on an ordinary manifold. Therefore Theorem 4.1 is a generalization of the result on an ordinary manifold (cf.[3]).

By the conformal change of $g_Q$, we can obtain the sharper inequality than Theorem 4.1. (see [6] for details)

**Theorem 4.2.** Under the same conditions as in Theorem 4.1 except for $q \geq 3$, we have

$$\lambda^2 \geq \frac{1}{4} q - 1 (\mu_1 + \min|\kappa|^2),$$

where $\mu_1$ is the smallest eigenvalue of the basic Yamabe operator $Y_0 = 4^{\frac{q-1}{q-2}} \Delta_B + \sigma^\nabla$. In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian with constant transversal scalar curvature.

On the other hand, on a Kähler spin foliation, the basic 2-form $\Omega$ play an important roles to estimate the eigenvalue of the transversal Dirac operator $D_{\text{tr}}$. In fact, $\Omega$ acts on $S$ and then $S$ splits into the orthogonal direct sum

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_n \quad (q = 2n).$$

By using this decomposition, we obtain the following theorem (see [7]).

**Theorem 4.3.** Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with Kähler spin foliation $\mathcal{F}$ of codimension $q = 2n$ and a bundle-like metric $g_M$ with $\kappa \in \Omega_B$. Assume that the mean curvature form $\kappa$ satisfies $\delta \kappa = 0$ and transversally holomorphic. If $\sigma^\nabla + |\kappa|^2 \geq 0$, then the eigenvalue $\lambda$ of $D_{\text{tr}}$ satisfies

$$\lambda^2 \geq \frac{q+2}{4q} \min(\sigma^\nabla + |\kappa|^2).$$

In the limiting case, $\mathcal{F}$ is minimal, transversally Einsteinian of odd complex codimension $n$ with nonnegative constant transversal scalar curvature.
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Problem. The estimates in Theorem 4.1, 4.2 and 4.3 imply that if the foliation $\mathcal{F}$ is not minimal, the inequality is strict. So, we can obtain a shaper estimates than the ones in theorems in case of non-minimal.

References


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