HEREDITARY PROPERTIES OF CERTAIN IDEALS OF COMPACT OPERATORS

Chong-Man Cho and Eun Joo Lee

Abstract. Let $X$ be a Banach space and $Z$ a closed subspace of a Banach space $Y$. Denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from $X$ to $Y$ and by $\mathcal{K}(X, Y)$ its subspace of compact linear operators. Using Hahn-Banach extension operators corresponding to ideal projections, we prove that if either $X^{**}$ or $Y^*$ has the Radon-Nikodým property and $\mathcal{K}(X, Y)$ is an $M$-ideal (resp. an $HB$-subspace) in $\mathcal{L}(X, Y)$, then $\mathcal{K}(X, Z)$ is also an $M$-ideal (resp. $HB$-subspace) in $\mathcal{L}(X, Z)$. If $\mathcal{K}(X, Y)$ has property $SU$ instead of being an $M$-ideal in $\mathcal{L}(X, Y)$ in the above, then $\mathcal{K}(X, Z)$ also has property $SU$ in $\mathcal{L}(X, Z)$. If $X$ is a Banach space such that $X^*$ has the metric compact approximation property with adjoint operators, then $M$-ideal (resp. $HB$-subspace) property of $\mathcal{K}(X, Y)$ in $\mathcal{L}(X, Y)$ is inherited to $\mathcal{K}(X, Z)$ in $\mathcal{L}(X, Z)$.

1. Introduction

A closed subspace $E$ of a Banach space $X$ is called an ideal in $X$ if $E^\perp$, the annihilator of $E$ in $X^*$, is the kernel of a norm one projection $P$ on $X^*$. In this case $P$ is called the ideal projection. The notion of an ideal in a Banach space was introduced by Godefroy, Kalton and Shaper [4] in 1993.

Let $E$ be an ideal in $X$ with the ideal projection $P$ on $X^*$, let $x^* \in X^*$ and consider the following norm conditions:

\begin{align*}
(1.1) \quad \|x^*\| &= \|Px^*\| + \|(I - P)x^*\|, \\
(1.2) \quad \|x^*\| &> \|Px^*\| \quad \text{if } x^* \neq Px^*, \\
(1.3) \quad \|x^*\| &\geq \|x^* - Px^*\|,
\end{align*}

Received September 9, 2003.
2000 Mathematics Subject Classification: 46B20, 46B28.
Key words and phrases: ideal, $M$-ideal, $HB$-subspace, property $SU$, compact operator.

The first named author was supported by Hanyang University, Korea, in the program year of 2002.
\[ \|x^*\| \geq \|x^* - 2Px^*\|. \]  

(1.4)

An ideal \(E\) is called an \(M\)-ideal if the condition (1.1) holds for all \(x^* \in X^*\). An \(M\)-ideal was introduced by Alfsen and Effros [1] in 1972 and has been studied seriously by many authors [5].

Following Hennefeld [6], an ideal \(E\) is called an \(HB\)-subspace if conditions (1.2) and (1.3) hold for all \(x^* \in X^*\). It is easy to see that an \(HB\)-subspace has property \(U\) in the sense of Phelps. According to Phelps [17], a subspace \(E\) of a Banach space \(X\) is said to have property \(U\) in \(X\) if every \(e^* \in E^*\) has a unique norm-preserving extension \(x^* \in X^*\). E. Oja [15] defined property \(SU\) which is an intermediate property between property \(U\) and \(HB\)-subspace. A subspace \(E\) is said to have property \(SU\) in \(X\) if \(E\) is an ideal in \(X\) and the condition (1.2) holds for all \(x^* \in X^*\).

An ideal \(E\) is called a \(u\)-ideal if condition (1.4) holds. A \(u\)-ideal was introduced by Casazza and Kalton [2].

An ideal is closely linked with a Hahn-Banach extension operator. For a closed subspace \(E\) of a Banach space \(X\) a linear operator \(\phi : E^* \to X^*\) is called a Hahn-Banach extension operator if \(\phi(e^*)\) is a norm preserving extension of \(e^*\) for all \(e^* \in E^*\). It is well known that there exists a Hahn-Banach extension operator \(\phi : E^* \to X^*\) if and only if \(E\) is an ideal in \(X\). In this case, the Hahn-Banach extension operator \(\phi\) and the corresponding ideal projection \(P : X^* \to X^*\) are related by \(Px^* = \phi(x^*|_E)\), where \(x^*|_E\) is the restriction of \(x^*\) to \(E\). Therefore, if a subspace \(E\) is an ideal with property \(U\) in \(X\), then the ideal projection is unique.

Let \(X\) and \(Y\) be Banach spaces. We denote by \(\mathcal{L}(X, Y)\) the space of all bounded linear operators from \(X\) to \(Y\) and by \(\mathcal{K}(X, Y)\) its subspace of compact operators.

In 1994, Lima, Oja, Rao and Werner [14] proved a sort of hereditary property of an \(M\)-ideal for \(\mathcal{K}(X, Y)\). More specifically, they proved the following results.

**Theorem 1.1.** Suppose that \(X^{**}\) or \(Y^*\) has the Radon-Nikodým property and that \(\mathcal{K}(X, Y)\) is an \(M\)-ideal in \(\mathcal{L}(X, Y)\).

(a) If \(X^*\) has the bounded compact approximation property with adjoint operators and \(Z\) is a closed subspace of \(Y\), then \(\mathcal{K}(X, Z)\) is an \(M\)-ideal in \(\mathcal{L}(X, Z)\).

(b) If \(Y^*\) has the bounded compact approximation property with adjoint operators and \(E\) is a closed subspace of \(X\), then \(\mathcal{K}(X/E, Y)\) is an \(M\)-ideal in \(\mathcal{L}(X/E, Y)\).
In this paper, we will investigate various ideal properties of $\mathcal{K}(X, Z)$ in $\mathcal{L}(X, Z)$ inherited from those of $\mathcal{K}(X, Y)$ in $\mathcal{L}(X, Y)$ for a closed subspace $Z$ of $Y$.

In Theorem 3.3, we will assume that $X$ and $Y$ are Banach spaces such that either $X^{**}$ or $Y^*$ has the Radon-Nikodým property and show that if $\mathcal{K}(X, Y)$ is an $M$-ideal (resp. an $HB$-subspace) in $\mathcal{L}(X, Y)$, then $\mathcal{K}(X, Z)$ is an $M$-ideal (resp. an $HB$-subspace) in $\mathcal{L}(X, Z)$. If $\mathcal{K}(X, Y)$ has property $SU$ in $\mathcal{L}(X, Y)$, then $\mathcal{K}(X, Z)$ also has property $SU$ in $\mathcal{L}(X, Z)$. The idea of proofs is using suitable Hahn-Banach extension operators corresponding to ideal projections and using Feder-Saphar representation of the dual space of certain space of compact operators (Theorem 2.1).

In Theorem 3.5 we prove that if $X^*$ has the metric compact approximation property with adjoint operators, then $M$-ideal (resp. $HB$-subspace) property of $\mathcal{K}(X, Y)$ in $\mathcal{L}(X, Y)$ is inherited to $\mathcal{K}(X, F)$ in $\mathcal{L}(X, F)$, where $F$ is a closed subspace of $Y$. The same properties are inherited to $\mathcal{K}(X/E, Y)$ in $\mathcal{L}(X/E, Y)$ if $Y$ has the metric compact approximation property, where $E$ is a closed subspace of $X$.

2. Preliminaries

A Banach space $X$ is said to have the compact approximation property if there exists a net $(K_\alpha)$ in $\mathcal{K}(X)$ such that $K_\alpha x \to x$ for all $x \in X$. If the net $(K_\alpha)$ in $\mathcal{K}(X)$ above can be chosen to be $\|K_\alpha\| \leq 1$ for all $\alpha$, then we say that $X$ has the metric compact approximation property. The dual space $X^*$ of $X$ is said to have the compact approximation property with adjoint operators if there exists a net $(K_\alpha)$ in $\mathcal{K}(X)$ such that $K_\alpha^* x^* \to x^*$ for all $x^* \in X^*$. We say that $X^*$ has the metric compact approximation property with adjoint operators if the net $(K_\alpha)$ above can be taken to be $\|K_\alpha\| \leq 1$ for all $\alpha$.

Let $X \hat{\otimes} Y$ be the projective tensor product of Banach spaces $X$ and $Y$. If $v \in X \hat{\otimes} Y$, then there exist sequences $(x_n)$ in $X$ and $(y_n)$ in $Y$ such that $v = \sum_{n=1}^{\infty} x_n \otimes y_n$, and $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$. Moreover, $\|v\| = \inf \{\sum_{n=1}^{\infty} \|x_n\| \|y_n\| \}$ with infimum being taken over all representations $v = \sum_{n=1}^{\infty} x_n \otimes y_n$, $x_n \in X$, $y_n \in Y$.

Let $v = \sum_{n=1}^{\infty} y_n^* \otimes x_n^* \in Y^* \hat{\otimes} X^{**}$ with $\sum_{n=1}^{\infty} \|y_n^*\| \|x_n^*\| < \infty$. For any $T \in \mathcal{L}(X, Y)$, we define $T^{**} v = \sum_{n=1}^{\infty} y_n^* \otimes T^{**} x_n^*$. Then $T^{**} v \in Y^* \hat{\otimes} Y^{**}$ and the map $T \to \text{trace}(T^{**} v) = \sum_{n=1}^{\infty} (T^{**} x_n^*)(y_n^*)$ defines a bounded linear functional on $\mathcal{L}(X, Y)$ with norm no larger than $\|v\|$.
The following theorem is originally due to M. Feder and P. Sapher [3] and slightly modified by Lima, Nygaard and Oja [9].

**Theorem 2.1.** Let $X$ and $Y$ be Banach spaces such that either $X^{**}$ or $Y^*$ has the Radon-Nikodým property. Let $\Psi : Y^* \widehat{\otimes} X^{**} \to \mathcal{L}(X,Y)^*$ be defined by
\[
(\Psi v)(T) = \text{trace}(T^{**}v)
\]
for all $T \in \mathcal{L}(X,Y)$ and $v \in Y^* \widehat{\otimes} X^{**}$. Then $\tau^* \Psi : Y^* \widehat{\otimes} X^{**} \to \mathcal{K}(X,Y)^*$ is a quotient map, where $\tau : \mathcal{K}(X,Y) \to \mathcal{L}(X,Y)$ is the inclusion map. Moreover, for each $f \in \mathcal{K}(X,Y)^*$, there exists $v \in Y^* \widehat{\otimes} X^{**}$ such that $f = (\Psi v)|_{\mathcal{K}(X,Y)}$ and $\|f\| = \|\Psi v\|$. 

3. Hereditary properties of ideals of compact operators.

Let $X$ and $Y$ be Banach spaces, and let $E \subset X$ and $F \subset Y$ be closed subspaces. In this section we will investigate various ideal properties of $\mathcal{K}(X/E, F)$ in $\mathcal{L}(X/E, F)$ inherited from the corresponding ideal properties of $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$.

Let $\pi : X \to X/E$ be the canonical projection and $i : F \to Y$ the inclusion mapping. Define $I : \mathcal{L}(X/E, F) \to \mathcal{L}(X,Y)$ and $J : \mathcal{K}(X/E, F) \to \mathcal{K}(X,Y)$ by $I(T) = i \circ T \circ \pi$ and $J(K) = i \circ K \circ \pi$ for $T \in \mathcal{L}(X/E, F)$ and $K \in \mathcal{K}(X/E, F)$, respectively. Then $I$ and $J$ are isometries into $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$, respectively. By a diagram chase we can easily check the following Lemma.

**Lemma 3.1.** Let $X$ and $Y$ be Banach spaces and let $E \subset X$ and $F \subset Y$ be closed subspaces. If $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ is a Hahn-Banach extension operator and $\phi_2 : \mathcal{K}(X/E, F)^* \to \mathcal{L}(X/E, F)^*$ is a linear operator such that $I^* \circ \phi_1 = \phi_2 \circ J^*$, then $\phi_2$ is a Hahn-Banach extension operator.

**Proposition 3.2.** Let $X$ and $Y$ be Banach spaces and let $E \subset X$ and $F \subset Y$ be closed subspaces. Suppose that $\mathcal{K}(X,Y)$ is an $M$-ideal (resp. an HB-subspace, or a u-ideal) in $\mathcal{L}(X,Y)$ with an ideal projection $P$. If $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ is the Hahn-Banach extension operator associated with $P$ and $\phi_2 : \mathcal{K}(X/E, F)^* \to \mathcal{L}(X/E, F)^*$ is a linear operator such that $I^* \circ \phi_1 = \phi_2 \circ J^*$,
then $K(X/E, F)$ is an $M$-ideal (resp. an $HB$-subspace, or a $u$-ideal) in $\mathcal{L}(X/E, F)$. If $K(X, Y)$ has property $SU$ in $\mathcal{L}(X, Y)$, then $K(X/E, F)$ also has property $SU$ in $\mathcal{L}(X/E, F)$.

**Proof.** By Lemma 3.1 $\phi_2$ is a Hahn-Banach extension operator, and so $K(X/E, F)$ is an ideal in $\mathcal{L}(X/E, F)$. Let $Q$ be the ideal projection on $\mathcal{L}(X/E, F)^*$ induced by $\phi_2$. Then for $f \in \mathcal{L}(X, Y)^*$ and $g \in \mathcal{L}(X/E, F)^*$ we have $Pf = \phi_1(f)$ and $Qg = \phi_2(g)$, where $\bar{f}$ and $\bar{g}$ are the restrictions of $f$ and $g$ to $K(X, Y)$ and $K(X/E, F)$, respectively.

If $g \in \mathcal{L}(X/E, F)^*$, then there exists $f \in \mathcal{L}(X, Y)^*$ such that $\|g\| = \|f\|$, $g = I^*f$, and so $\bar{g} = J^*\bar{f}$. Therefore, $Qg = \phi_2(\bar{g}) = \phi_2(J^*\bar{f}) = I^*\phi_1\bar{f} = I^*Pf$.

If $K(X, Y)$ is an $M$-ideal in $\mathcal{L}(X, Y)$, then
\[
\|g\| \leq \|I^*Pf\| + \|I^*f - I^*Pf\|
\leq \|Pf\| + \|f - Pf\|
= \|f\| = \|g\|
\]
and hence
\[
\|g\| = \|I^*Pf\| + \|I^*f - I^*Pf\|
= \|Qg\| + \|g - Qg\|.
\]
Therefore, $K(X/E, F)$ is an $M$-ideal in $\mathcal{L}(X/E, F)$.

If $K(X, Y)$ has property $SU$ in $\mathcal{L}(X, Y)$ and $g \neq Qg$, then $f \neq Pf$ and so
\[
\|g\| = \|f\| > \|Pf\| \geq \|I^*Pf\| = \|Qg\|.
\]
Therefore, $K(X/E, F)$ has property $SU$ in $\mathcal{L}(X/E, F)$.

The inequalities,
\[
\|g\| = \|f\| \geq \|f - Pf\| \geq \|I^*f - I^*Pf\| = \|g - Qg\|
\]
and
\[
1 \geq \|f - 2Pf\| \geq \|I^*f - 2I^*Pf\| = \|g - 2Qg\|
\]
prove $HB$-subspaces and a $u$-ideal cases.

In the above Proposition the existence of a linear operator $\phi_2 : K(X/E, F)^* \to \mathcal{L}(X/E, F)^*$ such that $I^* \circ \phi_1 = \phi_2 \circ J^*$ plays a key role. An interesting question is when such an operator exists. Theorem 3.3 and Theorem 3.5 are cases in which such operators exist.

**Theorem 3.3.** Let $X$ and $Y$ be Banach spaces such that either $X^{**}$ or $Y^*$ has the Radon-Nikodým property. If $K(X, Y)$ is an $M$-ideal (resp. an $HB$-subspace) in $\mathcal{L}(X, Y)$, then for every closed subspace $Z$ of $Y$ $K(X, Z)$ is an $M$-ideal (resp. an $HB$-subspace) in $\mathcal{L}(X, Z)$. If $K(X, Y)$
has property \( SU \) in \( \mathcal{L}(X,Y) \), then \( \mathcal{K}(X,Z) \) also has property \( SU \) in \( \mathcal{L}(X,Z) \) for every closed subspace \( Z \) of \( Y \).

**Proof.** Let \( Z \) be a closed subspace of \( Y \). Observe that if \( Y^* \) has the Radon-Nikodým property, then \( Z^* \) also has the Radon-Nikodým property. Let \( \Psi : Z^* \otimes X^{**} \to \mathcal{L}(X,Z)^* \) be defined as in Theorem 2.1. Since \( \mathcal{K}(X,Z)^* = Z^* \otimes X^{**}/\ker \Psi \), there exists a bounded linear operator \( \phi_2 : \mathcal{K}(X,Z)^* \to \mathcal{L}(X,Z)^* \) such that \( \Psi = \phi_2 \circ \rho \), where \( \rho : Z^* \otimes X^{**} \to Z^* \otimes X^{**}/\ker \Psi \) is the canonical projection. Then we have that

\[
\phi_2(z^* \otimes x^{**}) = z^* \otimes x^{**} \quad \text{for all } x^{**} \in X^{**} \text{ and } z^* \in Z^*.
\]

Let \( \phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^* \) be the Hahn-Banach extension operator corresponding to the ideal projection \( P \) on \( \mathcal{L}(X,Y)^* \) with \( \ker P = \mathcal{K}(X,Y)^\perp \). By Proposition 3.2, it suffices to show that \( I^* \circ \phi_1 = \phi_2 \circ J^* \).

Since either \( X^{**} \) or \( Y^* \) has the Radon-Nikodým property, by Theorem 2.1, every \( f \in \mathcal{K}(X,Y)^* \) has a representation

\[
f = \sum_n y_n^* \otimes x_n^{**}, \quad \sum_n ||x_n^{**}|| ||y_n^*|| < \infty, \quad x_n^{**} \in X^{**}, \quad y_n^* \in Y^*.
\]

Therefore, it is sufficient to show that for each \( x^{**} \in X^{**} \) and \( y^* \in Y^* \),

\[
J^* \circ \phi_1(x^* \otimes x^{**}) = \phi_2 \circ J^*(y^* \otimes x^{**}).
\]

Since \( \mathcal{K}(X,Y) \) has property \( U \) in \( \mathcal{L}(X,Y) \), we have

\[
\phi_1(y^* \otimes x^{**}) = y^* \otimes x^{**} \quad \text{for all } x^{**} \in X^{**} \text{ and } y^* \in Y^*.
\]

Therefore, we have

\[
I^* \circ \phi_1(y^* \otimes x^{**}) = I^*(y^* \otimes x^{**}) = i^*y^* \otimes x^{**} = \phi_2(i^*y^* \otimes x^{**}) = \phi_2 \circ J^*(y^* \otimes x^{**}).
\]

\( \square \)

Since a reflexive Banach space has the Radon-Nikodým property, we have the following corollary.

**Corollary 3.4.** Let \( X \) be a reflexive Banach space and \( Z \) a closed subspace of a Banach space \( Y \). If \( \mathcal{K}(X,Y) \) is an \( M \)-ideal (resp. an \( HB \)-subspace) in \( \mathcal{L}(X,Y) \), then \( \mathcal{K}(X,Z) \) is an \( M \)-ideal (resp. an \( HB \)-subspace) in \( \mathcal{L}(X,Z) \). If \( \mathcal{K}(X,Y) \) has property \( SU \) in \( \mathcal{L}(X,Y) \), then \( \mathcal{K}(X,Z) \) also has property \( SU \) in \( \mathcal{L}(X,Z) \).
Hereditary properties of certain ideals of compact operators 463

In 1979, J. Johnson [7] proved that if \( X \) and \( Y \) are Banach spaces, and \( Y \) has the metric compact approximation property, then there exists a Hahn Banach extension operator \( \phi : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^* \). His construction of \( \phi \) easily proves the following Theorem (cf. [14, Proposition 2.9, Corollary 2.10]).

**Theorem 3.5.** Let \( X \) and \( Y \) be Banach spaces. Suppose that \( \mathcal{K}(X,Y) \) is an \( M \)-ideal (resp. an HB-subspace) in \( \mathcal{L}(X,Y) \).

(a) If \( X^* \) has the metric compact approximation property with adjoint operators, then \( \mathcal{K}(X,F) \) is an \( M \)-ideal (resp. an HB-subspace) in \( \mathcal{L}(X,F) \) for every closed subspace \( F \) of \( Y \).

(b) If \( Y \) has the metric compact approximation property, then \( \mathcal{K}(X/E,Y) \) is an \( M \)-ideal (resp. an HB-subspace) in \( \mathcal{L}(X/E,Y) \) for every closed subspace \( E \) of \( X \).

**Proof.** (a) Let \((K_\alpha)\) be a net in \( K(X) \) such that \( \|K_\alpha\| \leq 1 \) for all \( \alpha \) and \( K_\alpha x^* \to x^* \) for all \( x^* \in X^* \). Then, by passing to a subnet of \((K_\alpha)\), which we still denote by \((K_\alpha)\), we can define Hahn-Banach extension operators [14, Lemma 1] \( \phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^* \) and \( \phi_2 : \mathcal{K}(X,F)^* \to \mathcal{L}(X,F)^* \) by

\[
\phi_1(f)(T) = \lim_\alpha f(T \circ K_\alpha), \quad f \in \mathcal{K}(X,Y)^*, T \in \mathcal{L}(X,Y)
\]

and

\[
\phi_2(g)(T) = \lim_\alpha g(T \circ K_\alpha), \quad g \in \mathcal{K}(X,F)^*, T \in \mathcal{L}(X,F).
\]

Then we can easily check that \( I^* \circ \phi_1 = \phi_2 \circ J^* \). Now we appeal to Proposition 3.2 to finish the proof.

(b) We choose a suitable net \((S_\alpha)\) of compact operators on \( Y \) such that \( \|S_\alpha\| \leq 1 \) for all \( \alpha \) and \( S_\alpha y \to y \) for all \( y \in Y \); we can define Hahn-Banach extension operators [14, Lemma 1] \( \phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^* \) and \( \phi_2 : \mathcal{K}(X/E,Y)^* \to \mathcal{L}(X/E,Y)^* \) by

\[
\phi_1(f)(T) = \lim_\alpha f(S_\alpha \circ T), \quad f \in \mathcal{K}(X,Y)^*, T \in \mathcal{L}(X,Y)
\]

and

\[
\phi_2(g)(T) = \lim_\alpha g(S_\alpha \circ T), \quad g \in \mathcal{K}(X/E,Y)^*, T \in \mathcal{L}(X/E,Y).
\]

Then \( I^* \circ \phi_1 = \phi_2 \circ J^* \) and another appeal to Proposition 3.2 finishes the proof.
References