Bull. Korean Math. Soc. **41** (2004), No. 2, pp. 299–306

# ON TWISTED GENERALIZED EULER NUMBERS

Yilmaz Simsek

Abstract. In this paper, we shall construct generating function of twisted generalized Euler numbers. By using this function, we shall define twisted generalized Euler polynomials and numbers. We shall give some basic properties of these polynomials and numbers.

## 1. Introduction

In the case of Euler numbers, which are classical and important in number theory, we consider the coefficients of the expansion of $\frac{2}{e^t+1}$ and $\frac{1}{\cosh t}$ [10]

$$e^{Ht} = \frac{2}{e^t+1} = \sum_{k=0}^{\infty} H_k \frac{t^k}{k!},$$
$$e^{Tt} = \frac{1}{\cosh t} = \sum_{k=0}^{\infty} T_k \frac{t^k}{k!},$$

where the symbols $H_k$ and $T_k$ are interpreted to mean that $H^k$ (respectively $T^k$) must be replaced by $H_k$ (respectively $T_k$ ) when we expand the one on the left. We easily see that

$$T_k = (2H+1)^k.$$

The recurrence formula for the Euler numbers has the form

$$(T+1)^n + (T-1)^n = 0, \text{ for } n \geq 1$$
$$T_0 = 1, \text{ for } n = 0.$$

Consequently, $T_{2n+1} = 0, T_{4n}$ are positive and $T_{4n+2}$ are negative integers for all $n = 0, 1, 2, \ldots; T_2 = -1, T_4 = 5, T_6 = -61, T_8 = 1385, T_{10} = -50521 \ldots$ .

Received December 12, 2002.
2000 Mathematics Subject Classification: Primary 11B68; Secondary 11M41.
Key words and phrases: Euler polynomials and numbers, generalized Euler polynomials and numbers, twisted generalized Euler polynomials and numbers.

These numbers are connected with the Bernoulli numbers. They are also used in the summation of infinite series.

Frobenius (1910) extended $H_n$ to the Euler numbers $H^n(u)$ belonging to an algebraic number $u$, and many authors (e.g. Carlitz [1], [2], Shiratani [7], Tsumura [11], Son and Kim [10], Uehara [12]) investigated their properties. Recently, Shiratani and Yamamoto [8] and Tsumura [11] constructed a $p$-adic interpolation function of the Euler numbers, $H^n(u)$, and their applications. They also obtained an explicit formula for $p$-adic $L$ functions, $L_p^{'}(0,\chi)$, with any Dirichlet character $\chi$. Uehara [12] gave a relation between Euler numbers and $p$-adic $L$-functions.

In [3], by using the $p$-adic $q$-integral and p-adic measure, Kim and Son gave a different construction of Euler numbers and generalized Euler numbers. Satoh [6] constructed a complex and a $p$-adic $q$-$l$-series which interpolate $q$-Bernoulli numbers and $q$-Euler numbers, respectively. Satoh also gave generalized Euler numbers.

In [11], Tsumura defined the generalized Euler numbers $H_{\chi}^n(u)$ for any Dirichlet character, $\chi$, which are analogues to the generalized Bernoulli numbers, and he constructed their $p$-adic interpolation function of generalized Euler numbers.

The aim of this paper is to define twisted generalized Euler polynomials and numbers which are used in the summation of infinite series and twisted Dirichlet $L$ series.

In Section 2, relations between generating function of Euler polynomials and character analogue of generating function of generalized Euler polynomials and numbers are defined. By using these generating functions, proof of generalized Euler numbers and polynomials are given.

In Section 3, generating functions of twisted generalized Euler numbers and polynomials are constructed. Some properties of these numbers and polynomials are given.

## 2. Generalized euler polynomials and numbers

Generalized Euler numbers are given as follows [11]: Let $u \neq 0$ be an algebraic number. The number $H^n(u)$ defined by

$$F_u(t) = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H^n(u) \frac{t^n}{n!} \tag{2.1}$$

is called the $n$-th Euler number belonging to $u$. The polynomial $E_n(u,x) \in \mathbb{Q}(u)[x]$ defined by

$$F_u(t,x) = \frac{(1-u)e^{tx}}{e^t - u} = \sum_{n=0}^{\infty} E_n(u,x)\frac{t^n}{n!} \tag{2.2}$$

is called the $n$-th Euler polynomial belonging to $u$. Thus we have

$$F_u(t,x) = e^{tx}F_u(t).$$

As is well known,

$$E_n(u,1-x) = (-1)^n E_n(u^{-1},x). \tag{2.3}$$

Lemma 1.

$$E_n(u,x) = \sum_{k=0}^{n} \binom{n}{k} H^k(u)x^{n-k}. \tag{2.4}$$

*Proof of Lemma 1.* By using (2.2), we have

$$\sum_{n=0}^{\infty} E_n(u,x)\frac{t^n}{n!} = (\sum_{n=0}^{\infty} \frac{x^n t^n}{n!})(\sum_{n=0}^{\infty} H^n(u)\frac{t^n}{n!}).$$

By applying Cauchy product in the above, we obtain

$$\sum_{n=0}^{\infty} E_n(u,x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} \binom{n}{k} H^k(u)x^{n-k})\frac{t^n}{n!}.$$

By comparing coefficient of $\frac{t^n}{n!}$ in the above series, we obtain the desired result. □

The recurrence formula for the $n$-th Euler number belonging to $u$, $H^n(u)$ has the form:

$$H_0(u) = 1, (H(u)+1)^k - uH_k(u) = 0, (k \geq 1),$$

with the usual convention of replacing $H^k$ by $H_k$. Consequently,

$$uH_k(u) = \sum_{m=0}^{k} \binom{k}{m} H^m(u),$$

and

$$(u-1)H_k(u) = \sum_{m=0}^{k-1} \binom{k}{m} H^m(u),$$

for $u \neq 1$ and $k \geq 1$. (For detail see [7], [10], [11]).

Let $\chi$ be a primitive Dirichlet character with conductor $f$. The $n$-th generalized Euler number $H_{\chi}^{n}(u)$ belonging to $u$ is defined by [11]

$$F_{u,\chi}(t)=\sum_{a=0}^{f-1}\frac{(1-u^{f})\chi(a)e^{at}u^{f-a-1}}{e^{ft}-u^{f}}=\sum_{n=0}^{\infty}H_{\chi}^{n}(u)\frac{t^{n}}{n!}. \tag{2.5}$$

Note that when $\chi=1$, we have $H_{1}^{n}(u)=H^{n}(u)$, for $n\geq 0$.The $n$-th generalized Euler polynomial $E_{\chi}^{n}(u,x)$ belonging to $u$ is defined by

$$F_{u,\chi}(t,x)=\sum_{a=0}^{f-1}\frac{(1-u^{f})\chi(a)e^{(a+x)t}u^{f-a-1}}{e^{ft}-u^{f}}=\sum_{n=0}^{\infty}E_{\chi}^{n}(u,x)\frac{t^{n}}{n!}. \tag{2.6}$$

Thus we have

$$F_{u,\chi}(t,x)=F_{u,\chi}(t)e^{tx}.$$

Note that when $\chi=1$, we have $E_{1}^{n}(u,x)=E^{n}(u,x)$, for $n\geq 0$.

By using (2.2) in (2.6), we arrive at the following

Lemma 2. *Let $\chi$ be a primitive Dirichlet character with conductor $f$.*

$$F_{u,\chi}(t,x)=\sum_{a=0}^{f-1}\chi(a)u^{f-a-1}F_{u^{f}}(ft,\frac{a+x}{f}),$$

*where $F_{u}(t,x)$ is defined in* (2.2).

Generalized Euler polynomials are given explicitly as follows:

Proposition 1. *Let $\chi$ be a primitive Dirichlet character with conductor $f$. For $n\geq 0$,*

$$E_{\chi}^{n}(u,x)=f^{n}\sum_{a=0}^{f-1}\chi(a)u^{f-a-1}E_{n}(u^{f},\frac{a+x}{f}).$$

*Proof.* By using Lemma 2 and (2.2) we have

$$F_{u,\chi}(t,x)=\sum_{n=0}^{\infty}(\sum_{a=0}^{f-1}\chi(a)u^{f-a-1}f^{n}E_{n}(u^{f},\frac{a+x}{f}))\frac{t^{n}}{n!}.$$

By comparing coefficient of $\frac{t^{n}}{n!}$ in both sides of (2.6) and the above series, we have the desired result. □

We shall give some proposition without proof. Their proofs are quite similar to those of Lemma 1 and Proposition 1.

Proposition 2. *Let $\chi$ be a primitive Dirichlet character with conductor $f$. For $n \geq 0$,*

$$H^n_\chi(u) = f^n \sum_{a=0}^{f-1} \chi(a) u^{f-a-1} E_n(u^f, \frac{a}{f}) \tag{2.7}$$
$$= \sum_{a=0}^{f-1} \chi(a) u^{f-a-1} \sum_{m=0}^{n} \binom{n}{m} H^m(u^f) a^{n-m} f^m.$$

Proposition 3. *Let $\chi$ be a primitive Dirichlet character with conductor $f$. For $n \geq 0$,*

$$E^n_\chi(u, x) = \sum_{k=0}^{n} \binom{n}{k} H^n_\chi(u) x^{n-k}.$$

In [5], by applying the $n$-th generalized Euler numbers and $p$-adic analytic function, Kozuka studied on $p$-adic Dedekind sums.

## 3. Twisted generalized Euler numbers and polynomials

In this section we define generating function of twisted Euler numbers. By using this function, we define generating function of twisted Euler polynomials. We investigate some basic properties of these numbers and polynomials. We shall explicitly determine the generating function $F_{u,\chi,\xi}(t)$ of twisted generalized Euler numbers, $H^n_{\chi,\xi}(u)$. Generating function of twisted generalized Euler numbers are given as follows:

Lemma 3. *Let $\chi$ be a primitive Dirichlet character with conductor $f$ and let $\xi$ be $r$-th root of 1. Then*

$$F_{u,\chi,\xi}(t) = \sum_{a=0}^{f-1} \frac{(1-u^{rf})\chi(a)\xi^a e^{at} u^{f-a-1}}{\xi^f e^{ft} - u^f} = \sum_{n=0}^{\infty} H^n_{\chi,\xi}(u) \frac{t^n}{n!},$$

*where $H^n_{\chi,\xi}(u)$ is called $n$-th twisted generalized Euler number belonging to $u$ and $\xi$.*

*Proof.* Replacing $\chi(a)$ by $\chi(a)\xi^a$ in (2.5), we have

$$\sum_{a=0}^{f-1} \frac{(1-u^f)\chi(a)\xi^a e^{at} u^{f-a-1}}{e^{ft} - u^f}.$$

Replacing $f$ by $fr$ in the above series, we obtain

$$F_{u,\chi,\xi}(t) = \sum_{a=0}^{rf-1} \frac{(1-u^{rf})\chi(a)\xi^a e^{at} u^{rf-a-1}}{e^{rft}-u^{rf}}$$
$$= \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \frac{(1-u^{rf})\chi(a)\xi^{a+bf} e^{(a+bf)t} u^{rf-a-fb-1}}{e^{rft}-u^{rf}}$$
$$= \sum_{a=0}^{f-1} \frac{(1-u^{rf})\chi(a)\xi^a e^{at} u^{rf-a-1}}{e^{rft}-u^{rf}} \sum_{b=0}^{r-1} (\frac{\xi^f e^{ft}}{u^f})^b.$$

Applying geometric progression in the second sum in the above and after some calculations, we obtain the desired result. □

By using definition of $F_u(t,x)$ in (2.2) and Lemma 3, we arrive at the following corollary:

Corollary 1. *Let $\chi$ be a primitive Dirichlet character with conductor $f$ and let $\xi$ be $r$-th root of 1.*

$$F_{u,\chi,\xi}(t,x) = \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \chi(a)\xi^{a+bf} u^{rf-a-bf-1} F_{u^{rf}}(trf, \frac{a+bf+x}{rf}). \tag{3.1}$$

By using (3.1), we have

$$F_{u,\chi,\xi}(t,x) = F_{u,\chi,\xi}(t)e^{tx} = \sum_{n=0}^{\infty} E^n_{\chi,\xi}(u,x)\frac{t^n}{n!}. \tag{3.2}$$

We shall explicitly determine twisted generalized Euler numbers,

$$E^n_{\chi,\xi}(u,x)$$

as follows:

Theorem 1. *Let $\chi$ be a primitive Dirichlet character with conductor $f$ and let $\xi$ be $r$-th root of 1. Then we have*

$$E^n_{\chi,\xi}(u,x) = (rf)^n \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \chi(a)\xi^{a+bf} u^{rf-a-bf-1} E_n(u^{rf}, \frac{a+bf+x}{rf}).$$

*Proof.* Replacing $u$ by $u^{rf}$ and $x$ by $\frac{a+bf+x}{rf}$ and $t$ by $trf$ in (2.2), we have

$$F_{u^{rf}}(trf, \frac{a+bf+x}{rf}) = \sum_{n=0}^{\infty} E^n(u^{rf}, \frac{a+bf+x}{rf})(rf)^n \frac{t^n}{n!}. \tag{3.3}$$

Substituting (3.3) into (3.1), we obtain

(3.4) $F_{u,\chi,\xi}(t,x)$

$$= \sum_{n=0}^{\infty}((fr)^n \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \chi(a)\xi^{a+bf}u^{rf-a-bf-1}E^n(u^{rf}, \frac{a+bf+x}{rf}))\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ in both sides of (3.2) and (3.4), we obtain the desired result. □

COROLLARY 2. *Let $\chi$ be a primitive Dirichlet character with conductor $f$ and let $\xi$ be $r$-th root of 1. Then*

$$H^n_{\chi,\xi}(u) = (rf)^n \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \chi(a)\xi^{a+bf}u^{rf-a-bf-1}E_n(u^{rf}, \frac{a+bf}{rf}).$$

Substituting $x=0$ into Theorem 1, we obtain the above corollary.

We shall next describe some simple properties of $H^n_{\chi,\xi}(u)$ and $E^n_{\chi,\xi}(u,x)$ as follows:

i) If $\chi = 1$, the principal character ($f=1$), and $r=1$, then $F_{u,\chi,\xi}(t) = F_u(t)$, so that $H^n_{1,1}(u) = H^n(u)$.

ii) If $r=1$, then $F_{u,\chi,\xi}(t) = F_{u,\chi}(t)$, so that $H^n_{\chi,1}(u) = H^n_\chi(u)$ and $E^n_{\chi,1}(u,x) = E^n_\chi(u,x)$.

iii) $E^n_{\chi,\xi}(u,0) = H^n_{\chi,\xi}(u), n \geq 0$.

iv)

$$H^0_{\chi,\xi}(u) = \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \chi(a)\xi^{a+bf}u^{rf-a-bf-1}E_0(u^{rf}, \frac{a+bf}{rf}).$$

By using (2.4), we have $E_0(u,x)=1$. Thus we get

$$H^0_{\chi,\xi}(u) = \sum_{a=0}^{f-1}\sum_{b=0}^{r-1} \chi(a)\xi^{a+bf}u^{rf-a-bf-1} = \sum_{a=0}^{f-1}\chi(a)\xi^a u^{rf-a-1}\sum_{b=0}^{r-1}(\frac{\xi^f}{u^f})^b.$$

By applying geometric progression in the second sum in the above and after some calculations, we obtain

$$H^0_{\chi,\xi}(u) = \frac{u^{rf}-1}{u^f-\xi^f}\sum_{a=0}^{f-1}\frac{\chi(a)\xi^a}{u^{a+1-f}}.$$

If $r=1$, then

$$H^0_{\chi,1}(u) = u^{f-1}\sum_{a=0}^{f-1}\frac{\chi(a)}{u^a}.$$

## References


[1] L. Carlitz, *A note on Euler numbers and polynomials*, Nagoya Math. J. **7** (1954), 35–43.

[2] ———, *Arithmetic properties of generalized Bernoulli numbers*, J. Reine. Angew. Math. **202** (1959), 174–182.

[3] M. -S. Kim and J. -W. Son, *Some remarks on a q-analogue of Bernoulli numbers*, J. Korean Math. Soc. **39** (2002), no. 2, 221–236.

[4] N. Koblitz, *A new proof of Certain formulas for p-adic L-functions,* Duke Math. J. **46** (1979), no. 2, 455–468.

[5] K. Kozuka, *On linear combinations of p-adic interpolating functions for the Euler numbers*, Kyushu J. Math. **54** (2000), no. 2, 403–421.

[6] J. Satoh, *q-analogue of Riemann's $\zeta$-function and q-Euler numbers*, J. Number Theory **31** (1989), no. 3, 346–362.

[7] K. Shiratani, *On Euler numbers*, Mem. Fac. Sci. Kyushu Univ. **27** (1975), 1–5.

[8] K. Shiratani and S. Yamamota, *On a p-adic interpolation function for the Euler numbers and its derivative*, Mem. Fac. Sci. Kyushu Univ. **39** (1985), no. 1, 113–125.

[9] Y. Simsek, *Analytic properties of twisted L-functions and Twisted Bernoulli numbers,* Submitted.

[10] J. -W. Son and M. -S. Kim, *On poly-Eulerian numbers*, Bull. Korean Math. Soc. **36** (1999), no. 1, 47–61.

[11] H. Tsumura, *On a p-adic interpolation of the generalized Euler numbers and its applications*, Tokyo J. Math. **10** (1987), no. 2, 281–293.

[12] T. Uehara, *On p-adic continuous functions determined by Euler numbers*, Rep. Fac. Sci. Engrg. Saga Univ. **8** (1980), no. 8, 1–8.



Mersin Universitesi, Fen Edebiyat Fakultesi Matematik, Bolumu Ciftlikkoy, Kampusu 33343, Mersin/Turkey

*E-mail*: ysimsek@mersin.edu.tr, yilmazsimsek@hotmail.com