REPRESENTING NATURAL NUMBERS AS UNIQUE SUMS OF
POSITIVE INTEGERS

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Abstract. It is known that each natural number can be written uniquely as a sum of Fibonacci numbers with suitably increasing indices. In 1960, Daykin showed that the sequence of Fibonacci numbers is the only sequence with this property. Consider here the problem of representing each natural number uniquely as a sum of positive integers taken from certain sequence allowing a fixed number, $l \geq 2$, of repetitions. It is shown that the $(l + 1)$-adic expansion is the only such representation possible.

1. Introduction

The problem of representing natural numbers by sums of Fibonacci numbers was first reported in 1952 by Lekkerkerker [2], who attributed it to E. Zeckendorf, with the result:

I. For each natural number $N$, there is one, and only one system of natural numbers $i_1, i_2, \ldots, i_d$ such that

$$N = u_{i_1} + u_{i_2} + \cdots + u_{i_d} \quad \text{and} \quad i_{\gamma+1} \geq i_{\gamma} + 2 \quad \text{for} \quad 1 \leq \gamma < d,$$

when $(u_n)$ is the sequence of Fibonacci numbers.

Later in 1960, Daykin [1] established the following converse of I.

II. The Fibonacci numbers form the only sequence of natural numbers $(u_n)$ for which I holds.

In so doing, Daykin introduced the concepts of Property P and the $(h, k)$-th Fibonacci sequence as follows: let $(a_n), (k_n)$ be two sequences of natural numbers.

Property P. For each natural number $N$, there is one and only one system
of natural numbers $i_1, i_2, \ldots, i_d$ such that

(1) \[ N = a_{i_1} + a_{i_2} + \cdots + a_{i_d}, \quad \text{with} \quad i_{\gamma+1} \geq i_{\gamma} + k \quad \text{for} \quad 1 \leq \gamma < d. \]

If $h, k$ are natural numbers such that $h \leq k \leq h + 1$, then the $(h, k)$-th Fibonacci sequence $(v_n)$ is defined by

\[ v_n = n \quad \text{for} \quad 1 \leq n \leq k, \]
\[ v_n = v_{n-1} + v_{n-h} \quad \text{for} \quad k < n < h + k, \]
\[ v_n = v_{n-1} + v_{n-k} + (k - h) \quad \text{for} \quad n \geq h + k. \]

With these concepts, Daykin also proved:

III. If $(v_n)$ is the $(h, k)$-th Fibonacci sequence, then for each natural number $N$, there is one and only one system of natural numbers $i_1, i_2, \ldots, i_d$ such that

(2) \[ N = v_{i_1} + v_{i_2} + \cdots + v_{i_d}, \]

where $i_2 \geq i_1 + h$ and $i_{\gamma+1} \geq i_{\gamma} + k$ for $2 \leq \gamma < d$.

In this representation, we also have $v_{i_d} \leq N < v_{i_d+1}$.

IV. If $(a_n), (k_n)$ are two sequences of natural numbers with Property P, and $(a_n)$ increasing, then $k_1 \leq k_2 \leq k_1 + 1$, $k_n = k_2$ for $n \geq 3$, and $(a_n)$ is the $(h, k)$-th Fibonacci sequence with $h = k_1$ and $k = k_2$.

V. Let $(v_n)$ be the $(h, k)$-th Fibonacci sequence. For each natural number $n$, let $\Psi_n$ be the average number of summands required in the representation (2) for all those natural numbers $N$ such that $v_n \leq N < v_{n+1}$. Then

\[
\frac{v_{n+1}}{v_n} \to \theta \quad \text{as} \quad n \to \infty,
\]

for $k \geq 1$;

\[
\frac{\Psi_n}{n} \to \frac{\theta - 1}{1 + k(\theta - 1)} \quad \text{as} \quad n \to \infty,
\]

where $\theta = \theta(k)$ is the positive real solution of the equation $z - 1 = z^{1-k}$.

In this paper, we complement the results in I, II and V by considering representations which allow repetition of each constituent up to a fixed number, $l$, of times. It turns out that such representation coincides with the $(l+1)$-adic representation. The difficult part of the proof is that of showing uniqueness.
2. Preliminary

Our results are based on two notions complementing Property $P_l$ which we now describe.

Let $(v_n)$ and $(k_n)$ be two sequences of natural numbers. We say that the two sequences $(v_n), (k_n)$ have Property $P_l, l \geq 2,$ when:

Property $P_l$. Each natural number $N$ can be written uniquely in the form

\[ N = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_d v_{i_d}, \]

where $\alpha_\gamma$ and $\alpha_d \in \{1, 2, \ldots, l\}, i_{\gamma + 1} \geq i_\gamma + k_\gamma$ and $1 \leq \gamma < d$.

Lemma 1. If $(v_n), (k_n)$ are two sequences of natural numbers with Property $P_l$ and $(v_n)$ is increasing, then $v_n$ is the least natural number which cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{n-1}$.

Proof. By Property $P_l$, the first $l$ natural numbers are uniquely represented in the form (M1) as $v_1 = 1, 2v_1 = 2, 3v_1 = 3, \ldots, lv_1 = l$. Since $l + 1$ cannot be written in the form (M1) using only $v_1$, then $v_2 = l + 1$. Suppose $n > 2$. If $v_n$ could be written in the form (M1) using only $v_1, v_2, \ldots, v_{n-1}$, then $v_n$ could be written by two distinct forms as $v_n = v_n$ and as a combination of $v_1, v_2, \ldots, v_{n-1}$, which contradicts the uniqueness of the representation (M1). If we have a number between $v_{n-1}$ and $v_n$ which cannot be written in the form (M1) using only $v_1, v_2, \ldots, v_{n-1}$, then this number cannot be written in the form (M1) at all. \(\square\)

Let $(v_n)$ be a sequence of natural numbers and $(k^{(1)}_n)$ be the sequence $\{1, 1, 1, 1, \ldots\}$ all of whose elements are 1. We say that the two sequences $(v_n), (k^{(1)}_n)$, or simply the sequence $(v_n)$, have Property $P^*_l, l \geq 2,$ when:

Property $P^*_l$. Each natural number $N$ can be written uniquely in the form

\[ N = \alpha_1 v_{i_1} + \alpha_2 v_{i_2} + \cdots + \alpha_d v_{i_d}, \]

where $i_1 < i_2 < \cdots < i_d$ and $\alpha_i \in \{1, 2, \ldots, l\}, 1 \leq i \leq d$.

Note that $P^*_l$ is just $P_l$ when the sequence $(k_n)$ is specialized as $(k^{(1)}_n)$; we define it separately for emphasis.

3. Results

Theorem 1. If $(v_n)$ is an increasing sequence of natural numbers with Property $P^*_l$, then $v_n = (l + 1)^{n-1}$. 
Proof. By Property \( P_l^* \), the first \( l \) natural numbers are uniquely represented in the form (M2) as \( v_1 = 1, 2v_1 = 2, 3v_1 = 3, \ldots, lv_1 = l \). Since \( l + 1 \) is the least natural number not representable in the form (M2) using only \( v_1 \), then by Lemma 1, \( v_2 = l + 1 \). Assume the results hold for \( v_1, v_2, \ldots, v_k \). By Lemma 1, \( v_{k+1} \) is the least natural number not representable in the form (M2) using only \( v_1 = 1, v_2 = l + 1, \ldots, v_k = (l + 1)^{k-1} \). This is just the \((l + 1)\)-adic expansion, i.e., expansion in base \((l + 1)\), and so \( v_{k+1} = (l + 1)^k \). \( \square \)

**Theorem 2.** Let \((v_n)\) and \((k_n)\) be two sequences of natural numbers with \((v_n)\) increasing. If \((v_n), (k_n)\) have Property \( P_l \), \( l \geq 2 \), then \((k_n) = (k_n^{(1)}) = (1, 1, 1, 1, \ldots)\) and \((v_n) = ((l + 1)^n - 1)\).

Proof. The shape of \( v_n \) follows from Theorem 1 if we show that \((k_n) = (k_n^{(1)}) = (1, 1, 1, 1, \ldots)\). We proceed to derive contradiction in each possible case assuming \((k_n) \neq (1, 1, 1, 1, \ldots)\). There are two possibilities.

**Case I.** \( l \geq 3 \).

**Subcase I.1.** \((k_n) = (h \geq 2, k_2, k_3, \ldots)\).

**Claim 1.** For \( h = 2 \), we have \( v_1 = 1, v_2 = l + 1, \) and \( v_3 = l + 2 \).

*Proof of Claim 1.* Clearly \( v_1 = 1 \). By Lemma 1, \( v_2 = l + 1 \). Since \( i_2 \geq i_1 + 2, l + 2 \) cannot be written in the form (M1) using only \( v_1, v_2 \), then by Lemma 1, \( v_3 = l + 2 \), and Claim 1 is proved.

The case \( h = 2 \) is now eliminated by noting that \( 2v_2 = 2l + 2 = lv_1 + v_3 \) contradicting the uniqueness of (M1).

For \( h = 3 \), by the same reasoning as in Claim 1, we have \( v_1 = 1, v_2 = l + 1, v_3 = l + 2, v_4 = l + 3 \).

The case \( h = 3 \) is now eliminated by noting that \( 2v_2 = 2l + 2 = (l - 1)v_1 + v_4 \) contradicting the uniqueness of (M1).

For \( h = 4 \), by the same reasoning as in Claim 1, we have \( v_1 = 1, v_2 = l + 1, v_3 = l + 2, v_4 = l + 3, v_5 = l + 4 \).

The case \( h = 4 \) is now eliminated by noting that \( 2v_2 = 2l + 2 = (l - 2)v_1 + v_5 \) contradicting the uniqueness of (M1).

**Claim 2.** For \( h \geq 5 \), we have

\[
\begin{align*}
v_1 &= 1, & v_2 &= l + 1, \\
v_3 &= l + 2, \ldots, & v_{h-2} &= l + h - 3, \\
v_{h-1} &= l + h - 2, & v_h &= l + h - 1.
\end{align*}
\]
\[ v_{h+1} = \begin{cases} 
    l + h + 1 & \text{for } l = h - 2 \\
    l + h & \text{for } l > h - 2 
\end{cases} \]

(Note that the values of \( v_{h+1} \) for \( 3 \leq l < h - 2 \) are not explicitly stated because they somewhat fluctuate and to obtain a desired contradiction later, their information is not needed.)

**Proof of Claim 2.** Clearly \( v_1 = 1 \). By Lemma 1, \( v_2 = l + 1 \). Since \( i_2 \geq i_1 + 5 \), \( l + 2 \) cannot be written in the form (M1) using only \( v_1, v_2 \), then \( v_3 = l + 2 \). Similarly \( v_4 = l + 3 \) and \( v_5 = l + 4 \).

For \( l = 3 \), \( l + 5 \) can be written in the form (M1) using only \( v_1, v_2, v_3, v_4, v_5 \), viz., \( l + 5 = 2(l + 1) = 2v_2 \). Thus \( v_6 \neq l + 5 \) for \( l = 3 \). For \( l > 3, l + 5 \) cannot be written in the form (M1) using only \( v_1, v_2, v_3, v_4, v_5 \), so \( v_6 = l + 5 \). To find \( v_6 \) for \( l = 3 \), by direct checking, \( l + 6 \) cannot be written in the form (M1) using only \( v_1, v_2, v_3, v_4, v_5 \), so \( v_6 = l + 6 \).

To sum up for \( l = 3 \), we have
\[
\begin{align*}
  v_1 &= 1, & v_2 &= l + 1, & v_3 &= l + 2, \\
  v_4 &= l + 3, & v_5 &= l + 4, & v_6 &= l + 6
\end{align*}
\]

and for \( l > 3 \),
\[
\begin{align*}
  v_1 &= 1, & v_2 &= l + 1, & v_3 &= l + 2, \\
  v_4 &= l + 3, & v_5 &= l + 4, & v_6 &= l + 5;
\end{align*}
\]

so Claim 2 holds for \( h = 5 \).

Assuming the results for \( h, \ i. \ e., \)
\[
\begin{align*}
  v_1 &= 1, & v_2 &= l + 1, \\
  v_3 &= l + 2, \ldots, & v_{h-1} &= l + h - 2, \\
  v_h &= l + h - 1, & v_{h+1} &= \begin{cases} 
    l + h + 1 & \text{for } l = h - 2 \\
    l + h & \text{for } l > h - 2 
\end{cases}
\end{align*}
\]

we proceed to prove the result for \( h + 1 \). Observe that we must start with
\[
\begin{align*}
  v_1 &= 1, & v_2 &= l + 1, & v_3 &= l + 2, \ldots, & v_h &= l + h - 1, & v_{h+1} &= l + h.
\end{align*}
\]

Consider the next value \( l + h + 1 \). If \( l = h - 1 \), then \( l + h + 1 = 2(l + 1) = 2v_2 \), so \( v_{h+2} \neq l + h + 1 \) in this case. By direct checking, \( l + h + 2 \) cannot be written in the form (M1) using only \( v_1, v_2, \ldots, v_{h+1} \) and \( v_{h+2} = l + h + 2 \). But for \( l > h - 1 \) by direct checking, \( l + h + 1 \) cannot be written in the form (M1) using only \( v_1, v_2, \ldots, v_{h+1} \) and \( v_{h+2} = l + h + 1 \) in this case, which ends the proof of Claim 2.
The case $h \geq 5$ is now eliminated by noting that $2v_3 = 2l + 4 = (l - (h - 3))v_1 + v_{h+1}$ for $l = h - 2$ and $2v_2 = 2l + 2 = (l - (h - 2))v_1 + v_{h+1}$ for $l > h - 2$, both of which contradict the uniqueness of representation.

**Subcase I.2.** $(k_n) = \left( \underbrace{1,1,\ldots,1}_{f-1 \text{ terms } (f \geq 2)}, h \geq 2, \ldots \right)$, where $f \geq 2$.

The next Claim works for all $l \geq 2$ which will also be needed later.

**Claim 3.** Let $l \geq 2$. Then

$$v_1 = 1, v_2 = l + 1, v_3 = (l + 1)^2, \ldots, v_{f+1} = (l + 1)^f, v_{f+2} = \frac{(l + 1)^{f+1} - 1}{l}.$$  

Proof of Claim 3. Clearly $v_1 = 1$, $v_2 = l + 1$ and the integers $l+2, \ldots, l+l(l+1) = l^2 + 2l$ can be written in the form (M1) using only $v_1$, $v_2$, but $(l + 1)^2$ cannot, so $v_3 = (l + 1)^2$. The values $v_4, v_5, v_6, \ldots, v_{f+1}$ are found analogously. The integers with values

$$v_{f+1} + 1 = (l + 1)^f + 1, (l + 1)^f + 2, \ldots, v_{f+1} + v_f + \cdots + v_2$$

can be written in the form (M1) using only $v_1, v_2, v_3, \ldots, v_{f+1}$, but the next integer

$$v_{f+1} + v_f + \cdots + v_2 + v_1 = (l + 1)^f + (l + 1)^{f-1} + \cdots + (l + 1) + 1$$

$$= \frac{(l + 1)^{f+1} - 1}{l}$$

cannot because $i_2 \geq i_1 + 1, i_3 \geq i_2 + 1, \ldots, i_f \geq i_{f-1} + 1, i_{f+1} \geq i_f + h \geq i_f + 2$, and so

$$v_{f+2} = \frac{(l + 1)^{f+1} - 1}{l}.$$  

This proves Claim 3.

The case $l \geq 2$ is now eliminated by noting that

$$lv_1 + v_{f+2} = 2(l + 1) + (l + 1)^2 + \cdots + (l + 1)^f = 2v_2 + v_3 + \cdots + v_{f+1},$$

contradicting the uniqueness of (M1).

**Case II.** $l = 2$.

Define $(P_2, h)$ to be a set of all possible pairs of sequences $\{(v_n), (k_n)\}$ with Property $P_2$ and $(k_n) = (h, k_2, k_3, \ldots)$.

**Subcase II.1.** $(k_n) = (h \geq 2, k_2, k_3, \ldots)$.

**Claim 4.** For fixed $H \in \mathbb{N}$, if $h \geq H$, each pair of sequences in $(P_2, h)$ has the same first $H + 1$ values in $(v_n)$, i.e., the same value of $v_1, v_2, \ldots, v_H$.

Proof of Claim 4. For $H = 1$, for each pair in $(P_2, h)$, we clearly have $v_1 = 1, v_2 = 3$ (because $2 = 2v_1$). For $H = 2$ in $(P_2, h \geq 2)$, direct computation gives
$v_1 = 1, v_2 = 3$ and $v_3 = 4$ (because $4 \neq 2v_1, 2v_2$). For general $H$, we determine $v_1, v_2, \ldots, v_{H+1}$ in $(P_2, h)$ by using Lemma 1. Observe that in $(P_2, h)$, where $h \geq H$, again using Lemma 1, the first $H + 1$ values must be the same as in $(P_2, h)$ because $i_2 \geq i_1 + h \geq i_1 + H$, and Claim 4 is proved.

**Claim 5.** Let $\{(v_n), (k_n)\} \in (P_2, h)$ with $h \geq 2$. Then $v_1, v_2, \ldots, v_{h+1}$ can be explicitly determined using 2-adic expansion as follows:

(i) Each positive integer written 2-adically ending with an odd number of 0’s cannot represent any of $v_1, v_2, \ldots, v_{h+1}$.

In the 2-adic expansion, for $n = 1, 2, \ldots, h$,

(ii) if $v_n = \overline{100 \ldots 00}$, then $v_{n+1} = \overline{100 \ldots 01}$,

(iii) if $v_n = \overline{011 \ldots 11}$, then $v_{n+1} = \overline{100 \ldots 00}$, and

(iv) if $v_n = \overline{0 \ 11 \ldots 11}$, then $v_{n+1} = \overline{1 \ 00 \ldots 01}$.

**Proof of Claim 5.** From numerical values displayed in the following table, we see that

<table>
<thead>
<tr>
<th>valid for $h \geq$</th>
<th>base 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1</td>
<td>$v_1 = 1$</td>
</tr>
<tr>
<td>1 11</td>
<td>$v_2 = 3$</td>
</tr>
<tr>
<td>2 1</td>
<td>$v_3 = 4$</td>
</tr>
<tr>
<td>2 100</td>
<td>$v_4 = 5$</td>
</tr>
<tr>
<td>3 11</td>
<td>$v_5 = 7$</td>
</tr>
<tr>
<td>3 101</td>
<td>$v_6 = 9$</td>
</tr>
<tr>
<td>4 111</td>
<td>$v_7 = 11$</td>
</tr>
<tr>
<td>4 1100</td>
<td>$v_8 = 12$</td>
</tr>
<tr>
<td>5 111</td>
<td>$v_9 = 13$</td>
</tr>
<tr>
<td>5 1111</td>
<td>$v_{10} = 15$</td>
</tr>
<tr>
<td>6 10000</td>
<td>$v_{11} = 16$</td>
</tr>
<tr>
<td>6 10001</td>
<td>$v_{12} = 17$</td>
</tr>
<tr>
<td>7 10111</td>
<td>$v_{13} = 19$</td>
</tr>
<tr>
<td>7 10101</td>
<td>$v_{14} = 20$</td>
</tr>
<tr>
<td>8 11011</td>
<td>$v_{15} = 21$</td>
</tr>
<tr>
<td>8 10011</td>
<td>$v_{16} = 23$</td>
</tr>
<tr>
<td>9 11111</td>
<td>$v_{17} = 25$</td>
</tr>
<tr>
<td>9 11011</td>
<td>$v_{18} = 27$</td>
</tr>
<tr>
<td>10 11100</td>
<td>$v_{19} = 28$</td>
</tr>
<tr>
<td>10 11111</td>
<td>$v_{20} = 29$</td>
</tr>
<tr>
<td>11 11111</td>
<td>$v_{21} = 31$</td>
</tr>
</tbody>
</table>
the results (i)-(iv) hold for \( v_n \) written 2-adically with \( \leq 5 \) digits. Note that (i) follows from (ii)-(iv), so we need only check (ii)-(iv). Assume the 2-adic shape of \( v_1, v_2, \ldots, v_n \). Since \( 1 \leq n \leq h \), by Property P₂, then \( v_{n+1} \) is the first integer greater than \( v_n \) which is not a double of \( v_1, v_2, \ldots, v_n \). The three assertions in (ii)-(iv) follows immediately by induction, which ends the proof of Claim 5.

The subcase II.1 is now eliminated by the following claim:

**Claim 6.**

(i) If \( v_{h+1} = \overline{\Xi 100 \ldots 00} \), then \( 2v_1 + v_{h+1} = 2v_n \) for some \( 1 < n \leq h + 1 \).

(ii) If \( v_{h+1} = \overline{\Xi 0 11 \ldots 11} \), then \( v_1 + v_{h+1} = 2v_n \) for some \( 1 < n \leq h + 1 \).

(iii) If \( v_{h+1} = \overline{\Xi 011 \ldots 11} \), then \( v_2 + v_{h+2} = 2v_n \) for some \( 1 < n \leq h + 1 \).

**Proof of Claim 6.**

(i) Writing in base 2, we have

\[
2v_1 + v_{h+1} = 2(01) + \overline{\Xi 100 \ldots 00} = \overline{\Xi 100 \ldots 010} = 2 \left( \overline{\Xi 1 00 \ldots 01} \right) = 2v_n
\]

for some \( 1 < n \leq h + 1 \).

(ii) Writing in base 2, we have

\[
v_1 + v_{h+1} = 01 + \overline{\Xi 0 11 \ldots 11} = \overline{\Xi 1 00 \ldots 00} = 2 \left( \overline{\Xi 1 00 \ldots 00} \right) = 2v_n
\]

for some \( 1 < n \leq h + 1 \).

(iii) We first show that

\[
v_{h+2} = \overline{\Xi 100 \ldots 011}.
\]

Recall from Lemma 1 that \( v_{h+2} \) is the least natural number which cannot be written in the form (M1) using only \( v_1, v_2, \ldots, v_{h+1} \) with the difference in the first two indices being \( \geq h \). Since

\[
v_1 + v_{h+1} = 01 + \overline{\Xi 011 \ldots 11} = \overline{\Xi 100 \ldots 00},
\]

and

\[
2v_1 + v_{h+1} = 2(01) + \overline{\Xi 011 \ldots 11} = \overline{\Xi 100 \ldots 01},
\]
Then \( v_{h+2} \) is not equal in value to \( v_1 + v_{h+1}, \ 2v_1 + v_{h+1} \). Now

\[ v_2 + v_{h+1} = 11 + \underbrace{011 \ldots 11}_{2m \text{ terms}} = \underbrace{100 \ldots 010}_{2m \text{ terms}} = 2 \left( \underbrace{1 00 \ldots 01}_{2m-1 \text{ terms}} \right). \]

Thus \( v_2 + v_{h+1} \) is equal to a double of some previous \( v_n \), so that \( v_{h+2} \) is not equal in value to \( v_2 + v_{h+1} \). Since

\[ v_3 + v_{h+1} = 100 + \underbrace{011 \ldots 11}_{2m \text{ terms}} = \underbrace{100 \ldots 011}_{2m \text{ terms}}, \]

then \( v_3 + v_{h+1} \) is not equal to a double of some previous \( v_n \). Thus the value of

\[ v_{h+2} (= v_3 + v_{h+1}) = \underbrace{100 \ldots 011}_{2m \text{ terms}}, \]

and so

\[ v_2 + v_{h+2} = 11 + \underbrace{100 \ldots 011}_{2m \text{ terms}} = \underbrace{100 \ldots 010}_{2m \text{ terms}} = 2 \left( \underbrace{100 \ldots 011}_{2m-1 \text{ terms}} \right) = 2v_n \]

for some \( 1 < n \leq h + 1 \), which completes the proof of Claim 6.

**Subcase II.2.**

\[ (k_n) = \left( \underbrace{1, 1, \ldots, 1}_{f-1 \text{ terms } (f \geq 2)}, \ h \geq 2, \ldots \right), \]

where \( f \geq 2 \).

This subcase is eliminated by Claim 3 in subcase I.2.

To sum up, we must have \((k_n) = (k_n^{(1)}) = (1, 1, 1, 1, \ldots)\) and by Theorem 1, \((v_n) = ((l + 1)^{n-1})\). \(\square\)

**Theorem 3.** Let the increasing sequence \((v_n)\) have the Property \( P_1 \). For each natural number \( n \), let \( \Psi_n \) be the average number of summands required in the representation (M2) for all natural numbers \( N \) such that \( v_n \leq N < v_{n+1} \). Then

\[ \Psi_n = \frac{nl + 1}{l + 1} \quad \text{and} \quad \lim_{n \to \infty} \frac{\Psi_n}{n} = \frac{l}{l + 1}. \]

**Proof.** For each natural number \( N \), let \( f(N) \) be the unique number of summands \( d \) used in representing \( N \) in the form (M2). Since \( \Psi_n \) is defined as the average value of \( f(N) \) for those numbers \( N \) such that \( v_n \leq N < v_{n+1} \), we have

\[ \Psi_n = \frac{\sum_{v_n \leq N < v_{n+1}} f(N)}{v_{n+1} - v_n}. \]
Let us first compute a few values of $n$.

$$
\Psi_1 = \frac{\sum_{1 \leq N < l+1} f(N)}{l + 1 - 1} = \frac{l}{l} = 1.
$$

For

$$
\Psi_2 = \frac{\sum_{l+1 \leq N < (l+1)^2} f(N)}{(l + 1)^2 - (l + 1)},
$$

$N$ is of the form $av_2 + bv_1 = a(l + 1) + b; a \in \{1, 2, \ldots, l\}, b \in \{0, 1, \ldots, l\}$, so

$$
\Psi_2 = \frac{l + 2 \left( \frac{1}{1} \right) l^2}{l^2 + l}.
$$

For arbitrary $\Psi_n$, $N$ is of the form $a_n v_n + \cdots + a_1 v_1; a_n \in \{1, 2, \ldots, l\}; a_{n-1}, \ldots, a_1 \in \{0, 1, \ldots, l\}$. Each summand of $k$ terms, $1 \leq k \leq n$, contributes to $\Psi_n$ with

$$
k \left( \frac{n - 1}{k - 1} \right) l^k.
$$

Hence

$$
\Psi_n = \frac{\sum_{k=1}^{n} k \left( \frac{n - 1}{k - 1} \right) l^k}{(l + 1)^{n-1} l} = \frac{nl + 1}{l + 1},
$$

and $\frac{\Psi_n}{n} \to \frac{l}{l+1} (n \to \infty)$.  

\[\square\]

REFERENCES
