NORM OF THE COMPOSITION OPERATOR MAPPING BLOCH SPACE INTO HARDY OR BERGMAN SPACE

ERN GUN KWON AND JINKEE LEE

Abstract. Let $1 \leq p < \infty$ and $\alpha > -1$. If $f$ is a holomorphic self-map of the open unit disc $U$ of $\mathbb{C}$ with $f(0) = 0$, then the quantity

$$\int_U \frac{|f'(z)|}{1 - |f(z)|^2} |f(z)|^{\alpha + p} dA$$

is equivalent to the operator norm of the composition operator $C_f : B \rightarrow A_{p,\alpha}$ defined by $C_fh = h \circ f - h(0)$, where $B$ and $A_{p,\alpha}$ are the Bloch space and the weighted Bergman space on $U$ respectively.

1. Introduction

Consider holomorphic mappings $f$ of the unit ball of $\mathbb{C}^n$ into the unit disc $U$ of $\mathbb{C}$. It is said that $f$ has the pull-back property if $h \circ f \in BMOA$ whenever $h$ belongs to the Bloch space $B$ on $U$. Since the pull-back property was first studied for monomials in [1], there have been several examples and conditions for $f$ to have the pull-back property ([1], [2], [7]). When $n = 1$, if $f$ is a function of Yamashita’s hyperbolic $BMOA$ class then the composition operator $C_f$ defined by $C_fh = h \circ f - h(0)$ takes $B$ into $BMOA$ ([6], [7]). In view of a known parallelism between the Hardy space $H^p$ and the Yamashita hyperbolic Hardy class $H_{p}^{\sigma}$, the first author gave a necessary and sufficient condition for $C_f$ to take $B$ into $H^{2p}$ ([6]).

We, in this paper, restrict ourselves to $n = 1$ and give a quantity equivalent to the operator norm $\|C_f\|$ of the composition operator $C_f$ that takes $B$ boundedly into the weighted Bergman space $A_{p,\alpha}$.
Theorem 1. Let $f : U \to U$ be a holomorphic function with $f(0) = 0$. For $1 \leq p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C_0^f : B \to A^{p,\alpha}$ defined by $C_0^f h = h \circ f - h(0)$ has its operator norm equivalent to the quantity

\[
\left\{ \int_U (1 - |z|)^{\alpha+p} \left( \frac{|f'(z)|}{1 - |f(z)|^2} \right)^p dx dy \right\}^{1/p}.
\] (1.1)

By the lemma of Schwarz-Pick, it is easy to see that (1.1) remains bounded for any holomorphic self map $f$ of $U$. What Theorem 1 expresses is that there are positive constants $C_1$ and $C_2$ independent of $f$ such that

$$C_1 \|C_0^f\| \leq (1.1) \leq C_2 \|C_0^f\|.$$

Corollary 2. Let $f : U \to U$ be a holomorphic function. For $1 \leq p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C_0^f : B \to A^{p,\alpha}$ defined by $C_0^f h = h \circ \varphi_f \circ f - h(0)$ has its operator norm equivalent to the quantity (1.1).

2. Preliminaries

We introduce a few facts that we need in the sequel, most of which are well known.

The group of automorphisms of $U$ will be denoted by $\mathcal{M}$. It is known that it consists of functions of the form $e^{i\beta} \varphi_a$, where $\beta$ is a real number and

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in U.$$

For $1 \leq p < \infty$ and for $f$ subharmonic in $U$, we set

$$\|f\|_p := \sup_r \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}.$$

Then the class $H^p = H^p(U)$ consists of those $f$ holomorphic in $U$ for which $\|f\|_p < \infty$.

The Yamashita hyperbolic Hardy class $H^p_{\sigma}$ is defined as the set of those holomorphic self-maps $f$ of $U$ for which $\|\sigma(f)\|_p < \infty$, where $\sigma(z)$ denotes the hyperbolic distance of $z$ and 0 in $U$, namely,

$$\sigma(z) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}.$$
We set, following Yamashita,
\[ \lambda(f) = \log \frac{1}{1 - |f|^2} \quad \text{and} \quad f^\sharp = \frac{|f'|}{1 - |f|^2} \]
for holomorphic self-maps \( f \) of \( U \). It is obvious that \( f \in H^p \) if and only if \( ||\lambda(f)||_p < \infty \) and that \( f^\sharp \) is \( \mathcal{M} \)-invariant in the sense that \( f^\sharp = (\varphi \circ f)^\sharp \) for any \( \varphi \in \mathcal{M} \).

The Bloch space \( \mathcal{B} \) consists of holomorphic functions \( h \) in \( U \) for which
\[ \sup_{z \in U} |h'(z)|(1 - |z|^2) < \infty. \]
This is a Banach space, if the norm \( ||h||_B \) of \( h \in \mathcal{B} \) is defined to be the sum of \( |h(0)| \) and the left side of above inequality. A pair of Bloch functions \( h_j, j = 1, 2 \) are constructed such that
\[ (2.1) \quad (1 - |z|^2)(|h_1'(z)| + |h_2'(z)|) \geq 1, \quad z \in U \]
(\([7]\)). Then it follows that
\[ (2.2) \quad \frac{1}{1 - |f|^2} \leq |h_1' \circ f| + |h_2' \circ f| \leq \frac{C}{1 - |f|^2} \]
for holomorphic self-maps \( f \), where \( C = 2 \max(||h_1||_B, ||h_2||_B) \). For \( h \in \mathcal{B} \), it follows from Schwarz-Pick’s Lemma (\([5]\)) that
\[ (2.3) \quad |(h \circ f)'(z)| \leq ||h||_B f^\sharp(z) \leq ||h||_B \frac{1}{1 - |z|^2}, \quad z \in U. \]

For \(-1 < \alpha < \infty \) and \( 0 < p < \infty \), let \( A^{p,\alpha} \) denote the weighted Bergman space of holomorphic functions on \( U \), that is,
\[ A^{p,\alpha} = \left\{ f \text{ holomorphic on } U : ||f||_{A^{p,\alpha}} \right\} 
\equiv \left\{ \left( \int_U |f(z)|^p (1 - |z|)^\alpha \, dx \, dy \right)^{1/p} < \infty \right\}. \]
We note that \( H^p \) is the limiting space of \( A^{p,\alpha} \) as \( \alpha \to -1 \).

For \( h \) holomorphic in \( U \), \( g \)-function of Paley defined by
\[ g(\theta) := g(h)(\theta) = \left( \int_0^1 |h'(re^{i\theta})|^2 (1 - r) \, dr \right)^{1/2}, \quad 0 \leq \theta < 2\pi, \]
satisfies
\begin{equation}
\|g(h)\|_{L^p} \sim \|h\|_p \quad \text{if} \quad h(0) = 0,
\end{equation}
for \(1 \leq p < \infty\) (see [4] and [8]). Here and after \(\psi \sim \phi\) means the equivalence of two quantities in the sense that either both sides are zeroes or the quotient \(\psi/\phi\) lies between two positive constants depending only on \(p\).

The hyperbolic version of \(g\)-function is defined as
\[
g_\sigma(\theta) := g_\sigma(f)(\theta) = \int_0^1 \left(f^2(r e^{i\theta})\right)^2 (1-r) \, dr, \quad 0 \leq \theta < 2\pi,
\]
and then for \(1 \leq p < \infty\) it is satisfied that
\begin{equation}
\|\lambda(f)\|_p \sim \|g_\sigma(f)\|_{L^p} \quad \text{if} \quad f(0) = 0.
\end{equation}
See [6].

3. Proof of the results

For functions holomorphic in \(U\) and for \(0 < p < \infty\), \(0 \leq r < 1\), \(M_p(r, f)\) is defined as usual by
\[
M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}.
\]
For simplicity, we denote \(\psi \lesssim \phi\) meaning that either \(\psi \sim \phi\) or the quotient \(\psi/\phi\) is bounded by a positive constant depending only on \(p\).

**Lemma.** Let \(f\) be holomorphic in \(U\). Then, for \(1 \leq p < \infty\) and \(-1 < \alpha < \infty\),
\[
\int_0^1 (1-r)^\alpha M_p^\alpha(r, f) \, dr \sim \int_0^1 (1-r)^{\alpha+p} M_p^\alpha(r, f') \, dr + |f(0)|^p.
\]
**Proof.** Applying the same process as in the proof of [3, Theorem 5.6] to \(1 \leq p < \infty\), we can obtain
\[
\int_0^1 (1-r)^\alpha M_p^\alpha(r, f) \, dr \lesssim \int_0^1 (1-r)^{\alpha+p} M_p^\alpha(r, f') \, dr + |f(0)|^p.
\]
Conversely, when \( \rho = \frac{1}{2}(1 + r) \) we see in the proof of [3, Theorem 5.5]

\[
M_p(r, f') \leq \frac{M_p(\rho, f)}{\rho^2 - r^2}.
\]

If we integrate both sides of this inequality with respect to \( dr \) after multiplying them by \((1 - r)^{\alpha + p}\), then we obtain

\[
\int_0^1 (1 - r)^{\alpha + p} M_p(r, f') \, dr \lesssim \int_0^1 (1 - r)^{\alpha} M_p\left(\frac{1 + r}{2}, f\right) \, dr,
\]

whence a change of variable completes the proof. \(\square\)

**Proof of Theorem 1.** We show that

\[
\|C^0_f\| \sim (1.1).
\]

By (3.1) and (2.3), we have

\[
\|C^0_f\| = \sup_{h \in \mathcal{B}} \left\{ \int_U (1 - |z|)^\alpha \left| (h \circ f)(z) - h(0) \right|^p \, dxdy \right\}^{1/p}
\]

\[
\sim \sup_{h \in \mathcal{B}} \left\{ \int_U (1 - |z|)^{\alpha + p} |(h \circ f)'(z)|^p \, dxdy \right\}^{1/p}
\]

\[
\leq \left\{ \int_U (1 - |z|)^{\alpha + p} \left( f^\sharp(z) \right)^p \, dxdy \right\}^{1/p}.
\]

Conversely, using Minkowski’s inequality with those \( h_j, \ j = 1, 2 \), of (2.1) and using (3.1), we obtain

\[
\left\{ \int_U (1 - |z|)^{\alpha + p} \left( f^t(z) \right)^p \, dxdy \right\}^{1/p}
\]

\[
\leq \left\{ \int_U (1 - |z|)^{\alpha + p} \left( \sum_{j=1}^2 |(h_j \circ f)'(z)| \right)^p \, dxdy \right\}^{1/p}
\]

\[
\leq \sum_{j=1}^2 \left\{ \int_U (1 - |z|)^{\alpha + p} |(h_j \circ f)'(z)|^p \, dxdy \right\}^{1/p}
\]

\[
\leq \sum_{j=1}^2 \left\{ \int_U (1 - |z|)^\alpha \left| (h_j \circ f)(z) - h_j(0) \right|^p \, dxdy \right\}^{1/p}.
\]

(3.2)
Since
\[
\left\{ \int_U (1 - |z|)^\alpha \left| (h_j \circ f)(z) - h_j(0) \right|^p \, dx \, dy \right\}^{1/p} \leq \|h_j\|_B \sup_{h \in B, \|h\|_{A^\alpha} \leq 1} \left\{ \int_U (1 - |z|)^\alpha \left| (h \circ f)(z) - h(0) \right|^p \, dx \, dy \right\}^{1/p}, \quad j = 1, 2,
\]
from (3.2) we have
\[
\left\{ \int_U (1 - |z|)^{\alpha + p} \left( f^2(z) \right)^p \, dx \, dy \right\}^{1/p} \lesssim \sup_{h \in B, \|h\|_{A^\alpha} \leq 1} \left\{ \int_U (1 - |z|)^\alpha \left| (h \circ f)(z) - h(0) \right|^p \, dx \, dy \right\}^{1/p}
\]
\[
= \sup_{h \in B, \|h\|_{A^\alpha} \leq 1} \|h \circ f - h(0)\|_{A^p, \alpha}
\]
\[
= \|C^0_f\|.
\]

\[\square\]

**Proof of Corollary 2.** The result follows from $\mathcal{M}$-invariance of $f^2$ and Theorem 1. \[\square\]

4. Limiting case $H^p$

As is well-known, we may regard $A^{p,-1} = H^p$. Using (2.1), (2.2), (2.3), (2.4) and (2.5), a method similar to Proof of Theorem 1 gives that the quantity $\|\lambda(f)\|_{p,2}^{1/2}$ is equivalent to the norm of Bloch-$H^p$ pullback operator. This fact will be discussed extensively in the coming paper of the first author.

**References**


Department of Mathematics Education  
Andong National University  
Andong 760-749, Korea  
*E-mail*: egkwon@andong.ac.kr  
leejin@hyowon.pusan.ac.kr