COMPACT INTERPOLATION FOR VECTORS IN TRIDIAGONAL ALGEBRA

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ABSTRACT. Given vectors $x$ and $y$ in a Hilbert space, an interpolating operator is a bounded operator $T$ such that $Tx = y$. An interpolating operator for $n$ vectors satisfies the equation $Tx_i = y_i$, for $i = 1, 2, \cdots, n$. In this article, we investigate compact interpolation problems in tridiagonal algebra: Given vectors $x$ and $y$ in a Hilbert space, when is there a compact operator $A$ in a tridiagonal algebra such that $Ax = y$?

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let $x$ and $y$ be vectors on $\mathcal{H}$. An interpolation question for $\mathcal{C}$ asks for which $x$ and $y$ is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the ‘$n$-vector interpolation problem’, asks for an operator $T$ such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The $n$-vector interpolation problem was considered for a $C^*$-algebra $\mathcal{U}$ by Kadison [10]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [11]: his result was extended by Hopenwasser [5] to the case that $\mathcal{U}$ is a CSL-algebra. Recently, Munch [12] obtained conditions for interpolation in case $T$ is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser [6] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra.

First, we establish some notations and conventions. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and $I$ lie in $\mathcal{L}$. We usually identify

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projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If each pair of projections in \( L \) commutes, then \( L \) is called a commutative subspace lattice, or CSL. If \( L \) is CSL, \( \text{Alg} L \) is called a CSL-algebra. The symbol \( \text{Alg} L \) is the algebra of all bounded linear operators on \( H \) that leave invariant all the projections in \( L \). Let \( x \) and \( y \) be two vectors in a Hilbert space. Then \( \langle x, y \rangle \) means the inner product of the vectors \( x \) and \( y \). Let \( M \) be a subset of a Hilbert space. Then \( M^\perp \) is the orthogonal complement of \( M \). Let \( \mathbb{N} \) be the set of all natural numbers and let \( \mathbb{C} \) be the set of all complex numbers.

### 2. Results

Let \( H \) be a separable complex Hilbert space with a fixed orthonormal basis \( \{ e_1, e_2, \cdots \} \). Let \( x_1, x_2, \cdots, x_n \) be vectors in \( H \). Then \( [x_1, x_2, \cdots, x_n] \) means the closed subspace generated by the vectors \( x_1, x_2, \cdots, x_n \). Let \( L \) be the subspace lattice generated by the subspaces \( [e_{2k-1}], [e_{2k-1}, e_{2k}, e_{2k+1}] \) \((k = 1, 2, \cdots)\). Then the algebra \( \text{Alg} L \) is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [3].

Let \( A \) be the algebra consisting of all bounded operators acting on \( H \) of the form

\[
\begin{pmatrix}
* & * & & & \\
& * & & & \\
& & * & * & \\
& & & * & * \\
& & & & * & \\
& & & & & \ddots
\end{pmatrix}
\]

with respect to the orthonormal basis \( \{ e_1, e_2, \cdots \} \), where all non-starred entries are zero. It is easy to see that \( \text{Alg} L = A \). Let \( D = \{ A : A \text{ is a diagonal operator on } H \} \). Then \( D \) is a masa(maximal abelian subalgebra) of \( \text{Alg} L \) and \( D = (\text{Alg} L) \cap (\text{Alg} L)^* \), where \( (\text{Alg} L)^* = \{ A^* : A \in \text{Alg} L \} \).

Let \( B(H) \) be the set of all bounded operators acting on \( H \).

In this paper, we use the convention \( 0_0 = 0 \), when necessary.

The following theorem is well-known.

**Theorem 1** [4]. Let \( A \) be a diagonal operator in \( B(H) \) with diagonal \( \{ a_n \} \). Then \( A \) is compact if and only if \( a_n \to 0 \) as \( n \to \infty \).

**Theorem 2.** Let \( x = (x_i) \) and \( y = (y_i) \) be two vectors in \( H \) such that \( x_i \neq 0 \) for all \( i = 1, 2, \cdots \). Then the following statements are equivalent.
(1) There exists an operator $A$ in Alg$\mathcal{L}$ such that $Ax = y$, $A$ is compact and every $E$ in $\mathcal{L}$ reduces $A$.

$$\sup \left\{ \frac{\| \sum_{k=1}^{l} \alpha_k E_{k,y} \|}{\| \sum_{k=1}^{l} \alpha_k E_{k,x} \|} : l \in \mathbb{N}, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L} \right\} < \infty \text{ and } y_n x_n^{-1} \to 0 \text{ as } n \to \infty.$$

**Proof.** If $\sup \left\{ \frac{\| \sum_{k=1}^{l} \alpha_k E_{k,y} \|}{\| \sum_{k=1}^{l} \alpha_k E_{k,x} \|} : l \in \mathbb{N}, \alpha_k \in \mathbb{C} \text{ and } E_k \in \mathcal{L} \right\} < \infty$, then, there is an operator $A$ in Alg$\mathcal{L}$ such that $Ax = y$ and every $E$ in $\mathcal{L}$ reduces $A$ by Theorem 1 ([9]). Since every $E$ in $\mathcal{L}$ reduces $A$, $A$ is diagonal. Let $A = (a_{ii})$. Since $A = (a_{ii})$ is diagonal and $Ax = y$, $a_{ii} x_i = y_i$ for all $i = 1, 2, \ldots$. Since $y_n x_n^{-1} \to 0$ as $n \to \infty$, $A$ is compact.

Conversely, since $Ax = y$ and every $E$ in $\mathcal{L}$ reduces $A$, $AE_{x} = E_{y}$ for every $E$ in $\mathcal{L}$. So $A(\sum_{k=1}^{l} \alpha_k E_{k,x}) = \sum_{k=1}^{l} \alpha_k E_{k,y}$ for every $l \in \mathbb{N}$, every $\alpha_k \in \mathbb{C}$ and every $E_k \in \mathcal{L}$. Thus $\| \sum_{k=1}^{l} \alpha_k E_{k,y} \| \leq \| A \| \| \sum_{k=1}^{l} \alpha_k E_{k,x} \|$. If $\| \sum_{k=1}^{l} \alpha_k E_{k,x} \| \neq 0$, then $\| \sum_{k=1}^{l} \alpha_k E_{k,y} \| \leq \| A \| \| \sum_{k=1}^{l} \alpha_k E_{k,x} \|$. Since every $E$ in $\mathcal{L}$ reduces $A$, $A$ is diagonal. Let $A = (a_{ii})$. Since $Ax = y$, $y_i = a_{ii} x_i$ and hence $a_{ii} = y_i x_i^{-1}$ for all $i = 1, 2, \ldots$. Since $A$ is compact, $y_i x_i^{-1} \to 0$ as $i \to \infty$. \qed

**Theorem 3.** Let $x_p = (x_{pi})$ and $y_p = (y_{pi})$ be vectors in $\mathcal{H}$ such that $x_{qi} \neq 0$ for some fixed $q$, all $i = 1, 2, \ldots$ and all $p = 1, 2, \ldots, n$. If there is an operator $A$ in Alg$\mathcal{L}$ such that $Ax_p = y_p$ for every $E$ in $\mathcal{L}$ reduces $A$ and $A$ is compact, then

$$\sup \left\{ \frac{\| \sum_{p=1}^{m_p} \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p,x_p} \|}{\| \sum_{p=1}^{m_p} \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p,y_p} \|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \right\} < \infty$$

and $y_{qi} x_{qi}^{-1} \to 0$ as $i \to \infty$.

**Proof.** Since $Ax_p = y_p$ and every $E$ in $\mathcal{L}$ reduces $A$, $AE_{x_p} = E_{y_p}$ for every $p = 1, 2, \ldots, n$. So $A(\sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p,x_p}) = \sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p,y_p}$, $m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L}$ and $\alpha_{k,p} \in \mathbb{C}$. Thus

$$\left\| \sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p,y_p} \right\| \leq \| A \| \left\| \sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p,x_p} \right\|.$$
If \( \| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p \| \neq 0 \), then
\[
\frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}y_p \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p \|} \leq \| A \|.
\]
Hence
\[
\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}y_p \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p \|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \} < \infty.
\]
Since every \( E \) in \( \mathcal{L} \) reduces \( A \), \( A \) is diagonal. Let \( A = (a_{ii}) \). Since \( Ax_p = y_p \), \( y_{pi} = a_{ii}x_{pi} \) for all \( p = 1, 2, \ldots, n \) and all \( i = 1, 2, \ldots \). Since \( x_{qi} \neq 0 \), \( a_{ii} = y_{qi}x_{qi}^{-1} \) \((i = 1, 2, \ldots)\). Since \( A \) is compact, \( y_{qi}x_{qi}^{-1} \to 0 \) as \( i \to \infty \).

**Theorem 4.** Let \( x_p = (x_{pi}) \) and \( y_p = (y_{pi}) \) be vectors in \( \mathcal{H} \) such that \( x_{qi} \neq 0 \) for some fixed \( q \), all \( i = 1, 2, \ldots \) and all \( p = 1, 2, \ldots, n \).

If \( \sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}y_p \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p \|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \} < \infty \)
and \( y_{qi}x_{qi}^{-1} \to 0 \) as \( i \to \infty \), then there is an operator \( A \) in \( \text{Alg}\mathcal{L} \) such that \( Ax_p = y_p \) for all \( p = 1, 2, \ldots, n \), every \( E \) in \( \mathcal{L} \) reduces \( A \) and \( A \) is compact.

**Proof.** Without loss of generality, we may assume that
\[
\sup \left\{ \frac{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}y_p \|}{\| \sum_{i=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p \|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \} = 1.
\]
So
\[
\sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}y_p \leq \sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p, m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \text{. (\ast).}
\]
Let \( \mathcal{M} = \left\{ \sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p : m_p \in \mathbb{N}, l \leq n, \alpha_{k,p} \in \mathbb{C} \text{ and } E_{k,p} \in \mathcal{L} \right\} \).

Then \( \mathcal{M} \) is a linear manifold. Define \( A : \mathcal{M} \to \mathcal{H} \) by \( A(\sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}x_p) = \sum_{p=1}^{l} \sum_{k=1}^{m_p} \alpha_{k,p}E_{k,p}y_p \). Then \( A \) is well-defined by \( (\ast) \).

Extend \( A \) to \( \mathcal{M} \) by continuity. Define \( A|_{\mathcal{M}^\perp} = 0 \). Clearly \( Ax_p = y_p \) \((p = 1, 2, \ldots, n)\) and \( \| A \| \leq 1 \). By an argument similar to that of the proof of Theorem 2, every \( E \) in \( \mathcal{L} \) reduces \( A \). So \( A \) is a diagonal operator. Let \( A = (a_{ii}) \). Since \( y_{pi} = Ax_{pi} \), \( a_{ii} = y_{pi}x_{pi}^{-1} \) \((i = 1, 2, \ldots)\). Since \( y_{qi}x_{qi}^{-1} \to 0 \) as \( i \to \infty \), \( A \) is compact.

If we summarize Theorems 3 and 4, then we can get the following theorem.
THEOREM 5. Let \( x_p = (x_{pi}) \) and \( y_p = (y_{pi}) \) be vectors in \( \mathcal{H} \) such that \( x_{qi} \neq 0 \) for some fixed \( q \) and all \( i = 1, 2, \ldots \). Then the following statements are equivalent.

1. There exists an operator \( A \in \text{AlgL} \) such that \( Ax_p = y_p \) \((p = 1, \cdots, n)\), every \( E \in \mathcal{L} \) reduces \( A \) and \( A \) is compact.

2. \[
\sup \left\{ \frac{\| \sum_{p=1}^l \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p} y_p \|}{\| \sum_{p=1}^l \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p} x_p \|} : m_p \in \mathbb{N}, l \leq n, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \right\} < \infty
\]
and \( y_{qi} x_{qi}^{-1} \to 0 \) as \( i \to \infty \).

If we modify the proof of Theorems 3 and 4, then we can get the following theorem.

THEOREM 6. Let \( x_p = (x_{pi}) \) and \( y_p = (y_{pi}) \) be vectors in \( \mathcal{H}(p = 1, 2, \cdots) \) such that \( x_{qi} \neq 0 \) for all \( i \) and for some fixed \( q \). Then the following statements are equivalent.

1. There exists an operator \( A \in \text{AlgL} \) such that \( Ax_p = y_p \) \((p = 1, \cdots)\) every \( E \in \mathcal{L} \) reduces \( A \) and \( A \) is compact.

2. \[
\sup \left\{ \frac{\| \sum_{p=1}^l \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p} y_p \|}{\| \sum_{p=1}^l \sum_{k=1}^{m_p} \alpha_{k,p} E_{k,p} x_p \|} : m_p, l \in \mathbb{N}, E_{k,p} \in \mathcal{L} \text{ and } \alpha_{k,p} \in \mathbb{C} \right\} < \infty
\]
and \( y_{qi} x_{qi}^{-1} \to 0 \) as \( i \to \infty \).

References

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