ON INJECTIVITY AND P-INJECTIVITY, IV

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ABSTRACT. This note contains the following results for a ring $A$:
(1) $A$ is simple Artinian if and only if $A$ is a prime right $YJ$-injective, right and left $V$-ring with a maximal right annihilator;
(2) if $A$ is a left quasi-duo ring with Jacobson radical $J$ such that $A/AJ$ is $p$-injective, then the ring $A/J$ is strongly regular;
(3) $A$ is von Neumann regular with non-zero socle if and only if $A$ is a left $p.p.$ ring containing a finitely generated $p$-injective maximal left ideal satisfying the following condition: if $e$ is an idempotent in $A$, then $eA$ is a minimal right ideal if and only if $Ae$ is a minimal left ideal;
(4) If $A$ is left non-singular, left $YJ$-injective such that each maximal left ideal of $A$ is either injective or a two-sided ideal of $A$, then $A$ is either left self-injective regular or strongly regular;
(5) $A$ is left continuous regular if and only if $A$ is right $p$-injective such that for every cyclic left $A$-module $M$, $AM/Z(M)$ is projective. (5) remains valid if $\ll$continuous$\gg$ is replaced by $\ll$self-injective$\gg$ and $\ll$cyclic$\gg$ is replaced by $\ll$finitely generated$\gg$). Finally, we have the following two equivalent properties for $A$ to be von Neumann regular:
(a) $A$ is left non-singular such that every finitely generated left ideal is the left annihilator of an element of $A$ and every principal right ideal of $A$ is the right annihilator of an element of $A$;
(b) Change $\ll$left non-singular$\gg$ into $\ll$right non-singular$\gg$ in (a).

Introduction

The concept of $p$-injective modules was introduced in 1974 to study von Neumann regular rings, $V$-rings, self-injective rings and their generalizations ([16], [17]). This was later generalized to $YJ$-injective modules [24]. Von Neumann regular rings are sometimes called absolutely flat rings in the sense that all left (right) modules are flat (M. Harada
Similarly, we may say that von Neumann regular rings are absolutely $p$-injective since all left (right) modules are $p$-injective (cf. [13], [14], [16], [30]). Many authors have studied injective modules over non-semi-simple Artinian rings and flat modules over non-von Neumann regular rings (cf. for example [1], [2], [3], [5], [6], [8], [11], [12]). We are thus motivated to study $p$-injective modules over rings which are not necessarily von Neumann regular. Throughout, $A$ denotes an associative ring with identity and $A$-modules are unital. $J$ will always denote the Jacobson radical of $A$. An ideal of $A$ will always mean a two-sided ideal of $A$. $A$ is called left quasi-duo (following S. H. Brown) if every maximal left ideal of $A$ is an ideal of $A$. $A$ is called reduced if it contains no non-zero nilpotent element. For a left $A$-module $M$, $Z(M) = \{ z \in M : l(z) \text{ is an essential left ideal of } A \}$ is called the left singular submodule of $M$. $M$ is called left non-singular (resp. singular) if $Z(M) = 0$ (resp. $Z(M) = M$).

The left singular ideal of $A$ is $Z(AA)$ which will be noted $Z$. $A$ is left non-singular if and only if $Z = 0$. It is well-known that $A$ is left non-singular if and only if every left annihilator is a complement left ideal of $A$. $J$ and $Z$ are fundamental concepts in ring theory ([3], [4], [5], [7], [9], [12]). Note that $A$ is von Neumann regular if and only if for every left $A$-module $M$, $Z(M)$ is flat [19, Theorem 5]. But if $Z(M)$ is injective for every left $A$-module $M$, $A$ needs not be semi-simple Artinian (cf. [3], [14] and the example below).

The study of non-singular rings has been motivated by the following well-known facts (among others) : (1) $A$ is left non-singular if and only if $A$ has a von Neumann regular maximal left quotient ring $Q$ (R. E. Johnson). In that case, $Q$ is a left self-injective ring and $AQ$ is the injective hull of $AA$. (2) The classes of non-singular rings include von Neumann regular rings, hereditary rings and prime rings with non-zero socle. In 1967, F. L. Sandomierski proved that if $A$ is left non-singular and has left finite Goldie dimension, then the homomorphic image of every injective left $A$-module contains its singular submodule as a direct summand (cf. the bibliography of [3]). Answering in the negative a question raised by Sandomierski, we showed (1969) that the hypothesis on Goldie dimension is superfluous (cf. Abraham ZAKS’ comment in Math. Reviews 40(1970)#5664 and [23]). A standard reference for non-singular rings and modules is K. R. Goodearl [4].

In [28], we prove the following results : (1) If $A$ is commutative, then every factor ring of $A$ is quasi-Frobeniusean if and only if every factor ring of $A$ is $p$-injective with maximum condition on annihilators; (2)
Every factor ring of $A$ is left self-injective regular with non-zero socle if and only if every factor ring of $A$ is semi-prime with an injective maximal left ideal. Non-singular rings and $p$-injectivity are concerned in some way or other in the results of the present sequel to [28]. In particular, we characterize von Neumann regular rings with non-zero socle, continuous regular rings and self-injective regular ring in terms of $p$-injectivity.

Recall that a left $A$-module $M$ is $p$-injective if for any principal left ideal $P$ of $A$, every left $A$-homomorphism of $P$ into $M$ extends to one of $A$ into $M$. ([3, p.122], [12, p.340], [13], [14], [16]). $A M$ is $\mathcal{Y} \mathcal{J}$-injective if, for any $0 \neq a \in A$, there exists a positive integer $n$ such that $a^n \neq 0$ and every left $A$-homomorphism of $Aa^n$ into $M$ extends to one of $A$ into $M$ ([13], [24], [26], [30]). $P$-injectivity and $\mathcal{Y} \mathcal{J}$-injectivity are similarly defined on the right side. $A$ is called left $p$-injective (resp. $\mathcal{Y} \mathcal{J}$-injective) if and only if $A A$ is $p$-injective (resp. $\mathcal{Y} \mathcal{J}$-injective). $P$-injectivity is also noted principal injectivity in the literature ([8], [10], [30]). (But the term ≪$p$-injective module≫ is used in R. Wisbauer [12] and C. Faith [3]).

We know that $A$ is a right $\mathcal{Y} \mathcal{J}$-injective ring if and only if for every $0 \neq b \in A$, there exists a positive integer $n$ such that $Ab^n$ is a non-zero left annihilator [24, Lemma 3]. Also if $A$ is right $\mathcal{Y} \mathcal{J}$-injective, then $Y$, the right singular ideal of $A$, coincides with $J$ [22, Proposition 1(1)]. Further, if $A$ is reduced right $\mathcal{Y} \mathcal{J}$-injective, then $A$ is strongly regular [22, Proposition 1(2)]. We first characterize simple Artinian rings in terms of $\mathcal{Y} \mathcal{J}$-injectivity and a maximal annihilator. $Y$ will stand for the right singular ideal of $A$.

**Theorem 1.** The following conditions are equivalent: (1) $A$ is simple Artinian; (2) $A$ is a prime right $\mathcal{Y} \mathcal{J}$-injective, right and left $V$-ring with a maximal right annihilator.

**Proof.** It is evident that (1) implies (2).

Assume (2). Let $R = r(c)$ be a maximal right annihilator, where $0 \neq c \in A$. Suppose that $Soc(A)$, the socle of $A$, is zero. Given any maximal left ideal $E$ of $A$, $E$ must be an essential left ideal of $A$. If $Ec = 0$, then $E = l(c)$ and $Ac$ is a minimal left ideal of $A$, contradicting $Soc(A) = 0$. Therefore $Ec \neq 0$. Let $0 \neq b \in E \cap Ec$. Since $A$ is right $\mathcal{Y} \mathcal{J}$-injective, there exists a positive integer $n$ such that $Ab^n$ is a non-zero left annihilator [24, Lemma 3]. Then $b^n = dc$ for some $d \in E$. Now $r(c) = r(dc)$ (in as much as $R = r(c)$ is a maximal right annihilator), which yields $r(b^n) = r(c)$. We have $c \in l(r(c)) = l(r(b^n)) = l(r(\text{Ann}(A))) = Ab^n$ (because $Ab^n$ is a left annihilator). But $Ab^n \subseteq E$ which proves that $c$ belongs to every maximal left ideal of $A$. Therefore $c \in J = Y$ by [22,
Proposition 1(1). Since $A$ is prime, $ckc \neq 0$ for some $k \in A$ and since $c \in Y$, $r(c)$ is an essential right ideal of $A$. Then $r(c) \cup kcA \neq 0$ and there exists $t \in A$ such that $0 \neq kct \in r(c)$. Therefore $t \in r(ckc) = r(c)$ (again because $r(c)$ is a maximal right annihilator), yielding $ct = 0$, which contradicts $kct \neq 0$. This proves that $Soc(A) \neq 0$. Since $A$ is a prime left and right $V$-ring, $A$ contains injective, projective, simple, faithful left and right modules, then $A$ is simple Artinian by a result of J.P. Jans [Pac. J. Math. 9(1959), 1103-1108 (Corollary 2.2)]. Thus (2) implies (1).

□

The next lemma is due to HuaPing Yu [15].

Lemma 2. If $A$ is left quasi-duo, then $A/J$ is a reduced ring.

Proposition 3. Let $A$ be a left quasi-duo ring such that $A/AJ$ is $p$-injective. Then $A/J$ is a strongly regular ring.

Proof. Let $B = A/J$, $b \in B$, $b = a + J$, $a \in A$, $f : Bb \to B$ a left $B$-homomorphism. Then $f : (Aa + J)/J \to A/J$ and $f(a + J) = d + J$ for some $d \in A$. Define a left $A$-homomorphism $g : Aa \to A/J$ by $g(ca) = cd + J$ for all $c \in A$. It is easily seen that $g$ is a well-defined left $A$-homomorphism. Since $A/AJ$ is $p$-injective, there exists $u \in A$ such that $g(ca) = cau + J$ for all $c \in A$. Therefore $f((ca + J) = (c + J)f(a + J) = (c + J)(d + j) = cd + J = g(ca) = cau + J = (ca + J)(u + J)$ for all $c \in A$. This proves that $B = A/J$ is a left $p$-injective ring. By Lemma 2, $B$ is a reduced ring. Therefore $B$ is a strongly regular ring by [22, Proposition 1(2)].

□

Remark 1. In Proposition 3, we do not have a von Neumann regular ring $A/J$ if left quasi-duo is withdrawn. Indeed, Beidar-Wisbauer [2, Example 4.8] showed that if $A$ is a semi-prime left (and right) $p$-injective, $P.I.$ ring, then $A$ is not necessarily von Neumann regular. (Note that $J = Z = 0$ here). They answered in the negative a question raised in 1981.

Question 1. Is it true that $Z \cap J = 0$ if every simple singular left $A$-module is $YJ$-injective? (The answer is yes if $YJ$-injective ring is replaced by $p$-injective (cf. [18, Proposition 3]).

The next remark improves [26, Proposition 6].

Remark 2. If $A$ is a semi-prime right $YJ$-injective ring, then the centre of $A$ is von Neumann regular. (But $A$ itself is not necessarily regular as confirmed by [2, Example 4.8]).
A well-known theorem of I. Kaplansky asserts that a commutative ring $A$ is von Neumann regular if and only if every simple $A$-module is injective. (This remains true if $\text{injective}$ is weakened to $\text{YJ-injective}$).

As usual, $A$ is called a right (resp. left) SF-ring if every simple right (resp. left) $A$-module is flat.

**Notation.** Write $A$ satisfies (*) if $A$ has a finite number of maximal right ideals whose product is contained in $J$.

**Proposition 4.** If $A$ is a right SF-ring, then

1. Any left regular element is right invertible in $A$; Consequently, $A$ coincides with its classical right (and left) quotient ring.
2. $Z \subseteq J$.

**Proof.** (1) Let $c \in A$ such that $l(c) = 0$. If we suppose that $cA \neq A$, let $M$ be a maximal right ideal containing $cA$. Then $A/M_A$ is flat which implies $c = dc$ for some $d \in M$. Therefore $1 - d \in l(c) = 0$ which yields $1 = d \in M$, contradicting $M \neq A$. This proves that $cu = 1$ for some $u \in A$. For any non-zero-divisor $c$, $c = cuc, u \in A$, and $1 - uc \in r(c) = 0$, whence $uc = cu = 1$, proving that every non-zero-divisor is invertible in $A$. It follows that $A$ coincides with its classical right (and left) quotient ring.

(2) was proved by YuFei Xiao [One sided SF-rings with certain chain conditions, Canad. Math. Bull. 37 (1994), 272–277].

**Corollary 5.** If $A$ is a right SF-ring whose simple left modules are either $p$-injective or projective, then $Z = 0$.

**Proof.** By [18, Proposition 3], $Z \cap J = 0$. Now apply Proposition 4 (2). 

**Proposition 6.** Let $A$ be a right SF-ring satisfying (*). Then $Z = J = 0$.

**Proof.** Let $M_1, \cdots, M_n$ be maximal right ideals of $A$ such that $M_1 M_2 \cdots M_{n-1} M_n \subseteq J$. If $u \in J$, since $u \in M_n$ and $A/M_{nA}$ is flat, then $u = u_n u$ for some $u_n \in M_n$. Since $u_n u \in J \subseteq M_{n-1}$ and $A/M_{n-1A}$ is flat, $u = u_n u = u_{n-1} u_n u$ for some $u_{n-1} \in M_{n-1}$ and so on. Finally, we have $u_i \in M_i$ for $1 \leq i \leq n$, such that $u_1 u_2 \cdots u_{n-1} u_n \in M_1 M_2 \cdots M_{n-1} M_n \subseteq J$ and $u = u_1 u_2 \cdots u_{n-1} u_n$. Now $v(1 - u_1 u_2 \cdots u_n) = 1$ for some $v \in A$ which yields $u = 1u = v(1 - u_1 u_2 \cdots u_n) u = 0$. Thus $J = 0, Z = 0$ by Proposition 4(2).
Corollary 7. If $A$ is a left or right self-injective, right SF-ring satisfying (*), then $A$ is von Neumann regular.

Now an example of a ring satisfying (*).

Example. Set $A = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$ where $K$ is a field. Then $A$ has only two maximal right 0 K ideals: $M = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$.

$M_A$ is injective while $L$ is the unique proper essential right ideal of $A$. Every simple right $A$-module is either injective or projective and the right (or left) singular ideal of $A$ is zero. But $J = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ and $J^2 = 0$.

$A$ is neither semi-prime nor right $p$-injective. $M, L$ are ideals of $A$ and $LM \subseteq J$ but $A$ is not right SF. Also, $A$ is a ring whose singular right modules are injective but $A$ is not von Neumann regular.

The above example shows that if $A$ is a ring such that each maximal right ideal is either injective or an ideal of $A$, $A$ needs neither be semi-prime nor right self-injective. It is well-known that if $A$ is semi-prime, for any idempotent $e$, $eA$ is a minimal right ideal if and only if $Ae$ is a minimal left ideal of $A$.

Theorem 8. Let $A$ be a left $p.p.$ ring containing a finitely generated $p$-injective maximal left ideal such that for any idempotent $e$ of $A$, $eA$ is a minimal right ideal if and only if $Ae$ is a minimal left ideal of $A$. Then $A$ is von Neumann regular with non-zero socle.

Proof. Let $M$ be a finitely generated $p$-injective maximal left ideal of $A$. Since $A M$ is $p$-injective, then $A A / M$ is flat. Since $M$ is finitely generated, then $A / M$ is a finitely related flat left $A$-module which is therefore projective (S. U. Chase). Consequently, $A M$ is a direct summand of $A A$. Then $A = M \oplus U$, where $U$ is a minimal left ideal of $A$, showing that $A$ has non-zero socle. If $U = A u$, $u = u^2 \in A$, by hypothesis, $u A$ is a minimal right ideal of $A$. First suppose that $M$ is an ideal of $A$. Then $M = A e$, where $e = 1 - u$, $M$ is a maximal right ideal of $A$ and $M \cap u A = 0$ (in as much as $u A$ is a minimal right ideal of $A$). Therefore $A A = M A \oplus u A A$ and $A / M A$ is simple projective. By [21, Lemma 1], $A A / M$ is $p$-injective and $A A = A M \oplus A U$ is $p$-injective. Now suppose that $M$ is not an ideal of $A$. Then $A = M \oplus U$, where $U = A u$, $u = u^2 \in A$, $M = A e$, $e = 1 - u$. If $M U = 0$, then $A = M A$ implies that $A u = M U = 0$, contradicting $u \neq 0$. Therefore $M U \neq 0$ and if $v \in U$ such that $M v \neq 0$, we have $M v = U$ and the map $g : M \to U$ defined by $g(m) = m v$ for all $m \in M$ yields $A M / \ker g \approx A U$. Since $A U$
is projective, \( \text{ker}g \) is a direct summand of \( \_AM \) which yields \( \_AM/\text{ker}g \) \( p \)-injective (because \( \_AM \) is \( p \)-injective). Thus \( \_AU \) is \( p \)-injective which implies that \( A = M \oplus U \) is left \( p \)-injective. In any case, \( A \) is left \( p \)-injective. Since \( A \) is a left \( p.p. \) ring, then every quotient of a \( p \)-injective left \( A \)-module is \( p \)-injective [17, p.176] (cf. also [12, p.340]). Since \( \_AA \) is \( p \)-injective, every cyclic left \( A \)-module is \( p \)-injective which proves that \( A \) is von Neumann regular [16, Lemma 2].

Since a semi-prime principal left ideal ring is left hereditary, we get

**Corollary 9.** \( A \) is semi-simple Artinian if and only if \( A \) is a semi-prime principal left ideal ring containing a \( p \)-injective maximal left ideal.

In connection with Theorem 1, we have

**Remark 3.** \( A \) is simple Artinian if and only if \( A \) is a prime right and left \( V \)-ring with a finitely generated \( p \)-injective maximal right ideal.

As before, \( A \) is called a left \( MI \)-ring if \( A \) contains an injective maximal left ideal. The proof of Theorem 8 yields the next lemma.

**Lemma 10.** Let \( A \) be a left \( MI \)-ring such that for any idempotent \( e \), \( eA \) is a minimal right ideal of \( A \) if and only if \( Ae \) is a minimal left ideal of \( A \). Then \( A \) is left self-injective.

**Proposition 11.** Let \( A \) be a left non-singular, left \( YJ \)-injective ring such that each maximal left ideal is either injective or an ideal of \( A \). Then \( A \) is either left self-injective regular or strongly regular.

**Proof.** Since \( A \) is left \( YJ \)-injective, then by [22, Proposition 1(1)], \( Z = J \). Therefore \( J = 0 \). First suppose that every maximal left ideal of \( A \) is an ideal of \( A \). Since \( J = 0 \), by Proposition 2, \( A \) is reduced. Since \( A \) is left \( YJ \)-injective, then \( A \) is strongly regular by [22, Proposition 1(2)]. Now suppose there exists a maximal left ideal \( M \) of \( A \) which is not an ideal of \( A \). By hypothesis, \( \_AM \) is injective and \( A \) is left \( MI \). Since \( A \) is semi-prime (because \( J = 0 \)), then by Lemma 11, \( A \) is left self-injective. Since \( J = 0 \), \( A \) is von Neumann regular.

The proof of Proposition 11 together with [17, Lemma 1] and [1, Theorem 3.1] yields

**Proposition 12.** Let \( A \) be a ring whose simple right modules are \( p \)-injective and such that each maximal right ideal is either injective or an ideal of \( A \). Then \( A \) is either right self-injective regular or strongly regular.
In connection with Sandomierski’s problem, we showed that if $A$ is left non-singular, then (1) $Z(M)$ is injective for every injective left $A$-module $M$ and (2) for any complement left ideal $C$ of $A$, $Z(A/C) = 0$. Recall that $A$ is left continuous (in the sense of Y. Utumi) if (a) every left ideal of $A$ isomorphic to a direct summand of $AA$ is a direct summand of $A$ and (b) every complement left ideal of $A$ is a direct summand of $AA$. In that case, $J = Z$ and $A/Z$ is von Neumann regular by a result of Y. Utumi (1965).

**Theorem 13.** The following conditions are equivalent: (1) $A$ is left continuous regular; (2) $A$ is a right $p$-injective ring such that for every cyclic left $A$-module $M$, $A/M/Z(M)$ is projective.

**Proof.** Assume (1). Then $Z = 0.$ In that case, for every cyclic left $A$-module, $Z(M/Z(M)) = 0.$ Write $C = M/Z(M).$ Then $C = Ac$ (being cyclic). For every essential extension $E$ of $l(c)$ in $AA,$ any $y \in E,$ there exists an essential left ideal $L$ of $A$ such that $Ly \subseteq l(c),$ which implies that $L \subseteq l yc,$ whence $yc \in Z(C) = 0.$ Therefore $y \in l(c)$ which yields $E = l(c)$ proving that $l(c)$ is a complement left ideal of $A.$ Since $A$ is left continuous, $l(c)$ is a direct summand of $AA.$ Then $AAc \approx A/l(c)$ is projective which means that $A/M/Z(M)$ is projective. Thus (1) implies (2). Conversely, assume (2). Then $AA/Z(A)$ is projective which implies that $Z = Z(A)$ is a direct summand of $AA,$ whence $Z = 0$ ($Z$ cannot contain a non-zero idempotent). Then for every complement left ideal of $K$ of $A,$ $Z(A/K) = 0.$ By hypothesis, $AA/K$ is projective which implies that $AA$ is a direct summand of $AA.$ Since $A$ is right $p$-injective, by Ikeda-Nakayama’s theorem, every principal left ideal $P$ of $A$ is a left annihilator. Since $Z = 0,$ $P$ is a complement left ideal of $A.$ In that case, $AP$ is a direct summand of $AA.$ This proves that $A$ is von Neumann regular. $A$ is therefore left continuous and (2) implies (1).

We may now have a nice characterization of left self-injective regular rings.

**Theorem 14.** The following conditions are equivalent:
(1) $A$ is a left self-injective regular ring;
(2) $A$ is a right $p$-injective ring such that for every finitely generated left $A$-module $M,$ $A/M/Z(M)$ is projective.

**Proof.** Assume (1). Since $Z = 0,$ for any finitely generated left $A$-module $M,$ $M/Z(M),$ is a non-singular left $A$-module. A finitely generated non-singular left $A$-module is projective by [29, Corollary 6]. Therefore $A/M/Z(M)$ is projective. Thus (1) implies (2). Assume (2).
Then $A$ is left continuous regular by Theorem 14. Let $AE$ denote the injective hull of $A$. For any $u \in E$, $B = A + Au$ is a finitely generated non-singular left $A$-module. By hypothesis, $AB$ is projective. Since the left annihilator of any proper finitely generated right ideal of $A$ is non-zero, by a well-known theorem of H. Bass, $AA$ is a direct summand of $AB$. But $AA$ is essential in $AB$ which yields $A = B$. This proves that $u \in A$ and hence $E = A$ is a left self-injective regular ring. Therefore (2) implies (1).

We know that if $A$ is left $p$-injective, then any left ideal isomorphic to a direct summand of $AA$ is itself a direct summand of $AA$. The proof of Theorems 13 and 14 then yield. □

Theorem 15. (1) $A$ is left continuous regular if and only if $A$ is a left $p$-injective ring such that for every cyclic left $A$-module $M$, $AM/Z(M)$ is projective; (2) $A$ is left self-injective regular if and only if $A$ is a left $p$-injective ring such that for every finitely generated left $A$-module $M$, $AM/Z(M)$ is projective.

We now turn to new characteristic properties of von Neumann regular rings.

Theorem 16. The following conditions are equivalent:
(1) $A$ is von Neumann regular;
(2) $A$ is a left non-singular ring such that every finitely generated left ideal is the left annihilator of an element of $A$ and every principal right ideal of $A$ is the right annihilator of an element of $A$;
(3) $A$ is a right non-singular ring such that every finitely generated left ideal is the left annihilator of an element of $A$ and every principal right ideal is the right annihilator of an element of $A$;
(4) $A$ is a right SF-ring whose divisible torsionfree right modules are $p$-injective;
(5) $A$ is a right SF-ring whose divisible torsionfree left modules are $p$-injective.

Proof. (1) implies (2) through (5) evidently.

Assume (2). Let $F$ be a finitely generated left ideal of $A$. Then $F = l(u), u \in A$, and $uA = r(w), w \in A$. Therefore $F = l(uA) = l(r(w)) = l(r(Aw))$. Since $Aw$ is a left annihilator, then $Aw = l(r(Aw))$ which implies that $F = Aw$ is a principal left ideal. Since every principal right ideal of $A$ is a right annihilator, by Ikeda-Nakayama’s theorem, $A$ is left $p$-injective. In that case, the left singular ideal $Z$ of $A$ coincides with the Jacobson radical $J$ of $A$ (cf. [22, Proposition 1(1)]). Therefore
$J = 0$ which implies $A$ semi-prime. Since every finitely generated left ideal of $A$ is principal, by [3, Theorem 7.5B], $A$ is left semi-hereditary. We have seen, in the proof of Theorem 9, that a left $p$-injective left $p.p.$ ring is von Neumann regular. Thus (2) implies (1). Similarly, (3) implies (1) (in as much as (3) implies that $A$ is left and right $p$-injective). Either (4) or (5) implies (1) by [26, Theorem 3] and Proposition 4(1).

**Question 2.** Is $A$ von Neumann regular if $A$ is left non-singular such that every principal one-sided ideal is the annihilator of an element of $A$?

A theorem of L. Levy (1963) and Proposition 4(1) yield an analogical result for semi-simple Artinian rings in terms of certain injective modules.

**Theorem 17.** The following conditions are equivalent:

1. $A$ is semi-simple, Artinian;
2. $A$ is a right SF-ring whose divisible torsionfree right modules are injective;
3. $A$ is a right SF-ring whose divisible torsionfree left modules are injective.

We now give a characteristic property of semi-prime rings.

**Proposition 18.** The following conditions are equivalent:

1. $A$ is a semi-prime ring;
2. Every essential left ideal of $A$ is a faithful left $A$-module.

**Proof.** Assume (1). Let $L$ be an essential left ideal of $A$. Since $A$ is semi-prime, $L \cap l(L) = 0$. Now $L$ essential implies that $l(L) = 0$. Thus (1) implies (2).

Assume (2). If $T$ is a non-zero ideal of $A$ such that $T^2 = 0$, then $T$ is not an essential left ideal by hypothesis. Let $K$ be a non-zero complement left ideal of $A$ such that $L = T \oplus K$ is an essential left ideal of $A$ ($K$ exists by Zorn’s Lemma). Then $TK \subseteq T \cap K = 0$ which implies that $TL = T(T \oplus K) = 0$, whence $l(L) \neq 0$, contradicting $A$L faithful. This proves that $A$ must be semi-prime and (2) implies (1).

**Corollary 19.** $A$ is a left non-singular semi-prime ring if and only if for every essential left ideal $L$ of $A$, $l(L) = r(L) = 0$.

**Remark 4.** If $A$ is a right non-singular ring such that every essential left ideal of $A$ is an essential right ideal, then $A$ is semi-prime left non-singular.
Remark 5. Let $A$ be a reduced ring having a classical left quotient ring $Q$. If $Q$ is a left $MI$-ring, then $Q$ is left and right self-injective strongly regular and $Q$ is also the classical right quotient ring of $A$.

Finally, we note that, answering positively two questions of the author, Zhang-Wu show that (1) Von Neumann regular rings are absolutely $YJ$-injective [30, Theorem 9] and (2) $A$ is a $\Pi$-regular ring if and only if every left $A$-module $M$ has the following property: for any $a \in A$, there exists a positive integer $n$ (depending on $a$) such that every left $A$-homomorphism of $Aa^n$ into $M$ extends to one of $A$ into $M$ [30, Theorem 3]. (Indeed, (2) confirms that $\Pi$-regular rings are absolutely $GP$-injective (cf. [30] for the definition of $GP$-injectivity)).

For other results on various generalizations of injectivity, consult, for example,


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References


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