AFFINENESS OF DEFINABLE $C^r$ MANIFOLDS AND ITS APPLICATIONS

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Abstract. Let $\mathcal{M}$ be an exponentially bounded o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. We prove that if $r$ is a non-negative integer, then every definable $C^r$ manifold is affine. Let $f: X \to Y$ be a definable $C^1$ map between definable $C^1$ manifolds. We show that the set $S$ of critical points of $f$ and $f(S)$ are definable and $\dim f(S) < \dim Y$. Moreover we prove that if $1 < s < r < \infty$, then every definable $C^s$ manifold admits a unique definable $C^r$ manifold structure up to definable $C^r$ diffeomorphism.

1. Introduction

Let $\mathcal{M}$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term “definable” means “definable with parameters in $\mathcal{M}$”, and any manifold in this paper does not have boundary, unless otherwise stated. Several properties of definable $C^r$ manifolds and definable $C^r$ maps are studied in [9], [10], [8]. The Nash category coincides with the definable $C^\infty$ category based on $\mathcal{R}$ [15], and definable $C^r$ categories based on $\mathcal{M}$ are generalizations of the $C^r$ Nash category. General references on o-minimal structures are [3], [5], see also [14]. Further properties and constructions of them are studied in [4], [6], [12].

We say that $\mathcal{M}$ is polynomially bounded if for every function $f: \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number $k$ and a real number $x_0$ such that $|f(x)| \leq x^k$ for any $x > x_0$. Otherwise, $\mathcal{M}$ is called exponential. One of typical examples of polynomially bounded structures is $\mathcal{R}$. By a result of C. Miller [11], if $\mathcal{M}$ is exponential,
then the exponential function $\mathbb{R} \to \mathbb{R}, x \mapsto e^x$ is definable. We call $\mathcal{M}$ \textit{exponentially bounded} if for every function $h : \mathbb{R} \rightarrow \mathbb{R}$ definable in $\mathcal{M}$, there exist a natural number $l$ and a real number $x_1$ such that $|h(x)| \leq \exp_l(x)$ for any $x > x_1$, where $\exp_l(x)$ denotes the $l$th iterate of the exponential function, e.g. $\exp_2(x) = e^{e^x}$. Note that the problem that every o-minimal expansion $\mathcal{M}$ of $\mathbb{R}$ is exponentially bounded is still open (e.g. [2]).

**Theorem 1.1.** If $\mathcal{M}$ is exponentially bounded and $0 \leq r < \infty$, then every definable $C^r$ manifold is affine.

Theorem 1.1 is a generalization of 1.1 [10] and an equivariant $C^\infty$ version of Theorem 1.1 is true if $\mathcal{M}$ is exponential and the manifold is compact (see 1.2 [10]). If $\mathcal{M} = \mathbb{R}$ and $r = \infty$, then Theorem 1.1 is not true [13].

As applications of Theorem 1.1, we have the following two results.

Let $A$ be a subset of an $n$-dimensional definable $C^r$ manifold $X$ with a definable $C^r$ atlas $\{(U_i, \phi_i : U_i \to \mathbb{R}^n)\}$, and $r > 0$. We say that $A$ has \textit{measure} 0 in $X$ if each $\phi_i(U_i \cap A) \subset \mathbb{R}^n$ has measure 0 (e.g. see P.68 [7]).

**Theorem 1.2.** Let $X$ and $Y$ be definable $C^1$ manifolds and $f : X \to Y$ a definable $C^1$ map. If $\mathcal{M}$ is exponentially bounded, then the set $S$ of critical points of $f$ and $f(S)$ are definable and $\text{dim } f(S) < \text{dim } Y$. In particular, the measure of $f(S)$ in $Y$ is 0.

Without assuming that $f$ is definable, there exists a $C^1$ map from $\mathbb{R}^2$ to $\mathbb{R}^1$ whose critical point set has positive measure [17]. Note that if $\text{dim } X < \text{dim } Y$ and $f$ is a definable $C^1$ imbedding, then $S = X$, in particular, $\text{dim } f(S) = \text{dim } X$. Thus in Theorem 1.2, one cannot replace $\text{dim } f(S) < \text{dim } Y$ by $\text{dim } f(S) < \text{min}(\text{dim } X, \text{dim } Y)$.

**Theorem 1.3.** If $\mathcal{M}$ is exponentially bounded and $1 < s < r < \infty$, then every definable $C^s$ manifold admits a unique definable $C^r$ manifold structure up to definable $C^r$ diffeomorphism.

By [13], there exists an uncountable family $\{X\}_{\lambda \in \Lambda}$ of Nash manifolds such that they are $C^2$ Nash diffeomorphic and that $X_\lambda$ is not Nash diffeomorphic to $X_\mu$ for $\lambda \neq \mu$. Thus if $\mathcal{M} = \mathbb{R}$ and $r = \infty$, then Theorem 1.3 does not hold.

2. Proof of results

To prove Theorem 1.1, we need the following three results.
**Proposition 2.1 (3.2 [10]).** Let $X$ be an affine definable $C^r$ manifold and $0 \leq r < \infty$. Then $X$ can be definably $C^r$ imbeddable into some $\mathbb{R}^n$ such that $X$ is closed in $\mathbb{R}^n$. Moreover it is possible to definably $C^r$ imbeddable into some $\mathbb{R}^k$ such that $X$ is bounded and $\overline{X} - X$ consists of at most one point, where $\overline{X}$ denotes the closure of $X$ in $\mathbb{R}^k$.

Let $e_n : \mathbb{R} \to \mathbb{R}, n \in \mathbb{N}$ be the function defined by

$$e_n(x) = \begin{cases} e^{-\exp_{n-1}(1/x^2)}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

where $\exp_0(x) = x$. Then elementary computations show the following proposition.

**Proposition 2.2.**

(1) For any polynomial function $P(x_1, \cdots, x_n)$ in $n$ variables,

$$\lim_{x \to 0} P \left( \frac{1}{x}, \exp_1 \left( \frac{1}{x^2} \right), \cdots, \exp_{n-1} \left( \frac{1}{x^2} \right) \right) e_n(x) = 0.$$

(2) Every $e_n$ is a $C^\infty$ function.

Since $M$ is exponentially bounded, in the proof of C.5 [5], we can take $\phi(t) = te_n(t)$ for some $n \in \mathbb{N}$. Hence a similar proof of C.14 [5] proves the following proposition.

**Proposition 2.3 ([5]).** Let $A$ be a non-empty compact definable subset of $\mathbb{R}^n$ and $f, g$ two continuous definable functions on $A$ such that $f^{-1}(0) \subset g^{-1}(0)$. If $M$ is exponentially bounded, then there exist a natural number $k$ and a positive constant $c$ such that

$$e_k(g) \leq c|f|$$
on $A$.

**Proof of Theorem 1.1.** Let $X$ be a definable $C^r$ manifold. If $\dim X = 0$, then $X$ consists of finitely many points. Thus the result holds.

Assume that $m := \dim X \geq 1$. Let $\{\phi_i : U_i \to \mathbb{R}^m \}_{i=1}^l$ be a definable $C^r$ atlas of $X$. Then each $\phi_i(U_i)$ is a noncompact definable $C^r$ submanifold of $\mathbb{R}^m$. Hence by Proposition 2.1, we have a definable $C^r$ imbedding $\phi'_i : \phi_i(U_i) \to \mathbb{R}^{m'}$ such that the image is bounded in $\mathbb{R}^{m'}$ and $\overline{\phi'_i \circ \phi_i(U_i)} = \phi'_i \circ \phi_i(U_i)$ consists of one point, say 0. For a sufficiently large positive integer $n$, set

$$\eta : \mathbb{R}^{m'} \to \mathbb{R}^{m'}, \eta(x_1, \cdots, x_{m'}) = \left( \sum_{j=1}^{m'} e_n(x_j)x_1, \cdots, \sum_{j=1}^{m'} e_n(x_j)x_{m'} \right),$$
Then $g_i$ is a definable $C^r$ imbedding of $U_i$ into $\mathbb{R}^{m'}$.

We now prove that the extension $\tilde{g}_i : X \rightarrow \mathbb{R}$ of $g_i$ is defined by $\tilde{g}_i = 0$ on $X - U_i$ is of class definable $C^r$. It is sufficient to see this on each definable $C^r$ coordinate neighborhood of $X$. Hence we may assume that $X$ is open and bounded in $\mathbb{R}^m$. We only have to prove that for any sequence $\{a_i\}_{i=1}^\infty$ in $U_i$ convergent to a point of $X - U_i$ and for any $\alpha \in (\mathbb{N} \cup \{0\})^m$ with $|\alpha| \leq r$, $\{D^\alpha g_i(a_i)\}_{i=1}^\infty$ converges to 0. On the other hand, $g_i = (\sum_{j=1}^{m'} e_n(\phi_{ij})\psi_{ij}, \cdots \sum_{j=1}^{m'} e_n(\phi_{ij})\psi_{im'})$, where $\psi_{ij}(\phi_{ij}) = (\phi_{ij}, \cdots, \phi_{im'})$. By the construction of $\psi_{ij}$, $\{\psi_{ij}(a_i)\}_{i=1}^\infty$ converges to 0. Hence for any natural number $k$, $\{e_k(\phi_{ij}(a_i))\psi_{is}(a_i)\}_{i=1}^\infty$ converges to 0. Assume that if $|\alpha| \leq r - 1$, then there exists some $K \in \mathbb{N}$ such that if $k \geq K$, then $D^\alpha(e_k(\phi_{ij}(a_i))\psi_{is}(a_i)) \rightarrow 0$ as $t \rightarrow \infty$. Let $D^\alpha(e_k(\phi_{ij}(x)))\psi_{is}(x) = F(x)e_k(\phi_{ij}(x))$. Then $F$ is a definable $C^{r-|\alpha|}$ map on $U$.

Let $\psi = \max\{1, |\partial F/\partial x_1|, |\partial \phi_{ij}/\partial x_1|\}$.

Define

$$\theta_{ij} = \begin{cases} \min\{|\phi_{ij}|, 1/\psi\} & \text{on } U_i \\ 0 & \text{on } X - U_i \end{cases}, \quad \tilde{\phi}_{ij} = \begin{cases} \phi_{ij} & \text{on } U_i \\ 0 & \text{on } X - U_i \end{cases}.$$

Then $\theta_{ij}$ and $\tilde{\phi}_{ij}$ are continuous definable maps on $X$ such that

$$X - U_i \subset (\theta_{ij})^{-1}(0) = (\tilde{\phi}_{ij})^{-1}(0).$$

Moreover by the construction of $\phi_{ij}$, $\theta_{ij}$ and $\tilde{\phi}_{ij}$, $\theta_{ij}$ and $\tilde{\phi}_{ij}$ are extendable to continuous definable maps on $\mathbb{R}^m$. Hence by Proposition 2.3, there exist a positive integer $a$, a positive number $b$ and a definable open neighborhood $V$ of $X - U_i$ in $X$ such that $e_a(\phi_{ij}) \leq b|\theta_{ij}|$ on $V$.

On the other hand, by the definition of $\theta_{ij}$, $|\psi \theta_{ij}| \leq 1$ on $U_i$. Thus $|\psi|e_a(\tilde{\phi}_{ij}) \leq b$. Hence if $n \geq N : = K + a + 1$, then

$$\frac{\partial}{\partial x_1} (D^\alpha(e_n(\phi_{ij})\psi_{is})) = \frac{\partial}{\partial x_1} (F e_n(\phi_{ij})) = \frac{\partial F}{\partial x_1} e_n(\phi_{ij}) + F R_1 e_n(\phi_{ij}) = \frac{\partial F}{\partial x_1} e_n(\phi_{ij}) + (F e_K(\phi_{ij}))(R_1 e_n(\phi_{ij})).$$
where $R_1 = 2(\frac{\partial \phi_{ij}}{\partial x_1}) \frac{\partial}{\partial x_1} \left( \frac{1}{\phi_{ij}} \right) \cdots \frac{\partial}{\partial x_1} \left( \frac{1}{\phi_{ij}} \right)$. Thus using the inductive hypothesis and Proposition 2.2,

$$|\frac{\partial}{\partial x_1} (D^n(\alpha(\phi_{ij})))|$$

$$\leq |\frac{\partial}{\partial x_1} e_n(\phi_{ij}) + |F\alpha_K(\phi_{ij})||R_1| e_n(\phi_{ij})$$

$$\leq b \frac{e_n(\phi_{ij})}{e_a(\phi_{ij})} + |F\alpha_K(\phi_{ij})| \frac{2b e_n(\phi_{ij})}{e_a(\phi_{ij}) e_K(\phi_{ij})} \frac{\exp(\frac{1}{\phi_{ij}}) \cdots \exp_{n-1}(\frac{1}{\phi_{ij}})}{|\phi_{ij}|} \rightarrow 0.$$ 

By the above argument, replacing some larger $N$, if $|\alpha| \leq r$ and $n \geq N$, then $|D^n(\alpha(\phi_{ij})))| \rightarrow 0$. Therefore if $n \geq N$, then each $\tilde{g}_i$ is a definable $C^r$ map and the function $h_i : X \rightarrow \mathbb{R}$ defined by $h_i = \sqrt{(\tilde{g}_i1)^2 + \cdots + (\tilde{g}_im')^2 + 1}$ is a definable $C^r$ function with $h_i(X - U_i) = 1$, $(1 \leq i \leq l)$, where $\tilde{g}_i = (\tilde{g}_i1, \cdots, \tilde{g}_im')$, $(1 \leq i \leq l)$. It is easy to see that

$$(\tilde{g}_i1, \cdots, \tilde{g}_i, h_1, \cdots, h_l) : X \rightarrow \mathbb{R}^{m'} \times \mathbb{R}^l$$

is a definable $C^r$ imbedding. \hfill \Box

**Proof of Theorem 1.2.** Since $\mathcal{M}$ is exponentially bounded and by Theorem 1.1, we may assume that $X$ and $Y$ are affine.

The first half of the theorem is obvious. We have only to prove the latter half. If $\dim X < \dim Y$, then $\dim f(S) \leq \dim f(X) \leq \dim X < \dim Y$. Thus we assume that $\dim Y \leq \dim X$.

By Sard’s theorem (e.g. 3.1.3 [7]), if $r > \max(0, \dim X - \dim Y)$, then the set of critical values of every $C^r$ map from $X$ to $Y$ has measure 0 in $Y$. Fix such an $r$.

By the definable $C^r$ cell decomposition theorem (e.g. 7.3.3 [3]), there exists a finite partition $\{C_i\}_i$ of $X$ into definable $C^r$ cells such that each $f|C_i : C_i \rightarrow Y$ is a definable $C^r$ map. Note that every $C_i$ is a definable $C^r$ submanifold of $X$ and that $C_i$ is open in $X$ if $\dim C_i = \dim X$.

Let $K_i$ denote the set of critical values of $f|C_i : C_i \rightarrow Y$ and let $K = f(S)$. Then by Sard’s theorem, each $K_i$ has measure 0 in $Y$. Thus $\dim K_i < \dim Y$. Hence $\dim \cup_i K_i < \dim Y$.

We now prove $K \subset \bigcup_i K_i \cup \dim C_i < \dim Y f(C_i)$. Let $y \in K$. Then there exists an $x \in X = \bigcup_i C_i$ such that $y = f(x)$ and the rank of the Jacobian of $f$ at $x$ is smaller than $\dim Y$. Assume that $x \in C_i$. If $\dim C_i < \dim Y$, then $y = f(x) \in \cup \dim C_i < \dim Y f(C_i)$. If $\dim C_i = \dim X$, then $y = f(x) \in K_i$ because $C_i$ is open in $X$. Assume that $\dim Y \leq \dim C_i < \dim X$. Since $C_i$ is a definable $C^r$ submanifold of $X$,
there exists a definable $C^r$ chart $\phi : U \to V \subset \mathbb{R}^k$ of $X$ around $x$ such that $\phi(x) = 0$ and $\phi(C_i \cap U) = V \cap \mathbb{R}^l$, where $k = \dim X, l = \dim C_i$ and $\mathbb{R}^l = \mathbb{R}^l \times 0 \subset \mathbb{R}^k$. The Jacobian $A$ of $((f|_{C_i}) \circ \phi^{-1})$ at $\phi(x)$ is a submatrix of the Jacobian $B$ of $f \circ \phi^{-1}$ at $\phi(x)$. Then the determinant of every minor of $B$ of degree $\dim Y$ at $x$ is 0 because $\dim Y \leq \dim C_i < \dim X$. Hence the rank of $A$ at $\phi(x)$ is smaller than $\dim Y$. Thus $y \in K_i$.

Since $\dim \cup_i K_i < \dim Y$ and $\dim f(C_i) \leq \dim C_i$, $\dim K = \dim f(S) < \dim Y$.

To prove Theorem 1.3, we need the following several results.

**Proposition 2.4** (1.3 [8]). Let $1 \leq r < \infty$. Then every definable $C^r$ submanifold $X$ of $\mathbb{R}^n$ has a definable $C^r$ tubular neighborhood $(U, p)$ of $X$ in $\mathbb{R}^n$, namely $U$ is a definable open neighborhood of $X$ in $\mathbb{R}^n$ and $p : U \to X$ is a definable $C^r$ map with $p|X = \text{id}_X$.

**Theorem 2.5** (1.2 [9]). If $0 < r < \infty$, then every noncompact affine definable $C^r$ manifold is definably $C^r$ diffeomorphic to the interior of some compact affine definable $C^r$ manifold with boundary.

**Theorem 2.6** (5.8 [8]). If $2 \leq r < \infty$, then every compact affine definable $C^r$ manifold with boundary admits a definable $C^r$ collar, namely there exists a definable $C^r$ imbedding $\phi : \partial X \times [0, 1] \to X$ such that $\phi((\partial X \times \{0\})$ is the inclusion $\partial X \to X$, where the action on $[0, 1]$ is trivial.

Note that Proposition 2.4, Theorem 2.5 and 2.6 are true in more general settings (see 1.3 [8], 1.2 [9] and 5.8 [8]).

The following two results are algebraic realizations of compact $C^\infty$ manifolds.

**Theorem 2.7** ([16]). Every compact $C^\infty$ manifold is $C^\infty$ diffeomorphic to a nonsingular algebraic set.

**Theorem 2.8** ([1]). Let $X'$ be a compact $C^\infty$ submanifold of a compact $C^\infty$ manifold $X$. Then there exist a nonsingular algebraic set $Y$ and its nonsingular algebraic subset $Y'$ such that $(X; X')$ is $C^\infty$ diffeomorphic to $(Y; Y')$.

The following is a result for raising differentiability of manifolds.

**Theorem 2.9** (2.2.9 [7]). If $1 \leq s < \infty$, then every $C^s$ manifold admits a compatible $C^\infty$ manifold structure. In other words, for any $C^s$ manifold $(X, \theta)$, there exists a $C^\infty$ structure $\theta'$ on $X$ such that $\text{id}_X : (X, \theta) \to (X, \theta')$ is a $C^s$ diffeomorphism.
Some refinement of the proof of 2.2.9 [7] proves the following relative version of it.

**Theorem 2.10.** Let \( X' \) be a compact \( C^s \) submanifold of a compact \( C^s \) manifold \( X \) and \( 1 \leq s < \infty \). Then there exist a compact \( C^\infty \) manifold \( Y \) and its compact \( C^\infty \) submanifold \( Y' \) such that \((X;X')\) is \( C^s \) diffeomorphic to \((Y;Y')\).

The following is useful to approximate a relative \( C^1 \) diffeomorphism by relative definable \( C^r \) diffeomorphisms.

**Theorem 2.11.** Let \( X \) and \( Y \) compact definable \( C^r \) manifolds and \( 1 \leq r < \infty \). Suppose that \( X' \) and \( Y' \) are compact definable \( C^r \) submanifolds of \( X \) and \( Y \), respectively, and that \( f : (X;X') \to (Y;Y') \) is a \( C^1 \) diffeomorphism. Then there exists a definable \( C^r \) diffeomorphism \( h : (X;X') \to (Y;Y') \) as an approximation of \( f \) in the \( C^1 \) Whitney topology.

**Proof.** Since \( X, Y \) are compact and by 1.1 [10] and 1.2 [10], we may assume that \( X \) and \( Y \) are definable \( C^r \) submanifolds of \( \mathbb{R}^n \) and \( \mathbb{R}^m \), respectively.

Since \( f|X' : X' \to Y' \) is a \( C^1 \) diffeomorphism and by the polynomial approximation theorem and Proposition 2.4, there exists a definable \( C^r \) diffeomorphism \( f_1 : X' \to Y' \) as an approximation of \( f|X' : X' \to Y' \) in the \( C^1 \) Whitney topology. Similarly, one can find a definable \( C^r \) diffeomorphism \( f_2 : X \to Y \) as an approximation of \( f : X \to Y \) in the \( C^1 \) Whitney topology.

By Proposition 2.4, there exists a definable \( C^r \) tubular neighborhood \((U,p)\) of \( X' \) in \( \mathbb{R}^n \) (resp. \((V,q)\) of \( Y \) in \( \mathbb{R}^m \)). Then \( U' := U \cap X \) is a definable open neighborhood of \( X' \) in \( X \). Thus we have a definable \( C^r \) map \( f_3 : U' \to Y' \) with \( f_3|X' = f_1 \). Take a definable open neighborhood \( U_1 \) of \( X' \) in \( U' \) such that the closure of \( U_1 \) in \( X \) is properly contained in \( U' \) and take a definable \( C^r \) function \( \lambda : X \to \mathbb{R} \) such that \( \lambda = 1 \) on \( U_1 \) and its support lies in \( U' \). Then we have a definable \( C^r \) map \( h : (X;X') \to (Y;Y'), h(x) = \lambda(x)f_3(x) + (1 - \lambda(x))f_2(x) \) as an approximation of \( f : (X;X') \to (Y;Y') \) in the \( C^1 \) Whitney topology. If our approximation is sufficiently close, then \( h \) is the required definable \( C^r \) diffeomorphism.

One can define the definable \( C^s \) topology on the set of definable \( C^s \) maps between affine definable \( C^s \) manifolds (see [9]). This definable \( C^s \) topology is different from the \( C^s \) Whitney topology in general, but they coincide if the domain manifold is compact.
Theorem 2.12 ([14], 4.11 [9]). Let $0 \leq s < r < \infty$. Then every definable $C^r$ map between affine definable $C^s$ manifolds is approximated in the definable $C^s$ topology by definable $C^r$ maps.

Note that Theorem 2.12 are true in a more general setting (see 1.1 [8]).

Proposition 2.13 ([14], 4.10 [9]). Let $X$ and $Y$ be definable $C^s$ submanifolds of $\mathbb{R}^n$ and $0 < s < \infty$. If $f : X \to Y$ is a definable $C^s$ diffeomorphism, then an approximation of $f$ in the definable $C^s$ topology is a definable $C^s$ diffeomorphism.

Proof of Theorem 1.3. Let $X$ be a definable $C^s$ manifold. Then by Theorem 1.1 and since $\mathcal{M}$ is exponentially bounded, $X$ is affine.

Assume that $X$ is compact. By Theorem 2.9, $X$ is $C^s$ diffeomorphic to a compact $C^\infty$ manifold $X'$. Thus by Theorem 2.7, $X'$ is $C^\infty$ diffeomorphic to a nonsingular algebraic set $X''$. In particular, $X$ is $C^s$ diffeomorphic to an affine definable $C^\infty$ manifold $X''$. By Theorem 2.11, $X$ is definably $C^s$ diffeomorphic to $X''$. Thus in this case, $X$ admits a definable $C^r$ manifold structure.

Assume that $X$ is not compact. By Theorem 2.5, $X$ is definably $C^s$ diffeomorphic to the interior of some compact affine definable $C^s$ manifold $Y$ with boundary $\partial Y$. Thus by Theorem 2.6, $Y$ admits a definable $C^s$ collar. Hence we have the double $D$ of $Y$. By Theorem 1.1, $D$ is affine and compact. Using Theorem 2.10, there exist a compact $C^\infty$ manifold $D'$ and a compact $C^\infty$ submanifold $Z$ of $D'$ such that $(D, \partial Y)$ is $C^s$ diffeomorphic to $(D', Z)$. By Theorem 2.8, one can find a nonsingular algebraic set $D''$ and a nonsingular algebraic subset $Z'$ of $D''$ such that $(D', Z)$ is $C^\infty$ diffeomorphic to $(D'', Z')$. In particular, $D''$ is an affine definable $C^\infty$ manifold, $Z'$ is a definable $C^\infty$ submanifold of $D''$ and $(D, \partial Y)$ is $C^s$ diffeomorphic to $(D'', Z')$. Using Theorem 2.11, $(D, \partial Y)$ is definably $C^s$ diffeomorphic to $(D'', Z')$. Thus $X$ is definably $C^s$ diffeomorphic to some union of connected components of $D'' - Z'$. Therefore $X$ admits a definable $C^r$ manifold structure.

Uniqueness follows from Theorem 1.1, Theorem 2.12 and Proposition 2.13.

Remark that the above proof shows that every definable $C^s$ manifold is definably $C^s$ diffeomorphic to an affine definable $C^\infty$ manifold.
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References


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