ORLICZ-TYPE INTEGRAL INEQUALITIES FOR OPERATORS

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Abstract. We examine Orlicz-type integral inequalities for operators and obtain as a corollary a characterization of such inequalities for the Hardy-Littlewood maximal operator extending the well-known $L^p$-norm inequalities.

1. Introduction

Let $f \to Tf$ be an operator and for $j = 1, 2, \ldots$, let

$$T^{(j)} f = \underbrace{T \cdots T f}_{j \text{ times}}$$

be the $j$-times iterated operator $T$. The problem which we will address in this note is to find conditions on $T$ and conditions on $\Phi, \Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$(1) \quad \int_{\mathbb{R}^n} \Phi(|T^{(j)} f|) \leq c_j \int_{\mathbb{R}^n} \Psi(c_j |f|)$$

with the constant $c_j$ independent of $f$. Conversely, we will also examine what the inequality (1) implies about $\Phi$ and $\Psi$.

In particular, if $T = M$, the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(t)| dt,$$

our results lead to a characterization of (1) with $T = M$. The case $T = M$ and $j = 1$ has been studied extensively when $\Phi = \Psi$ is non-decreasing [1]. In fact

$$(2) \quad \int_{\mathbb{R}^n} \Phi(Mf) \leq c \int_{\mathbb{R}^n} \Phi(c|f|)$$

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if and only if
\[ \int_0^s \frac{d\Phi(t)}{t} \leq \frac{c\Phi(cs)}{s}, \quad 0 \leq s < \infty. \]
In the general case, \( \Phi \neq \Psi \), the latest attempt to characterize \( (2) \) with \( \Psi \) in the right side is in [4]. However in [4] the well-known \( L^p \)-inequalities for \( M \) are excluded. Our results will restore the \( L^p \)-inequalities, and in doing this we were led to study general integral inequalities \( (1) \).

In section 2 we examine conditions on \( \Phi \) and \( \Psi \) which imply \( (1) \) and in section 3 we study when these conditions on \( \Phi \) and \( \Psi \) are implied by \( (1) \). All of this will give us a characterization of \( (1) \) with \( T = M \). Section 4 is devoted to \( L \log L \) and \( \exp(L) \) inequalities of \( T \), and in section 5 we outline similar results for the operator \( T_r f = [T(|f|^r)]^{1/r}, \ r > 0 \).

2. Integral inequalities

We will assume that \( \Phi \) and \( \Psi \) can be written as
\[ \Phi(t) = \int_0^t a(s) \, ds, \quad \Psi(t) = \int_0^t b(s) \, ds, \]
where \( a, b : \mathbb{R}_+ \to \mathbb{R}_+ \). Our results, as in [4], will be expressed in terms of \( a \) and \( b \).

Our first theorem deals with an operator \( T \) satisfying the following distributional inequality:

\[ |\{|Tf| > \lambda\}| \leq \frac{c_0}{\lambda} \int_{\lambda/c_0}^\infty |\{|f| > s\}| \, ds, \]
where \( c_0 > 1 \) is independent of \( \lambda > 0 \) and \( f \).

**Remark.** If \( T \) is sublinear, then \( (3) \) holds if and only if \( T \) is weak-type \((1, 1) \) and \((\infty, \infty) \) [3, p.91]. In particular, \( M \) satisfies \( (3) \). In Theorem 2 below we will see another characterization of \( (3) \).

We need a version of \( (3) \) for \( T^{(j)} \) and this is the content of Lemma 1.

**Lemma 1.** If \( T \) satisfies \( (3) \), then

\[ |\{|T^{(j)}f| > \lambda\}| \leq \frac{c_0^j}{(j-1)! \lambda} \int_{\lambda/c_0}^\infty |\{|f| > s\}| \log^{j-1} \left( \frac{c_0^j s}{\lambda} \right) \, ds \]
for \( j = 1, 2, \ldots \).
Proof. The proof is by induction. The case \( j = 1 \) is (3). Assume (4) holds up to \( j - 1 \), and let \( c = c_0 \). Then

\[
\left| \left\{ |T^{(j)}f| > \lambda \right\} \right| \leq \frac{c}{\lambda} \int_{\lambda/c}^{\infty} \left| \left\{ |T^{(j-1)}f| > s \right\} \right| ds
\]

\[
\leq \frac{c^j}{(j-2)!\lambda} \int_{\lambda/c}^{\infty} \frac{1}{s} \int_{s/c^{j-1}}^{\infty} \left| \left\{ |f| > t \right\} \right| \log^{j-2} \left( \frac{c^{j-1}t}{s} \right) dt ds
\]

\[
= \frac{c^j}{(j-2)!\lambda} \int_{\lambda/c}^{\infty} \int_{s/c^{j-1}}^{s} \left| \left\{ |f| > t \right\} \right| \frac{1}{s} \log^{j-2} \left( \frac{c^{j-1}t}{s} \right) ds dt.
\]

The integral in \( s \) is \( \frac{1}{j-1} \log^{j-1} \left( \frac{c^j t}{\lambda} \right) \).

The next theorem gives a characterization of condition (3) in terms of the integral inequality (1).

Theorem 2. The following are equivalent for an operator \( f \rightarrow Tf \).

(A) The integral inequality (1) holds for every \( a, b : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfying

\[
\int_0^s \frac{a(t)}{t} \log^{j-1} \left( \frac{s}{t} \right) dt \leq c' b(c'' s), \quad 0 \leq s < \infty.
\]

(B) \( T \) satisfies condition (3).

Remark. It is easy to verify that the left side of (5) is a \( j \)-times iterated integral:

\[
(j-1)! \int_0^s \frac{1}{s_{j-1}} \int_0^{s_{j-1}} \cdots \int_0^{s_1} \frac{a(t)}{t} dt ds_1 \cdots ds_{j-1}.
\]

Proof. \((B) \Rightarrow (A)\). From Lemma 1

\[
\int_{\mathbb{R}^n} \Phi(|T^{(j)}f|) = \int_0^\infty \left| \left\{ |T^{(j)}f| > t \right\} \right| a(t) dt
\]

\[
\leq c_j \int_0^\infty \frac{a(t)}{t} \int_{t/c_0}^{\infty} \left| \left\{ |f| > s \right\} \right| \log^{j-1} \left( \frac{c_0^j s}{t} \right) ds dt
\]

\[
= c_j \int_0^\infty \left| \left\{ |f| > s \right\} \right| \int_0^{c_0^j s} \frac{a(t)}{t} \log^{j-1} \left( \frac{c_0^j s}{t} \right) ds dt
\]

\[
c_j c' \int_0^\infty \left| \left\{ |f| > s \right\} \right| b(c'' c_0^j s) ds = C_j \int_{\mathbb{R}^n} \Psi(c'' c_0^j |f|).
\]
Since $c_j = c_0^j/(j-1)!$, the constant $C_j = c'/(c''(j-1)!)$.

(A)⇒(B). First we show that (5) implies

$$\int_0^s \frac{a(t)}{t} \, dt \leq c' b(c'' e s), 0 \leq s < \infty.$$ 

Simply note that the integral on the left is

$$\leq \int_0^s \frac{a(t)}{t} \log^{j-1} \left( \frac{es}{t} \right) \, dt \leq \int_0^s \frac{a(t)}{t} \log^{j-1} \left( \frac{es}{t} \right) \, dt \leq c' b(c'' e s).$$

Thus (A) implies (1) with $j = 1$, i.e.,

$$L \equiv \int_0^\infty \{||Tf| > \lambda\}|a(\lambda)\,d\lambda \leq c \int_0^\infty \{|c| > \lambda\}|b(\lambda)\,d\lambda \equiv R,$$

whenever $a, b$ satisfy (5) with $j = 1$. For $0 < \lambda_0 < \infty$ and $h > 0$, let

$$a(\lambda) = \frac{1}{h} \chi_{[\lambda_0, \lambda_0+h]}(\lambda)$$

and define

$$b(s) = \int_0^s \frac{a(t)}{t} \, dt = \begin{cases} 0, & 0 \leq s \leq \lambda_0 \\ \frac{1}{h} \log(s/\lambda_0), & \lambda_0 < s < \lambda_0 + h \\ \frac{1}{h} \log \frac{\lambda_0 + h}{\lambda_0}, & s \geq \lambda_0 + h. \end{cases}$$

Then

$$L = \frac{1}{h} \int_{\lambda_0}^{\lambda_0 + h} |\{\|Tf\| > \lambda\}|\,d\lambda \rightarrow |\{\|Tf\| > \lambda_0\}|,$$

as $h \rightarrow 0$. The right side

$$R = \frac{1}{h} \int_{\lambda_0}^{\lambda_0 + h} |\{c| > \lambda\}| \log(\lambda/\lambda_0)\,d\lambda$$

$$+ \frac{1}{h} \log \left( \frac{\lambda_0 + h}{\lambda_0} \right) \int_{\lambda_0}^{\infty} |\{c| > \lambda\}|\,d\lambda.$$ 

When $h \rightarrow 0$, the first term goes to 0 and the second term goes to $\frac{c}{\lambda_0} \int_{\lambda_0}^{\infty} |\{c| > \lambda\}|\,d\lambda$. Hence

$$|\{\|Tf\| > \lambda\}| \leq \frac{c^2}{\lambda_0} \int_{\lambda_0/c}^{\infty} |\{\|f\| > s\}|\,ds.$$ 

This is (3) with constant $c^2$. \qed
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Remarks. (i) In (A)⇒(B) we have shown that condition (5) implies the same condition with \( j = 1 \). The converse is not true as \( a(t) = \log^+ t \) and \( b(s) = \int_0^s \frac{a(t)}{t} dt \) shows. However, if \( a = b \) and (5) holds for \( j = 1 \), then it also holds for \( j = 2, 3, \cdots \). This can be seen from the Remark after the statement of Theorem 2. (ii) This is the special case \( j = 1 \) of Theorem 2.

If \( T \) satisfies condition (3) with constant \( c_0 \), and if

\[
\int_0^s \frac{a(t)}{t} \, dt \leq c'b(c''s), \quad 0 \leq s < \infty,
\]

then

\[
\int_{\mathbb{R}^n} \Phi(|Tf|) \leq \frac{c'}{c''} \int_{\mathbb{R}^n} \Psi(c''c_0|f|).
\]

Before we examine to what extent (1) implies (5) it will be useful to make an observation. Let us assume that \( T \) satisfies the following reverse weak-type inequality:

\[
\frac{c}{\lambda} \int_{\{|f| > \lambda\}} |f| \leq |\{|Tf| > \lambda\}|,
\]

with \( c \) independent of \( \lambda > 0 \) and \( f \).

Again we need a version of (6) for \( T^{(j)} \) and the next two lemmas deal with this.

**Lemma 3.** Assume \( T \) satisfies (6) and \( 0 \leq k < \infty \). Then there exists \( 0 < c < \infty \) such that

\[
c \int_{\{|f| > \lambda\}} |f| \log^{k+1} \left( \frac{|f|}{\lambda} \right) \leq \int_{\{|Tf| > \lambda\}} |Tf| \log^k \left( \frac{|Tf|}{\lambda} \right).
\]

**Proof.** Let \( \Phi(t) = 0, 0 \leq t \leq \lambda, \) and \( = t \log^k(t/\lambda), t > \lambda. \) Then

\[
\Phi(t) = \int_0^t a(s) ds, \quad \text{where} \ a(s) = 0, 0 \leq s \leq \lambda, \ \text{and} \ = \log^k(s/\lambda) + k \log^{k-1}(s/\lambda), \ s > \lambda.
\]

Let now \( b(s) = \int_0^s \frac{a(t)}{t} \, dt. \) Then \( b(s) = 0, 0 \leq s \leq \lambda, \) and is \( \geq c \log^{k+1}(s/\lambda), \ s > \lambda. \) Replace in (6) \( \lambda \) by \( s \), multiply by \( a(s) \) and integrate in \( s \) from \( \lambda \) to \( \infty. \) Interchange the order of integration to obtain the inequality. \( \square \)
Lemma 4. Assume $T$ satisfies (6). Then there exists $0 < c_j < \infty$ such that
\[
\frac{c_j}{\lambda} \int_{\{|f| > \lambda\}} |f| \log^{j-1} \left( \frac{|f|}{\lambda} \right) \leq \left| \{ |T^{(j)} f| > \lambda \} \right|
\]
for $j = 1, 2, \cdots$.

Proof. (6) gives us
\[
\left| \{ |T^{(j)} f| > \lambda \} \right| \geq \frac{c}{\lambda} \int_{\{|T^{(j-1)} f| > \lambda\}} |T^{(j-1)} f|
\]
and by Lemma 3 the last integral is
\[
\geq c_1 \int_{\{|T^{(j-2)} f| > \lambda\}} |T^{(j-2)} f| \log \left( \frac{|T^{(j-2)} f|}{\lambda} \right)
\]
\[
\geq \cdots \geq c_{j-2} \int_{\{|f| > \lambda\}} |f| \log^{j-1} \left( \frac{|f|}{\lambda} \right).
\]

Theorem 5. Assume $T$ satisfies (6). If $a : \mathbb{R}_+ \to \mathbb{R}_+$ and $b(s) = \int_0^s a(t) \log^{j-1} \left( \frac{t}{s} \right) dt$, then
\[
c \int_{\mathbb{R}^n} |f| b(|f|) \leq \int_{\mathbb{R}^n} \Phi(|T^{(j)} f|),
\]
where as before $\Phi(t) = \int_0^t a(s) ds$.

Proof. Multiply the inequality in Lemma 4 by $a(\lambda)$ and integrate in $\lambda$ from 0 to $\infty$. The right side is what we want, and the left side is obtained by an interchange of the order of integration.

Corollary 6. Assume $T$ satisfies both condition (3) and (6). If $a, b$ are as in Theorem 5 and $\Phi(t) = \int_0^t a(s) ds, \Psi(t) = \int_0^t b(s) ds$, then
\[
c \int_{\mathbb{R}^n} |f| b(|f|) \leq \int_{\mathbb{R}^n} \Phi(|T^{(j)} f|) \leq C \int_{\mathbb{R}^n} \Psi(C |f|).
\]

Proof. The left side is Theorem 5 and the right side is Theorem 2.

Remark. (i) The Corollary is especially interesting in the case where the left and right sides are essentially the same. An example is $a(t) = t^\alpha, \alpha > 0$. Other examples will be examined in section 4. (ii) An example of an operator satisfying both (3) and (6) is $M$ as we shall point out in the next section.
3. (1) implies (5)

For this implication we need to assume that the function $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies (i) $b(t) > 0, t > 0$, and (ii) $b$ is \textit{quasi-increasing}, i.e., there is a constant $0 < c_0 < \infty$ such that $b(t') \leq c_0 b(c_0 t'')$ for $0 \leq t' \leq t''$.

\textbf{Remark.} In [4] $b(t)$ was assumed to be positive for $t \geq 0$ thus not allowing $b(t) = t^\alpha, \alpha > 0$. (ii) is the same as in [4].

\textbf{Theorem 7.} Assume $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ with $b$ satisfying (i) and (ii). If $T$ satisfies the condition (6), then the integral inequality (1) implies (5).

\textbf{Proof.} We deny (5) and have then $0 < s_k < \infty$ such that

$$\int_0^{s_k} \frac{a(t)}{t} \log^{j-1} \left( \frac{s_k}{t} \right) dt > 2^k b(2^k k s_k), k = 1, 2, \ldots$$

Let $\{Q_k\}$ be a disjoint collection of cubes with

$$|Q_k| = \frac{1}{2^k \Psi(2^k s_k)}.$$

Let $0 < c_2 < \infty$ be given. We claim that there is $f : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\int_{\mathbb{R}^n} \Psi(c_2 f) < \infty \text{ and } \int_{\mathbb{R}^n} \Phi(|T(\cdot) f|) = \infty.$$

Define

$$f(x) = \frac{1}{c_2} \sum 2^k s_k \chi_{Q_k}(x).$$

Then

$$\int_{\mathbb{R}^n} \Psi(c_2 f) = \sum \int_{Q_k} \Psi(c_2 f) = \sum \Psi(2^k s_k) |Q_k| = \sum \frac{1}{2^k} < \infty.$$

Next, by Lemma 4,

$$\int_{\mathbb{R}^n} \Phi(|T(\cdot) f|) = \int_0^\infty |\{T(\cdot) f| > \lambda\}| a(\lambda) d\lambda \geq$$

$$c \int_0^\infty \left( \int_{\{|f| > \lambda\}} \log^{j-1} \left( \frac{f(x)}{\lambda} \right) dx \right) a(\lambda) \frac{d\lambda}{\lambda} =$$

$$c \int_{\mathbb{R}^n} f(x) \int_0^{f(x)} \frac{a(\lambda)}{\lambda} \log^{j-1} \left( \frac{f(x)}{\lambda} \right) d\lambda dx =$$

$$\frac{c}{c_2} \sum 2^k s_k \left( \int_0^{2^k s_k/c_2} \frac{a(\lambda)}{\lambda} \log^{j-1} \left( \frac{2^k s_k}{c_2 \lambda} \right) d\lambda \right) |Q_k| \geq$$

$$\frac{c}{c_2} \sum_{k \geq k_0} 2^k s_k 2^k b(2^k k s_k) \frac{1}{2^k \Psi(2^k s_k)} \equiv L,$$
where \( k_0 > c_0^2 \) is chosen so large that \( 2^{k_0} \geq c_2 \). Since \( b \) is quasi-increasing we see that

\[
\Psi(2^k s_k) = \int_0^{2^k s_k} b(s) ds \leq c_0 b(c_0 2^k s_k) 2^k s_k.
\]

Hence

\[
L \geq c \frac{c}{c_0 c_2} \sum_{k \geq k_0} b(2^k s_k) / b(c_0 2^k s_k) = \infty,
\]

since for \( k \geq c_0^2 \), \( c_0 2^k s_k \leq (k 2^k s_k) / c_0 \), from which we get \( b(c_0 2^k s_k) \leq c_0 b(k 2^k s_k) \) \( \square \)

**Remark.** The proof of Theorem 4 is along the lines of [4](see also [1, p.14]).

Theorems 2 and 7 imply the following characterization of (1) for \( T = M \), the Hardy-Littlewood maximal operator.

**Theorem 8.** Assume \( a, b : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( b \) satisfying (i) and (ii). Then

\[
\int_{\mathbb{R}^n} \Phi(M(j) f) \leq c_j \int_{\mathbb{R}^n} \Psi(c_j |f|)
\]

if and only if

\[
\int_0^s \frac{a(t)}{t} \log^j \left( \frac{s}{T} \right) dt \leq c' b(c'' s), 0 \leq s < \infty.
\]

**Proof.** Since \( M \) is sublinear, weak-type \((1,1)\), and \((\infty, \infty)\), \( M \) satisfies the condition (3)[3, p.91]. Hence Theorem 2 gives the sufficiency. For the necessity, simply observe that \( M \) also satisfies the condition (6). For \( f \in L^1(\mathbb{R}^n) \), this follows immediately from a Calderon-Zygmund decomposition at height \( \lambda \) [2, p.23]. For general \( f \in L^1_{loc}(\mathbb{R}^n) \), approximate by \( f_N(x) = f(x) \chi_{\{|x| \leq N\}}(x) \). \( \square \)

4. LlogL and exp(L) inequalities

In this section we will present some more examples for which the left and right sides in Corollary 6 are essentially the same. This leads then to a characterization for \( \int_{\mathbb{R}^n} \Phi(|T(j) f|) \) to be finite.

For the next lemma it is useful to use the notation \( f(s) \sim g(s) \) on \( E \) to mean: there are constants \( 0 < c < C < \infty \) such that \( c \leq f(s) / g(s) \leq C, s \in E \).
Lemma 9. For $0 < k < \infty$ and $j$ a positive integer, let

$$\phi(s) = \int_0^s \frac{\log^j(1 + t)}{t} \log^{j-1} \left( \frac{s}{t} \right) dt.$$ 

Then (i) $\phi(s) \sim s^k$ on $[0, 1]$ and (ii) $\phi(s) \sim \log^k(1 + s)$ on $[1, \infty)$.

Proof. (i) This follows from the fact that $\phi(s)$ can be written as a $j$-times iterated integral of $(\log^k(1 + t))/t$ (Remark after Theorem 2) and $\log(1 + t) \sim t$ on $[0, 1]$. For (ii) simply note that $\phi(s) > 0$ and continuous for $s \geq 1$. Thus all we have to verify is that $\phi(s)/\log^k(1 + s) \to L$ as $s \to \infty$ and $0 < L < \infty$. This follows from a $j$-times repeated application of L’Hôpital’s rule. □

Let for $0 < k < \infty$ and $j = 1, 2, \cdots$,

$$L_{kj}(f) = \int_{\{|f| \leq 1\}} |f|^{k+1} + \int_{\{|f| > 1\}} |f| \log^k(1 + |f|),$$ 

$$K_{kj}(f) = \int_{\mathbb{R}^n} |T^{(j)}f| \log^k(1 + |T^{(j)}f|).$$

Theorem 10. Let $0 < k < \infty$. If $T$ satisfies the conditions (3) and (6), then there are constants $0 < c_{kj} < C_{kj} < \infty$ such that

$$c_{kj} L_{kj}(f) \leq K_{kj}(f) \leq C_{kj} L_{kj}(f).$$

Proof. Let $\Phi(t) = t \log^k(1 + t)$. Then $\Phi(t) = \int_0^t a(s) ds$, where

$$a(s) = \log^k(1 + s) + \frac{ks}{1 + s} \log^{k-1}(1 + s).$$

Let $b(s) = \int_0^s \frac{a(t)}{t} \log^{j-1} \left( \frac{s}{t} \right) dt$. By Corollary 6,

$$c \int_{\mathbb{R}^n} |f(x)| b(|f(x)|) \leq \int_{\mathbb{R}^n} |T^{(j)}f| \log^k(1 + |T^{(j)}f|) \leq C \int_{\mathbb{R}^n} \Psi(C|f|),$$

where $\Psi(t) = \int_0^t b(s) ds$ and where we may take $C > 1$. By Lemma 9 the left side is

$$\geq c \left( \int_{\{|f| \leq 1\}} |f|^{k+1} + \int_{\{|f| > 1\}} |f| \log^{k+j}(1 + |f|) \right).$$
Again from Lemma 9 we see that \( \Psi(t) \sim t^{k+1} \) on \([0, 1]\) and \( \Psi(t) \sim t \log^{k+j}(1+t) \) on \([1, \infty)\). Since for \( C > 1 \), \( \log(1 + Ct) \leq C \log(1 + t) \), the right side is 
\[
\leq C'' \int_{\mathbb{R}^n} \Psi(|f|) \leq C'' L_{kj}(f).
\]
\( \square \)

The \( k = 0 \) case reads as follows:

**Theorem 11.** Assume \( T \) satisfies the conditions (3) and (6). Then there are constants \( 0 < c_j < C_j < \infty \) and there is a constant \( 0 < c_0 < 1 \) such that

\[
c_j \int_{\{|f| \geq 1\}} |f| \log^{j-1} |f| \log(1 + |f|) \leq \int_{\{|T^{(j)}f| \geq 1\}} |T^{(j)}f| \leq C_j \int_{\{|f| > c_0\}} |f| \log^{j-1} \left( \frac{|f|}{c_0} \right) \log \left( 1 + \frac{|f|}{c_0} \right).
\]

**Proof.** Let \( a(t) = \chi_{[1, \infty)}(t) \). Then \( \Phi(t) = \int_0^t a(s) ds = 0, 0 \leq t \leq 1, \) and \( = t - 1, t > 1 \). If \( b(s) = \int_0^s a(t) \log^{j-1} \left( \frac{t}{s} \right) dt \), then \( b(s) = 0, 0 \leq s \leq 1, \) and \( = (\log^j s)/j, s > 1 \). Hence from Corollary 6

\[
c \int_{\mathbb{R}^n} |f| b(|f|) \leq \int_{\{|T^{(j)}f| > 1\}} (|T^{(j)}f| - 1) \leq C \int_{\mathbb{R}^n} \Psi(C|f|),
\]

where again \( \Psi(t) = \int_0^t b(s) ds \). Add \( \{|T^{(j)}f| > 1\} \) to the inequality and let \( L \) be the left side and \( R \) the right side. By Lemma 4,

\[
L \geq c \left( \int_{\{|f| \geq 1\}} |f| \log^j |f| + \int_{\{|f| \geq 1\}} |f| \log^{j-1} |f| \right) \geq c \int_{\{|f| \geq 1\}} |f| \log^{j-1} |f|(1 + \log |f|) \geq c \int_{\{|f| \geq 1\}} |f| \log^{j-1} |f| \log(1 + |f|).
\]

By Lemma 1,

\[
R \leq C \left( \int_{\mathbb{R}^n} \Psi(C|f|) + \int_{\mathbb{R}^n} \chi_{\{|f| > s\}} |f| \log^{j-1} \left( \frac{s}{c_j} \right) ds \right)
\]
for some $0 < c' < 1$. Note that $\Psi(s) = \int_1^s \log^j t \, dt \leq cs \log^j s$, $s > 1$. Hence

$$\int_{\mathbb{R}^n} \Psi(|C|f|) \leq C \int_{\{|f| > 1/C\}} |f| \log^j (C|f|).$$

If we set $\Phi_j(s) = \int_s^\infty \log^j (t) \, dt$ for $s \geq c'$ and $= 0$ for $0 \leq s \leq c'$, then the second integral in $R$ is

$$\int_{c'}^\infty |\{|f| > s\}| d\Phi_j(s) = \int_0^\infty |\{\Phi_j(|f|) > t\}| dt = \int_{\{|f| > c'\}} \Phi_j(|f|).$$

Since $\Phi_j(s) \leq s \log^j (s/c')$, $s \geq c'$, we see that

$$\int_{c'}^\infty |\{|f| > s\}| d\Phi_j(s) \leq \int_{\{|f| > c'\}} |f| \log^j \left( \frac{|f|}{c'} \right).$$

Let now $c_0 = \min(1/C, c')$. Then

$$R \leq C \int_{\{|f| > c_0\}} |f| \log^j \left( \frac{|f|}{c_0} \right) \left( 1 + \log \left( \frac{|f|}{c_0} \right) \right).$$

Since $1 + \log s \leq 2 \log(1 + s)$, $s \geq 0$, the proof is complete. \[\Box\]

**Remarks.** (i) If $T$ only satisfies condition (6), then in Theorems 10 and 11 we only get the left inequality, whereas if $T$ only satisfies condition (3) we get only the right inequality. (ii) Since $M$ satisfies both (3) and (6), Theorems 10 and 11 are valid for $T = M$, and for $j = 1$ are the $\mathbb{R}^n$-versions of the well-known fact: if $B$ is a finite ball and $k \geq 0$, then $Mf \log^k (1 + Mf) \in L^1(B)$ if and only if $|f| \log^{k+1} (1 + |f|) \in L^1(B)$ [2, p.23].

The set-up for the $\exp(L)$ inequalities is the following: $a : [1, \infty) \to \mathbb{R}_+$, $b : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(t) = \int_1^t a(s) \, ds$, and $\Psi(t) = \int_0^t b(s) \, ds$.

**Theorem 12.** If

\[(*)\]

$$\int_1^{e^s} \frac{a(t)}{\log t} \, dt \leq c' b'(c''s), \quad 0 \leq s < \infty,$$

and if $T$ satisfies condition (3) with constant $c_0$, then

$$\int_{\mathbb{R}^n} \Phi(e^{\left|\mathit{T}|f|\right|}) \leq \frac{c'}{c''} \int_{\mathbb{R}^n} \Psi(c''c_0 |f|).$$
Proof. The substitution \( t = e^u \) in \((*)\) gives
\[
\int_0^s \frac{a(e^u)e^u}{u} du \leq c'b(c''s).
\]
Hence from Remark (ii) after Theorem 2,
\[
\int_{\mathbb{R}^n} \Phi_*(|Tf|) \leq \frac{e'}{e''} \int_{\mathbb{R}^n} \Psi(c''|f|),
\]
where \( \Phi_*(t) = \int_0^t a(e^u)e^u du = \int_1^t a(s)ds = \Phi(e^t). \) \qed

We will now briefly discuss two examples illustrating Theorem 12. In both examples we assume that \( T \) satisfies condition (3) with constant \( c_0. \)

(I) If \( 1 < p < \infty, \) then
\[
\int_{\mathbb{R}^n} \left( e^{|Tf|} - 1 \right)^p \leq C_p \left\{ \int_{\{|f| \leq 1/c_0\}} |f|^p + \int_{\{|f| > 1/c_0\}} (e^{c_0|f|} - 1)^p \right\}.
\]
Let \( a(t) = (t-1)^{p-1}, t \geq 1, \) and let \( b(s) = \int_1^s \frac{(t-1)^{p-1}}{\log t} dt. \) The left side is what we want, and for the right side note first that \( b(s) \leq c'_p s^{p-1}, 0 \leq s \leq 1, \) and \( b(s) \leq c''_p (e^s - 1)^p, 1 \leq s < \infty. \) This can be seen by showing that
\[
\lim_{s \to 0} \frac{b(s)}{s^{p-1}} = 1, \quad \lim_{s \to \infty} \frac{b(s)}{(e^s - 1)^p} = 0.
\]
From this we see that \( \Psi(t) \leq c'_p t^p/p, 0 < t \leq 1, \) and for \( t > 1, \)
\[
\Psi(t) \leq c'_p/p + c''_p \int_1^t (e^s - 1)^p ds.
\]
Since \( \int_1^t (e^s - 1)^p ds/(e^t - 1)^p \rightarrow 1/p \) as \( t \rightarrow \infty, \) the last integral is \( \leq c''_p (e^t - 1)^p. \) Consequently, \( \Psi(t) \leq c_p (e^t - 1)^p, t > 1 \) for some constant \( c_p. \)

(II) There is a version of the inequality in (I) for \( 0 < p \leq 1 \) and it reads as follows:
\[
\int_{\mathbb{R}^n} \left( e^{|Tf|} - 1 \right)^{p-1} \left( e^{|Tf|}(|Tf| - 1) + 1 \right) \leq C_p \left\{ \int_{\{|f| \leq 1/c_0\}} |f|^{p+1} + \int_{\{|f| > 1/c_0\}} (e^{c_0|f|} - 1)^p \right\}.
\]
To obtain this inequality we let
\[ a(t) = (t - 1)^{p-1} \log t, \quad t \geq 1, \]
and
\[ b(s) = \int_1^e (t - 1)^{p-1} dt = \frac{1}{p} (e^s - 1)^p. \]

Since \((e^s - 1)^p \leq c_p' s^p\) on \([0, 1]\), we see that \(\Psi(t) \leq c_p' t^{p+1}\) on \([0, 1]\), and for \(t > 1\), \(\Psi(t) \leq c_p (t^1 - 1)^p\) as in (I) above. This gives us the right side of the inequality. For the left side, since \(p - 1 < 0\), \(\Phi(t) \geq (t - 1)^{p-1} \int_1^t \log s \, ds = (t - 1)^{p-1} (t \log t - t + 1)\).

5. The operator \(T_r\)

For \(0 < r < \infty\) let \(T_r f(x) = [T(|f|^r)(x)]^{1/r}\), and let \(T_r^{(j)} f = [T^{(j)}(|f|^r)]^{1/r}\) be the operator \(T_r\) \(j\)-times iterated. Theorem 2 for \(T_r^{(j)}\) reads as follows:

**Theorem 13.** Assume that \(T\) satisfies condition (3) and that

\[ \int_0^s \frac{a(t)}{t^r} \log^{j-1} \left( \frac{s}{t} \right) dt \leq \frac{c' b(c'' s)}{s^{r-1}}, \quad 0 \leq s < \infty. \]  

If \(\Phi(t) = \int_0^t a(s) ds\) and \(\Psi(t) = \int_0^t b(s) ds\), then

\[ \int_{\mathbb{R}^n} \Phi(|T_r^{(j)} f|) \leq c_{j,r} \int_{\mathbb{R}^n} \Psi(c_{j,r} |f|). \]  

**Proof.** The reader will have no difficulty to adapt the proof of Theorem 2, (B)⇒(A), to \(T_r^{(j)}\). \(\square\)

**Remark.** The version of Theorem 2, (A)⇒(B), for \(T_r^{(j)}\) reads as follows: If (8) holds whenever \(a, b : \mathbb{R}_+ \to \mathbb{R}_+\) satisfies (7), then \(T\) satisfies condition (3) for non-negative functions. The proof is the same by first reducing (7) to \(j = 1\), and then letting \(a(\lambda) = \frac{1}{\lambda^r} \chi_{[\lambda_0, \lambda_0 + h]}(\lambda)\), and \(b(s) = s^{r-1} \int_0^s \frac{a(l)}{l^r} dl\).

**Theorem 14.** Assume \(a, b : \mathbb{R}_+ \to \mathbb{R}_+\) with \(b\) satisfying (i) and (ii) of section 3. Let \(\Phi\) and \(\Psi\) be as in Theorem 13. If \(T\) satisfies the condition (6) and if (8) holds, then we have (7).
Proof. With the choice of
\[
|Q_k| = \frac{1}{2^k r \psi(2^k s_k)}
\]
the proof is, apart from obvious modifications, the same as the proof of
Theorem 7. □

Since the Hardy-Littlewood maximal operator satisfies the conditions
(3) and (6) we have:

Corollary 15. If \(a, b : \mathbb{R}_+ \to \mathbb{R}_+\) with \(b\) satisfying (i) and (ii) of
section 3. If \(\Phi\) and \(\Psi\) are as in Theorem 12, then
\[
\int_{\mathbb{R}^n} \Phi(M_j(r)f) \leq c_{jr} \int_{\mathbb{R}^n} \Psi(c_{jr}|f|)
\]
if and only if
\[
\int_0^s \frac{a(t)}{t^r} \log^{j-1} \left( \frac{s}{t} \right) dt \leq \frac{c'b(c's)}{s^{r-1}}, \quad 0 \leq s < \infty.
\]

References


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