SPACE-LIKE SUBMANIFOLDS WITH CONSTANT SCALAR CURVATURE IN THE DE SITTER SPACES

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Abstract. Let $M^n$ be a space-like submanifold in a de Sitter space $M_p^{n+p}(c)$ with constant scalar curvature. We firstly extend Cheng-Yau’s technique to higher codimensional cases. Then we study the rigidity problem for $M^n$ with parallel normalized mean curvature vector field.

1. Introduction

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold of constant curvature $c$ whose index is $p$. It is called an indefinite space form of index $p$ and simply a space form when $p = 0$. If $c > 0$, we call it as a de Sitter space of index $p$. Akutagawa [3] and Ramanathan [11] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature $H$ satisfies $H^2 \leq c$ when $n = 2$ and $n^2 H^2 < 4(n-1)c$ when $n \geq 3$. Later, Cheng [4] generalized this result to general submanifolds in a de Sitter space.

To our best knowledge, there are almost no intrinsic rigidity results for the space-like submanifolds with constant scalar curvature in a de Sitter space until Zheng [15] obtained the following result.

Theorem. Let $M^n$ be an $n$-dimensional compact space-like hypersurface in $M_1^{n+1}(c)$ with constant scalar curvature. If $M^n$ satisfies

1. $K(M) > 0$,
2. $\text{Ric}(M) \leq (n-1)c$,
3. $R < c$,

where $R$ is the normalized scalar curvature of $M^n$, then $M^n$ is totally umbilical.

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In [5], Cheng-Yau firstly studied the rigidity problem for a hypersurface with constant scalar curvature in a space form by introducing a self-adjoint second order differential operator (See Theorems 1 and 2 in [5]). They proved that, for an $M^n$ in $M^{n+1}(c)$, if $R$ is constant and $R \geq c$, then $|\nabla \sigma|^2 \geq n^2|\nabla H|^2$ where $\sigma$ and $H$ denote the second fundamental form and the length of the mean curvature vector field of $M^n$ respectively. By using Cheng-Yau’s technique, Li [7] [8] studied the pinching problem and also proved some global rigidity theorems for hypersurfaces with constant scalar curvature.

In the present paper, we would like extend Cheng-Yau’s technique to higher codimensional cases and use this result to study the rigidity problem for space-like submanifolds in a de Sitter space with constant scalar curvature.

2. Preliminaries

Let $M_p^{n+p}(c)$ be an $(n + p)$-dimensional semi-Riemannian manifold of constant curvature $c$ whose index is $p$. Let $M^n$ be an $n$-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of $M^n$, $M^n$ is called a space-like submanifold. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $M_p^{n+p}(c)$ such that at each point of $M^n$, $e_1, \ldots, e_n$ span the tangent space of $M^n$ and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n + p; \quad 1 \leq i, j, k, \ldots \leq n; \quad n + 1 \leq \alpha, \beta, \gamma \leq n + p.$$ 

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $ds^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

(1) $d\omega_A = \sum_B \epsilon_B \omega_{AB} \land \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$

(2) $d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \land \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \land \omega_D,$

(3) $K_{ABCD} = c \epsilon_A \epsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}).$

Restrict these form to $M^n$, we have

(4) $\omega_\alpha = 0, \quad n + 1 \leq \alpha \leq n + p,$
the Riemannian metric of \( M^n \) is written as \( ds^2 = \sum_i \omega_i^2 \). From Cartan’s lemma we can write

\[
\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.
\]

From these formulas, we obtain the structure equations of \( M^n \):

\[
d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,
\]

\[
d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l,
\]

\[
R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),
\]

where \( R_{ijkl} \) are the components of the curvature tensor of \( M^n \). For indefinite Riemannian manifolds in detail, refer to O’Neill [9].

Denote \( L_\alpha = (h_{ij}^\alpha)_{n \times n} \) and \( H_\alpha = (1/n) \sum_i h_{ii}^\alpha \) for \( \alpha = n + 1, \ldots, n + p \). Then the mean curvature vector field \( \xi \), the mean curvature \( H \) and the square of the length of the second fundamental form \( S \) are expressed as

\[
\xi = \sum_\alpha H_\alpha e_\alpha, \quad H = |\xi|, \quad S = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2,
\]

respectively. Moreover, the normal curvature tensor \( \{ R_{\alpha \beta kl} \} \), the Ricci curvature tensor \( \{ R_{ik} \} \) and the normalized scalar curvature \( R \) are expressed as

\[
R_{\alpha \beta kl} = \sum_m (h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta),
\]

\[
R_{ik} = (n - 1) c \delta_{ik} - n \sum_\alpha (H_\alpha) h_{ik}^\alpha + \sum_{\alpha, j} h_{ij}^\alpha h_{jk}^\alpha,
\]

\[
R = c + \frac{1}{n(n - 1)} (S - n^2 H^2).
\]

Define the first and the second covariant derivatives of \( \{ h_{ij}^\alpha \} \), say \( \{ h_{ijk}^\alpha \} \) and \( \{ h_{ijkl}^\alpha \} \) by

\[
\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{kji}^\alpha \omega_{ki} + \sum_k h_{ikj}^\alpha \omega_k + \sum_\beta h_{ijk}^{\beta \alpha},
\]
(11) \[ \sum_i h_{ijkl}^\alpha \omega_l = \sum_i d h_{ijk}^\alpha + \sum_m h_{mjk}^\alpha \omega_m + \sum_m h_{imk}^\alpha \omega_m + \sum_m h_{ijm}^\alpha \omega_m + \sum_\beta h_{ij}^\beta \omega^\beta. \]

Then, by exterior differentiation of (5), we obtain the Codazzi equation

(12) \[ h_{ijk}^\alpha = h_{ikj}^\alpha. \]

It follows from Ricci’s identity that

(13) \[ h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{imk}^\alpha R_{mijkl} + \sum_m h_{im}^\alpha R_{mkjl} + \sum_\beta h_{ik}^\beta h_{mj}^\beta. \]

The Laplacian of \( h_{ij}^\alpha \) is defined by \( \Delta h_{ij}^\alpha = \sum_k h_{ikj}^\alpha \). From (13), we have

\[ \Delta h_{ij}^\alpha = n H_{a,ij} + \sum_{k,m} h_{km}^\alpha R_{mij} + \sum_{k,m} h_{im}^\alpha R_{mkj} + \sum_{k,\beta} h_{ik}^\beta R_{\beta ajk} - 2 \sum_{\beta,k,m} h_{ik}^\beta h_{km}^\beta h_{mj}^\beta + \sum_{m,\beta} S_{\beta} h_{ij}^\beta \]

where \( S_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \) for all \( \alpha \) and \( \beta \). Define \( N(A) = \sum_{i,j} a_{ij}^2 \) for any real matrix \( A = (a_{ij})_{n\times n} \). Then we have

(14) \[ \sum_{i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha = n \sum_{i,j} H_{a,ij} h_{ij}^\alpha + n c h_{ij}^\alpha - n \sum_{\beta} H_{\beta} h_{im}^\alpha h_{mj}^\beta + \sum_{\beta} S_{\alpha\beta} h_{ij}^\beta \]

where \( S_{\alpha} = \sum_{i,j} (h_{ij}^\alpha)^2 \), for every \( \alpha \).

Suppose \( H > 0 \) on \( M^n \) and choose \( e_{n+1} = \xi / H \). Then it follows that

(15) \[ H_{n+1} = H; \quad H_{a} = 0, \quad \alpha > n + 1. \]

From (10) and (15) we can see

(16) \[ H_{n+1,kl} \omega_k = dH, \quad H_{a,k} \omega_k = H \omega_{n+1} \quad \alpha > n + 1. \]

From (11), (15) and (16) we have

(17) \[ H_{n+1,kl} = H_{kl} - \frac{1}{H} \sum_{\beta > n+1} H_{\beta,k} H_{\beta,l}, \]

where \( dH = \sum_i H_i \omega_i \) and \( \nabla H_k = \sum_i H_{ki} \omega_i \equiv dH_k + H_{i} \omega_{ik} \) for all \( k \).
Using (14) and (17), we have
\[
\sum_{i,j} h_{ij}^{n+1} \Delta h_{ij}^{n+1} = n \sum_{i,j} H_{ij} h_{ij}^{n+1} - \frac{n}{H} \sum_{i,j} \sum_{\beta>n+1} H_{\beta,ij} h_{ij}^{n+1}
+ n c S_{n+1} - c n^2 H^2 - n H f_{n+1} + S_{n+1}^2 + \sum_{\beta>n+1} S_{\beta,n+1}^2
\]
\[
+ \sum_{\beta>n+1} \tilde{N}(L_{n+1} L_{\beta} - L_{\beta} L_{n+1}),
\]
where \( f_{n+1} = Tr(L_{n+1})^3 \).

M. Okumura [10] established the following lemma (see also [2]).

**Lemma 2.1.** Let \( \{a_i\}_{i=1}^n \) be a set of real numbers satisfying \( \sum_i a_i = 0 \), \( \sum_i a_i^2 = t^2 \), where \( t \geq 0 \). Then we have
\[
-\frac{n-2}{\sqrt{n(n-1)}} t^3 \leq \sum_i a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} t^3,
\]
and the equalities hold if and only if at least \((n-1)\) of the \( a_i \) are equal.

Denote the eigenvalues of \( L_{n+1} \) by \( \{\lambda_i\}_{i=1}^n \). Then we have
\[
n H = \sum_i \lambda_i, \quad S_{n+1} = \sum_i \lambda_i^2, \quad f_{n+1} = \sum_i \lambda_i^3.
\]
Set \( \tilde{L}_{n+1} = L_{n+1} - H I_n \), \( \tilde{f}_{n+1} = f_{n+1} - 3 H S_{n+1} + 2 n H^3 \), \( \tilde{S}_{n+1} = S_{n+1} - n H^2 \), and \( \tilde{\lambda}_i = \lambda_i - H \), where \( I_n \) denotes the identity matrix of degree \( n \). Then (19) changes into
\[
0 = \sum_i \tilde{\lambda}_i, \quad \tilde{S}_{n+1} = \sum_i \tilde{\lambda}_i^2, \quad \tilde{f}_{n+1} = \sum_i \tilde{\lambda}_i^3.
\]

By applying Okumura’s Lemma to \( \tilde{f}_{n+1} \), we have
\[
\tilde{f}_{n+1} \leq \frac{n-2}{\sqrt{n(n-1)}} \tilde{S}_{n+1} \sqrt{\tilde{S}_{n+1}} \iff
\]
\[
f_{n+1} \leq 3 H S_{n+1} - 2 n H^3 + \frac{n-2}{\sqrt{n(n-1)}} \tilde{S}_{n+1} \sqrt{\tilde{S}_{n+1}}.
\]
So we have
\[
n c S_{n+1} - c n^2 H^2 - n H f_{n+1} + S_{n+1}^2
\geq \tilde{S}_{n+1} \{ n c + \tilde{S}_{n+1} - n H^2 - n(n-2) H \sqrt{\frac{\tilde{S}_{n+1}}{n(n-1)}} \}.\]
It follows from (15) that

\[
\sum_{\beta>n+1} S^2_{n+1\beta} = \sum_{\beta>n+1} \left\{ \sum_{i,j} (h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\beta} \right\}^2.
\]

Denote \( S_I = \sum_{\beta>n+1} S_{\beta} \). From (22), we have

\[
\sum_{\beta>n+1} S^2_{n+1\beta} \leq S_{n+1} S_I.
\]

Let \( T = \sum_{i,j} T_{ij}\omega_i\omega_j \) be a symmetric tensor on \( M^n \) defined by

\[
T_{ij} = h_{ij}^{n+1} - nH\delta_{ij}.
\]

We introduce an operator \( \Box \) associated to \( T \) acting on \( f \in C^2(M^n) \) by

\[
\Box f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j} h_{ij}^{n+1}f_{ij} - nH\Delta f,
\]

where \( \Delta \) is the Laplacian. Since \( (T_{ij}) \) is divergence-free, it follows from [5] that the operator \( \Box \) is self-adjoint relative to the \( L^2 \)-inner product of \( M^n \).

Choosing \( f = H \) in above expression, we have

\[
\sum_{i,j} h_{ij}^{n+1}H_{ij} = \Box H + nH\Delta H.
\]

Denote \( \bar{S} = \bar{S}_{n+1} + S_I \). Substituting (21), (23) and (25) into (18), we get

\[
\sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1} \geq n\Box H + \frac{1}{2}n^2\Delta(H^2) - n^2|\nabla H|^2 \]

\[
- \frac{n}{H} \sum_{\beta>n+1} \sum_{i,j} H_{\beta,i}H_{\beta,j}h_{ij}^{n+1}
\]

\[
+ \sum_{\beta>n+1} N(L_{n+1}L_{\beta} - L_{\beta}L_{n+1})
\]

\[
+ \bar{S}_{n+1} \left\{ nc - nH^2 + \bar{S}_{n+1} - n(n-2)H \sqrt{\bar{S}_{n+1}} \right\}.
\]

3. An extension of Cheng-Yau’s technique

Cheng-Yau [5] gave a lower estimation for \( |\nabla \sigma|^2 \), the square of the length of the covariant derivative of \( \sigma \), which plays an important role in their discussion. They proved that, for a hypersurface in a space form of constant scalar curvature \( c \), if the normalized scalar curvature \( R \) is constant and \( R \geq c \), then \( |\nabla \sigma|^2 \geq n^2|\nabla H|^2 \).
For the space-like submanifolds in a de Sitter space, we can prove the following

**Theorem 3.1.** Let $M^n$ be a connected submanifold in $M^{n+p}_p(c)$ with nowhere zero mean curvature $H$. If $R$ is constant and $R < c$, then

$$|\nabla \sigma|^2 = \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 \geq n^2 |\nabla H|^2$$

and the symmetric tensor $T$ defined by (24) is negative semi-definite. Moreover, if the equality in (27) holds on $M^n$, then $H$ is constant and $T$ is negative definite.

**Proof.** From (9), we have

$$n^2 H^2 - S = n(n-1)(c-R) > 0.$$ Taking the covariant derivative on both sides of this equality, we get

$$n^2 H H_k = \sum_{i,j,\alpha} h^\alpha_{ij} h^\alpha_{ijk}, \quad k = 1, \ldots, n.$$ For every $k$, it follows from Cauchy-Schwarz's inequality that

$$n^4 H^2 H_k^2 = (\sum_{i,j,\alpha} h^\alpha_{ij} h^\alpha_{ijk})^2 \leq S \sum_{i,j,\alpha} (h^\alpha_{ijk})^2,$$

where the equality holds if and only if there exists a real function $c_k$ such that

$$h^\alpha_{ijk} = c_k h^\alpha_{ij}$$

for all $i, j$ and $\alpha$. Taking sum on both sides of (28) with respect to $k$, we have

$$n^4 H^2 |\nabla H|^2 = n^4 H^2 \sum_k H_k^2 \leq S \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 \leq n^2 H^2 \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2.$$

Therefore (27) holds on $M^n$.

Denote the eigenvalues of $L_{n+1}$ by $\{\lambda_i\}_{i=1}^n$. Then $\lambda_i^2 \leq S_{n+1} \leq S \leq n^2 H^2$ for all $i$. Hence $|\lambda_i| \leq nH$ for all $i$. Therefore $T = (T_{ij}) = L_{n+1} - nH I_n$ is negative semi-definite.

Suppose that $\sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 = n^2 |\nabla H|^2$ holds on $M^n$. It follows from (30) that

$$0 \leq n^3 (n-1)(c-R)|\nabla H|^2 \leq S \left( \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 - n^2 |\nabla H|^2 \right).$$

Hence $(c-R)|\nabla H|^2 = 0$ on $M^n$. Because $R < c$, $|\nabla H|^2 = 0$ on $M^n$. In this case, $|\lambda_i| \leq (S_{n+1})^{1/2} \leq S^{1/2} < nH$ for all $i$. Thus $T$ is negative definite. This completes the proof of Theorem 3.1. \qed
4. Submanifolds with flat normal bundle

In this section, we propose to use the extension of Cheng-Yau’s technique given in section 3 to study the rigidity problem for compact submanifolds in the de Sitter space \( M_{n+p}^p(c) \). We continue to use the same notations as in section 2. Let \( M^n \) be a compact submanifold in \( M_{n+p}^p(c) \) with nowhere zero mean curvature \( H \). Suppose that \( \xi/H \) is parallel and choose \( e_{n+1} = \xi/H \). Then \( \omega_{n+1} = 0 \) for all \( \alpha \). It follows from (11) and (16) that

\[
H_{\alpha,k} = 0, \quad H_{\alpha,kl} = 0,
\]

for all \( \alpha > n + 1 \) and \( k, l = 1, \ldots, n \).

Suppose in addition that the normal bundle of \( M^n \) is flat. Then

\[
\Omega_{\alpha\beta} = -\frac{1}{2} R_{\alpha\beta kl} \omega_k \wedge \omega_l = 0,
\]

for all \( \alpha \) and \( \beta \) on \( M^n \). For all \( \alpha \) and \( \beta \) we have \( L_{\alpha}L_{\beta} = L_{\beta}L_{\alpha} \), which is equivalent to that \( \{L_{\alpha}\}_{\alpha=n+1}^{n+p} \) can be diagonalized simultaneously.

We denote the eigenvalues of \( L_{\alpha} \) by \( \{\lambda_{\alpha}^1, \ldots, \lambda_{\alpha}^n\} \) for every \( \alpha \). It follows from [13] that

\[
\frac{1}{2} \Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n \sum_{i,j,\alpha} H_{\alpha,ij} h_{ij}^\alpha + \sum_{\alpha} \sum_{i<j} K_{ij}(\lambda_{\alpha}^i - \lambda_{\alpha}^j)^2,
\]

where \( K_{ij} = c + \sum_\beta \lambda_{\beta}^i \lambda_{\beta}^j \) denotes the sectional curvature of \( M^n \) corresponding to the plane section spanned by \( \{e_i, e_j\} \) for every pair of \( i < j \).

Assume that \( R \) is constant and \( R < c \). From (25) and (32), we have

\[
\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + n \sum_{i,j,\alpha} H_{\alpha,ij} h_{ij}^\alpha = n \square H + \frac{1}{2} \Delta (n^2 H^2) + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2 |\nabla H|^2.
\]

Note that \( \Delta S = \Delta (n^2 H^2) \). Therefore (34) turns into

\[
0 = n \square H + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2 |\nabla H|^2 + \sum_{\alpha} \sum_{i<j} K_{ij}(\lambda_{\alpha}^i - \lambda_{\alpha}^j)^2.
\]

Integrating the both sides of above equality on \( M^n \), we have

\[
0 = \int_M \left( \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - n^2 |\nabla H|^2 \right) + \sum_{i<j} \sum_{\alpha} \int_M K_{ij}(\lambda_{\alpha}^i - \lambda_{\alpha}^j)^2 * 1.
\]

If \( K_{ij} \geq 0 \) on \( M^n \), it follows from (27) and the above equality that

\[
\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \equiv n^2 |\nabla H|^2; \quad K_{ij}(\lambda_{\alpha}^i - \lambda_{\alpha}^j)^2 \equiv 0,
\]

for every \( \alpha \) and \( i < j \). Hence we can prove the following theorem.
Theorem 4.1. Let $M^n$ be a compact submanifold with non-negative sectional curvature in $M^{n+p}_c(c)$. Suppose that the normal bundle $N(M)$ is flat and the normalized mean curvature vector is parallel. If $R$ is constant and $R < c$, then $M^n$ is totally umbilical.

Proof. From the first equality of (35) and Theorem 3.1, we have that $H$ is constant on $M^n$, then $\xi$ is parallel. From Theorem 3 of [1] we know that $M^n$ is totally umbilical. \qed

Remark 4.1. In Theorem 4.1, we have used the assumptions that are different from that in Theorem 3 [1] to obtain the same result. Also, we need the following

Lemma 4.1 [12]. Let $A$ and $B$ be $n \times n$-symmetric matrices satisfying $\text{Tr} A = 0, \text{Tr} B = 0$ and $AB - BA = 0$. Then

$$-(n-2) \frac{(\text{Tr} A^2)(\text{Tr} B^2)^{1/2}}{\sqrt{n(n-1)}} \leq \text{Tr} A^2 B \leq \frac{(\text{Tr} A^2)(\text{Tr} B^2)^{1/2}}{\sqrt{n(n-1)}},$$

and the equality holds on the right (resp. left) hand side if and only if $n-1$ of the eigenvalues $x_i$ of $A$ and the corresponding eigenvalues $y_i$ of $B$ satisfy $|x_i| = \frac{(\text{Tr} A^2)^{1/2}}{\sqrt{n(n-1)}}$, $x_i x_j \geq 0$, $y_i = -\frac{(\text{Tr} B^2)^{1/2}}{\sqrt{n(n-1)}}$ (resp. $y_i = \frac{(\text{Tr} B^2)^{1/2}}{\sqrt{n(n-1)}}$).

Choose a suitable normal frame field $\{e_\beta\}_{\beta=1}^{n+p}$ such that $S_{\alpha \beta} = 0$ for all $\alpha \neq \beta$. Then

$$\sum_{\alpha,\beta > n+1} S_{\alpha \beta}^2 = \sum_{\beta > n+1} S_{\beta}^2 \leq S_I^2,$$

where the equality holds if and only if at least $p-2$ numbers of $S_{\alpha}$’s are zero.

Taking sum with respect to $\alpha > n+1$ on both-sides of (14), we have

$$\sum_{i,j,\alpha > n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha = (n c - n H^2)S_I - nH \sum_{\alpha > n+1} \text{Tr}(L_{\alpha L_{\alpha}}^{2n+1})$$

$$+ \sum_{\alpha > n+1} S_{\alpha}^2 + \sum_{\alpha > n+1} S_{\alpha}^2.$$\hspace{1cm} (38)

Using the left hand side of (36) to $\text{Tr}(L_{\alpha L_{\alpha}}^{2n+1})$, we have

$$\text{Tr}(L_{\alpha L_{\alpha}}^{2n+1}) \leq (n-2)S_{\alpha} \sqrt{\frac{S_{n+1}}{n(n-1)}}.$$
Substituting this into (38) and using (23) and (37), we have
\[ (39) \quad \sum_{i,j,\alpha>n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha \geq S_I \left\{ (nc - nH^2) - n(n-2)H \sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} + \bar{S}_{n+1} \right\}. \]

Substituting (32) into (26), we have
\[ (40) \quad \sum_{i,j} h_{n+1}^{ij} \Delta h_{n+1}^{ij} \geq n\Box H + \frac{1}{2} \Delta (n^2H^2) - n^2|\nabla H|^2 + \bar{S}_{n+1} \left\{ (nc - nH^2) - n(n-2)H \sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} + \bar{S}_{n+1} \right\}. \]

Note that \[ \Delta S = \Delta (n^2H^2) \] and
\[ \frac{1}{2} \Delta S = \sum_{i,j,k,\alpha} (h_{ij,k}^\alpha)^2 + \sum_{i,j} h_{n+1}^{ij} \Delta h_{n+1}^{ij} + \sum_{i,j,\alpha>n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha. \]

From (39) and (40), we obtain
\[ (41) \quad 0 \geq n\Box H + \sum_{(i,j,k,\alpha)} (h_{ij,k}^\alpha)^2 - n^2|\nabla H|^2 + \bar{S} \left\{ (nc - nH^2) - n(n-2)H \sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} + \bar{S}_{n+1} \right\}. \]

Consider the quadratic form \[ Q(u,t) = u^2 - \frac{n-2}{\sqrt{n-1}} ut - t^2. \] By the orthogonal transformation
\[ \begin{align*}
\bar{u} &= \frac{1}{\sqrt{2n}} \left\{ (1 + \sqrt{n-1})u + (1 - \sqrt{n-1})t \right\} \\
\bar{t} &= \frac{1}{\sqrt{2n}} \left\{ (\sqrt{n-1} - 1)u + (\sqrt{n-1} + 1)t \right\}
\end{align*} \]

\[ Q(u,t) \] turns into \[ Q(u,t) = \frac{n}{2\sqrt{n-1}} (\bar{u}^2 - \bar{t}^2), \] where \( \bar{u}^2 + \bar{t}^2 = u^2 + t^2. \)

Take \( u = \sqrt{\bar{S}_{n+1}}, t = \sqrt{n}H, \) then
\[ nc - nH^2 - n(n-2)H \sqrt{\frac{\bar{S}_{n+1}}{n(n-1)}} + \bar{S}_{n+1} \geq nc - n\bar{S}_{n+1} \frac{2\sqrt{n}}{2\sqrt{n-1}} + \frac{\bar{u}^2}{\sqrt{n-1}} \]
\[ \geq nc - \frac{n\bar{S}_{n+1}}{2\sqrt{n-1}} \]
\[ (43) \]
Note that
\( \bar{S}_{n+1} \leq \bar{S}_{n+1} + S_I = \bar{S}. \)

From (43), (44) and (27) we have
\( 0 \geq n \Box H + \bar{S} \left\{ nc - \frac{n \bar{S}}{2\sqrt{n - 1}} \right\}. \)

Integrating the both sides of (45) on \( M^n \), we have
\( 0 \geq \int_M \bar{S} \left\{ nc - \frac{n \bar{S}}{2\sqrt{n - 1}} \right\} * 1. \)

Therefore we can prove the following

**Theorem 4.2.** Let \( M^n \) \((n \geq 3)\) be a closed space-like submanifold with parallel normalized mean curvature vector field immersed into \( M_{p+p}(c) \). Suppose that \( R \) is constant and \( \bar{R} = c - R > 0 \). If the normal bundle \( N(M) \) is flat and
\( S < nH^2 + 2\sqrt{n - 1}c, \)
then \( S = nH^2 \) and \( M^n \) is umbilical (hence isometric to a sphere).

**Proof.** Denote \( \bar{R} = c - R \). Then \( \bar{S} = n(n - 1)(H^2 - \bar{R}) \) and \( S = n\bar{R} + n^2(H^2 - \bar{R}) \). Since \( n \geq 3 \), we have
\( nc - \frac{n \bar{S}}{2\sqrt{n - 1}} = n(c - \frac{n(n - 1)(H^2 - \bar{R})}{2\sqrt{n - 1}}) = n(c - \frac{S - nH^2}{2\sqrt{n - 1}}). \)

It is clear that the condition (47) is equivalent to
\( nc - \frac{\bar{S}}{2\sqrt{n - 1}} > 0. \)

From (46) and (49) we have \( \bar{S} = 0 \) on \( M^n \), so \( H^2 = \bar{R} \) and \( S = n\bar{R} \), that is \( S = nH^2 \). Since \( H \) is constant on \( M^n \), hence \( \xi \) is parallel, from Theorem 3 of [1] we know that \( M^n \) is totally umbilical.

**References**


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