\textbf{\textit{\omega}}-LIMIT SETS FOR MAPS OF THE CIRCLE

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Abstract. For a continuous map of the circle to itself, we give necessary and sufficient conditions for the \textit{\omega}-limit set of each nonwandering point to be minimal.

1. Introduction

Let $S^1$ be the circle. Throughout this paper $f$ will denote a continuous map of the circle to itself. For any positive integer $n$, we define $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let $f^0$ be the identity map of the circle. Let $AP(f), P(f), R(f), \Gamma(f), \Lambda(f)$ and $\Omega(f)$ denote the set of almost periodic points, periodic points, recurrent points, $\gamma$-limit points, $\omega$-limit points and nonwandering points of $f$, respectively.

A subset $Y$ in $S^1$ is called invariant if $f(Y) \subset Y$, and strongly invariant if $f(Y) = Y$. Suppose $Y \subset S^1$ is non-void, closed and invariant relative to $f$.

If $Y$ has no proper subset which is non-void and invariant relative to $f$, then $Y$ is said to be a minimal set.

J. C. Xiong [4,5] proved that for any continuous map $g$ of the interval, the following conditions are equivalent.

(1) $\Gamma(g) = AP(g)$.

(2) The period of each periodic point of $g$ is a power of 2.

In this paper, we obtain the following theorem for maps of the circle.

\textbf{Theorem 5.} Suppose that $f$ is a continuous map of the circle. Then the following conditions are equivalent:

(1) $\Gamma(f) = AP(f)$.

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(2) For every \( x \in \Omega(f) \), the \( \omega \)-limit set \( \omega(x, f) \) of \( x \) is minimal.

In 1986, L. Block and E. M. Coven [3] proved that for a continuous map \( g \) of the interval, if \( x \in \Lambda(g) \setminus \overline{R(g)} \), then \( \omega(x, g) \) is infinite minimal, and if \( x \in \Omega(g) \setminus \overline{R(g)} \), then \( \omega(x, g) \) need not be minimal. We have the following theorem for maps of the circle.

**Theorem 6.** Suppose that \( f \) is a continuous map of the circle. Let \( R(f) \) be closed, \( x \in \Omega(f) \), \( f^{kN}(x) = p \in F(f^N) \) and \( x \in \text{int}(W_i) \) for some \( i \). If \( x \in \Omega(f) \setminus \overline{R(f)} \), then \( \omega(x, f) \) is infinite minimal.

## 2. Preliminaries and Definitions

Let \( f \) be a continuous map of the circle \( S^1 \) to itself. The orbit \( \text{Orb}(x) \) of \( x \in S^1 \) is the set \( \{ f^k(x) | k = 1, 2, \cdots \} \). A point \( x \in S^1 \) is a fixed point of \( f \) if \( f(x) = x \) and we denote the set of fixed points by \( F(f) \). A point \( x \in S^1 \) is a periodic point of \( f \) provided that for some positive integer \( n \), \( f^n(x) = x \). The period of \( x \) is the least such integer \( n \). We denote the set of periodic points of \( f \) by \( P(f) \).

A point \( x \in S^1 \) is a recurrent point of \( f \) provided that there exists a sequence \( \{ n_i \} \) of positive integers with \( n_i \to \infty \) such that \( f^{n_i}(x) \to x \), or equivalently, \( f^n(x) \to x \). We denote the set of recurrent points of \( f \) by \( R(f) \).

A point \( x \in S^1 \) is called a nonwandering point of \( f \) provided that for every neighborhood \( U \) of \( x \), there exists a positive integer \( m \) such that \( f^m(U) \cap U \neq \emptyset \). We denote the set of nonwandering points of \( f \) by \( \Omega(f) \).

A point \( x \in S^1 \) is almost periodic point of \( f \) provided that for any \( \epsilon > 0 \) one can find an integer \( n > 0 \) with the following property that for any integer \( q > 0 \) there exists an integer \( r \) with \( q \leq r < q + n \) such that \( d(f^r(x), x) < \epsilon \), where \( d \) is the metric of \( S^1 \). We denote the set of almost periodic points of \( f \) by \( AP(f) \).

J. C. Xiong [4] investigated the set \( AP(g) \) of almost periodic points of a continuous map \( g \) of the interval and proved the followings.

\( AP(g) = P(g) \) if and only if \( \Omega(g) = P(g) \), and \( AP(g) \) is closed if and only if \( R(g) \) is closed. Also, if \( g \) has a periodic point of period which is not a power of 2, then \( AP(g) - P(g) \neq \emptyset \) and \( R(g) - AP(g) \neq \emptyset \), and if
the period of each periodic point of \( g \) is power of 2, then \( R(g) = AP(g) \). Therefore the period of each periodic point of \( g \) is power of 2 if and only if \( R(g) = AP(g) \).

A point \( y \in S^1 \) is called an \( \omega \)-limit point of \( x \in S^1 \) provided that there exists a sequence \( \{n_i\} \) of positive integers with \( n_i \to \infty \) such that \( f^{n_i}(x) \to y \). We denote the set of \( \omega \)-limit points of \( x \) by \( \omega(x, f) \). Define \( \Lambda(f) = \bigcup_{x \in S^1} \omega(x, f) \).

A point \( y \in S^1 \) is called an \( \alpha \)-limit point of \( x \in S^1 \) if there exist a sequence \( \{n_i\} \) of positive integers with \( n_i \to \infty \) and a sequence \( \{x_i\} \) of points in \( S^1 \) with \( x_i \to x \) such that \( f^{n_i}(x_i) = y \) for all \( i \geq 1 \). We denote the set of \( \alpha \)-limit points of \( x \) by \( \alpha(x, f) \).

A point \( x \in S^1 \) is called a \( \gamma \)-limit point of \( y \in S^1 \) if \( x \in \omega(y, f) \cap \alpha(y, f) \). Define \( \Gamma(f) = \bigcup_{x \in S^1} \{\omega(x, f) \cap \alpha(x, f)\} \).

For a fixed point \( p \) of \( f \) and a side \( S \), the one-side unstable set of \( p \) is

\[
W^u(p, f, s) = \cap_u \bigcup_{k \geq 0} f^k(U),
\]

where the intersection is taken over all \( s \)-half-neighborhoods \( U \) of \( p \). Let \( p \) be a fixed point of \( f^N \) and \( S_i \) a side at \( f^i(p) \) for each \( i \). We denote \( W_i \) by \( W^u(f^i(p), f^N, S_i) \) for each \( i \).

3. Main results

The following lemmas appear in [1], [2], [4] and [6].

**Lemma 1** [1]. Suppose that \( f \) is a continuous map of the circle \( S^1 \) to itself. Then

\[
P(f) \subset AP(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).
\]

**Lemma 2** [4]. Suppose that \( f \) is a continuous map of the circle \( S^1 \) to itself. Then \( x \in AP(f) \) if and only if \( x \in \omega(x, f) \) and \( \omega(x, f) \) is minimal.

**Lemma 3** [6]. Suppose that \( f \) is a continuous map of the circle. Then

\[
\Lambda(\Omega(f)) = \Lambda(\Gamma(f)) = \Gamma(f).
\]
Lemma 4 [2]. Suppose that $f$ is a continuous map of the circle. If $x \in \Omega(f)$ has a finite orbit, $f^{kN}(x) = p \in F(f^N)$ and $x \in \text{int}(W_i)$ for some $i$, then $x \in \overline{R(f)}$.

Proof of Theorem 5 (1) $\Rightarrow$ (2) : Suppose that $\Gamma(f) = AP(f)$. Let $x$ be any point in $\Omega(f)$, and let $y$ be arbitrary point in $\omega(x, f)$. Let $z \in \omega(y, f)$. Then there exists a sequence of positive integers $n_i \to \infty$ such that $f^{n_i}(y) \to z$. Since $y \in \omega(x, f)$, there exists a sequence of positive integers $m_i \to \infty$ such that $f^{m_i}(x) \to y$. Hence $f^{m_i+n_i}(x) \to z$. Thus $z \in \omega(x, f)$. Hence $\omega(y, f) \subset \omega(x, f)$.

Since $y$ is arbitrary point in $\omega(x, f)$, it suffices to show that $y \in \omega(y, f)$. Since $x \in \Omega(f)$, $\omega(x, f) \subset \Lambda(\Omega(f))$. By Lemma 3, $y \in \omega(x, f) \subset \Gamma(f)$. Since $\Gamma(f) = AP(f)$, $y \in AP(f)$. By Lemma 2, $y \in \omega(y, f)$. Hence $\omega(x, f) \subset \omega(y, f)$. Therefore $\omega(x, f) = \omega(y, f)$ and $\omega(x, f)$ is minimal.

(2) $\Rightarrow$ (1) : Suppose that for any $x \in \Omega(f)$, $\omega(x, f)$ is minimal. Let $y \in \Gamma(f)$. Then by Lemma 3, $y \in \Lambda(\Omega(f))$. There is $z \in \Omega(f)$ such that $y \in \omega(z, f)$. Since $\omega(z, f)$ is minimal, $\omega(y, f) = \omega(z, f)$. Hence $y \in \omega(y, f)$. By Lemma 1, $y \in \Omega(f)$. So $\omega(y, f)$ is minimal. Thus, by Lemma 2, $y \in AP(f)$. Therefore $\Gamma(f) \subset AP(f)$.

Corollary 1. Suppose that $f$ is a continuous map of the circle. Let $R(f)$ be closed. Then the following conditions are equivalent:

(1) $R(f) = AP(f)$.
(3) For every $x \in \Omega(f)$, the $\omega$-limit set $\omega(x, f)$ of $x$ is minimal.

Proof of Theorem 6 Suppose that $R(f)$ is closed. Let $x \in \text{int}(W_i)$ for some $i$ and $x \in \Omega(f) \setminus \overline{R(f)}$. Since $R(f)$ is closed, by Lemma 1, $\Gamma(f) = R(f)$. By Theorem 5, $\omega(x, f)$ is minimal. Now we show that $\omega(x, f)$ is infinite. Assume that $\omega(x, f)$ is finite. Since $\omega(x, f)$ is closed and invariant, $\omega(x, f) = \overline{\text{Orb}(x, f)}$ by definition. Then $\overline{\text{Orb}(x, f)}$ is finite. Hence $\text{Orb}(x, f)$ is finite, a contradiction.

The set $\Omega(f)$ of nonwandering points of $f$ is always closed and invariant and $P(f) = P(f^n) \subset \Omega(f^n) \subset \Omega(f)$ holds for all $n$. It is well known that $R(f) = R(f^n)$ for all $n$. Therefore we have the following corollary.
Corollary 2. Suppose that \( f \) is a continuous map of the circle. Let \( R(f) \) be closed, \( x \in \Omega(f) \), \( f^{kN}(x) = p \in F(f^N) \) and \( x \in \text{int}(W_i) \) for some \( i \). If \( x \in \Omega(f^n) \setminus R(f) \), then \( \omega(x, f) \) is minimal.

References


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