

EXAMPLE AND COUNTEREXAMPLES IN DOUBLE INTEGRAL AND ITERATED INTEGRAL

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ABSTRACT. We give an example and counterexamples in double integrals and iterated integrals

1. Multiple Riemann Integrals

Problem 1.1. Show that $\int_0^1 \left[\int_0^1 f(x, y) dy \right] dx = \int_0^1 \left[\int_0^1 f(x, y) dx \right] dy$

Counterexample. If p_k denotes the k -th prime number, let

$$S(p_k) = \left\{ \left(\frac{n}{p_k}, \frac{n}{p_k} \right); n = 1, 2, \dots, p_k - 1 \right\},$$

let $S = \cup_{k=1}^{\infty} S(p_k)$, and let $Q = [0, 1] \times [0, 1]$. Define f on Q as follows ; $f(x, y) = 1$ if $(x, y) \in S$, $f(x, y) = 0$ if $(x, y) \in Q - S$.

Solution. Let $y \in [0, 1]$.

Case i) $y \neq \frac{n}{p_k}$ for all $n < p_k$ and for all $k \in N$.

In this case $(x, y) \in Q - S$ for all $x \in [0, 1]$. Hence $f(x, y) = 0$ so that $\int_0^1 f(x, y) dx = 0$.

Case ii) $y = \frac{n}{p_k}$ for some $k \in N$ and $n < p_k$.

Since $S(p_1), S(p_2), \dots$ are disjoint, there are finitely many $x \in [0, 1]$ namely $x = \frac{1}{p_k}, \frac{2}{p_k}, \dots, \frac{p_k-1}{p_k}$ for which $(x, y) \in S$. Hence $f(x, y) = 1$ for all but finitely many $x \in [0, 1]$, so that $\int_0^1 f(x, y) dx = 1$.

Consequently, we have $\int_0^1 f(x, y) dx = 1$ for all $y \in [0, 1]$ and $\int_0^1 \int_0^1 f(x, y) dx dy = 1$. Symmetrically $\int_0^1 \int_0^1 f(x, y) dy dx = 1$.

To show that S is dense in Q , let $(x, y) \in Q$ and let $\epsilon > 0$ be given.

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There are natural numbers ϵ, r, s such that $\epsilon < s, r < s$ and $\left| \left(\frac{q}{s}, \frac{r}{s} \right) - (x, y) \right| < \epsilon$. Choose a $k \in \mathbb{N}$ such that $\frac{1}{p_k} < \epsilon, s < p_k$. Set $n = \left[\frac{qp_k}{s} \right], m = \left[\frac{rp_k}{s} \right]$ then $\left| \left(\frac{n}{p_k}, \frac{m}{p_k} \right) - \left(\frac{q}{s}, \frac{r}{s} \right) \right| < 2\epsilon$.

Thus $\left| \left(\frac{n}{p_k}, \frac{m}{p_k} \right) - (x, y) \right| < 3\epsilon$.

It is clear that $Q - S$ is dense in Q .

Since every rectangle $R \subset Q$ contains a point of S as well as a point of $Q - S$, $\int_Q f = 1, \int_{Q-S} f = 0$. Hence $\int_Q f(x, y) d(x, y)$ does not exist.

Problem 1.2. Show that $\int_0^1 f(x, y) dx = \int_0^1 \left[\int_0^1 f(x, y) dx \right] dy = \int_Q f(x, y) d(x, y) = 0$ but that $\int_0^1 f(x, y) dy$ does not exist for rational x .

Counterexample. Define f on the square $Q = [0, 1] \times [0, 1]$ as follows :

$$f(x, y) = \begin{cases} 0 & \text{if at least one of } x, y \text{ is irrational,} \\ 1 & \text{if } y \text{ is rational and } x = \frac{m}{n}, \\ & \text{where } m \text{ and } n \text{ are relatively prime integers, } n > 0 \end{cases}$$

Solution. Let $y \in [0, 1]$.

Case i) y is irrational.

$$\int_0^1 f(x, y) dx = \int_0^1 0 dx = 0$$

Case ii) y is rational.

$$f(x, y) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. Put $S = \{x \in [0, 1] | x = \frac{m}{n}, \gcd(m, n), n < N\}$. Then S is a finite set.

Take a $\delta > 0$ such that $\delta \sum_{\frac{m}{n} \in S} \frac{1}{n} < \epsilon$.

If $\Delta = \{\Delta_1, \Delta_2, \dots, \Delta_s\}$ is a partition of $[0, 1]$ into finite closed intervals, $\|\Delta\| < \delta$ and ξ is a choice function on Δ then

$$\begin{aligned}
 0 &\leq S_{\Delta\xi}(f(\cdot, y)) \\
 &= \sum_i f(\xi_i, y)\Delta_i x \\
 &= \sum_{\substack{\xi_i \text{ is} \\ \text{irrational}}} f(\xi_i, y)\Delta_i x + \sum_{\xi_i \in S} f(\xi_i, y)\Delta_i x + \sum_{\substack{\xi_i \text{ is rational} \\ \xi_i = \frac{m}{n} \\ n \geq N}} f(\xi_i, y)\Delta_i x \\
 &\leq 0 + \sum_{\frac{m}{n} \in S} \frac{1}{n}\Delta_i x + \sum_{\substack{\xi_i \text{ is rational} \\ \xi_i = \frac{m}{n} \\ n \geq N}} \frac{1}{n}\Delta_i x \\
 &< \epsilon + \epsilon = 2\epsilon.
 \end{aligned}$$

where $\Delta_i x$ is the length of the intervals Δ_i . Hence $\int_0^1 f(x, y)dx = 0$.

Let $\epsilon > 0, N \in \mathbb{N}, S, \delta$ be the same as above.

If $\Delta = \{\Delta_{ij} | i = 1, 2, \dots, s, j = 1, 2, \dots, t\}$ is a partition of Q into finite rectangles $\|\Delta\| = \max_{i,j} \text{dia } \Delta_{ij} < \delta$, and (ξ, η) is a choice function on Δ then

$$\begin{aligned}
 0 &\leq S_{\Delta\xi\eta}(f) \\
 &= \sum_{ij} f(\xi_{ij}, \eta_{ij})\Delta_{ij} A \\
 &= \sum_{\substack{\text{at least} \\ \text{one of } \xi_{ij}, \eta_{ij} \\ \text{is irrational}}} f(\xi_{ij}, \eta_{ij})\Delta_{ij} A + \sum_{\substack{\eta_{ij}, \text{ irrational} \\ \xi_{ij} \in S}} f(\xi_{ij}, \eta_{ij})\Delta_{ij} A \\
 &+ \sum_{\substack{\eta_{ij}, \text{ irrational} \\ \xi_{ij}, \text{ irrational} \\ \xi_{ij} \notin S}} f(\xi_{ij}, \eta_{ij})\Delta_{ij} A \\
 &\leq 0 + \epsilon + \epsilon \\
 &= 2\epsilon.
 \end{aligned}$$

Hence $\int_Q f(x, y)d(x, y) = 0$.

For a fixed rational $x = \frac{m}{n}, \text{gcd}(m, n) = 1$.

$$f(x, y) = \begin{cases} 0 & \text{if } y \text{ is irrational} \\ \frac{1}{n} & \text{if } y \text{ is rational} \end{cases}$$

$$\int_0^{\bar{1}} f(x, y) dy = \frac{1}{n}, \quad \int_{\underline{0}}^1 f(x, y) dy = 0.$$

Hence $\int_0^1 f(x, y) dy$ does not exist for rational x .

2. Multiple Lebesgue Integrals

Problem 2.1. Show that $f \in L(R^2)$ by the monotone convergence theorem and calculate the double integral $\int_{R^2} f(x, y) d(x, y)$

Example. : For fixed $c, 0 < c < 1$, define f on R^2 as follows;

$$f(x, y) = \begin{cases} \frac{(1-y)^c}{(x-y)^c} & \text{if } 0 \leq y < x, 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution. : Define

$$f_n(x, y) = \begin{cases} \frac{(1-y)^2}{(x-y)^c} & \text{if } 0 \leq y < x - \frac{1}{n}, 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then f_n is Riemann integrable and

$$\begin{aligned} \int \int_{R^2} f_n(x, y) d(x, y) &= \int_0^{1-\frac{1}{n}} \int_{\frac{1}{n}}^{1-v} \frac{(1-v)^c}{u^c} du dv \\ &= \frac{1}{1-c} \left(\left(1 - \frac{1}{n}\right) - \frac{1}{2} \left(1 - \frac{1}{n}\right)^2 \right) \\ &\quad + \frac{1}{1-c} \cdot \frac{1}{n^{1-c}} \cdot \left(\frac{1}{1+c} \cdot \frac{1}{n^{1+c}} \right) \\ &\xrightarrow{n \rightarrow \infty} \frac{1}{2(1-c)} \end{aligned}$$

where $x - y = u, y = v, x = u + v, y = v$.

By the monotone convergence theorem,

$$\int_{R^2} f(x, y) d(x, y) = \frac{1}{2(1-c)} < \infty.$$

Hence $f \in L(R^2)$ and $\int_{R^2} f(x, y) d(x, y) = \frac{1}{2(1-c)}$.

Problem 2.2. Show the both iterated integrals $\int_R \left[\int_R f(x, y) dx \right] dy$ and $\int_R \left[\int_R f(x, y) dy \right] dx$ exist and are equal, but that the double integral of f over R^2 does not exist.

Counterexample. : Let $f(x, y) = e^{-xy} \sin x \sin y$ if $x \geq 0, y \geq 0$ and let $f(x, y) = 0$ otherwise.

Solution. : $|f(x, y)| \leq e^{-xy}$ and for each y , the function $x \mapsto e^{-xy}$ is integrable by the bounded convergence theorem with integral $\frac{1}{y}$ if $y > 0$. Hence $x \mapsto f(x, y)$ is integrable. If $y = c$ then $f(x, y) = 0$ for all x so that $f(x, y)$ is integrable with integral 0. Hence for all $y \geq 0$, and calculate the double integral,

$$\int_R \left[\int_R f(x, y) dx \right] dy = \int_R \left[\int_R f(x, y) dy \right] dx = \frac{\pi}{2}.$$

By the Tonelli-Hobson test, $\int_{R^2} f(x, y) d(x, y)$ does not exist.

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REFERENCES

1. T. M. Apostol, *Mathematical analysis*, Second edition, Addison-Wesley Publishing Co., Mass.-London-Don Mills, Ont., 1974 MR **49**#9123.
2. E. Asplund and L. Bungart, *A first course in integration*, Holt, Rinehart and Winston, New York-Toronto, Ont.-London, 1966 MR **33**#2780.
3. R. G. Bartle, *The elements of integration*, John Wiley & Sons, Inc., New York-London-Sydney, 1966 MR **34**#293.
4. H. Kestelman, *Modern theories of integration*, 2nd revised ed., Dover Publications, Inc., New York, 1960 MR **23**#A282.
5. J. Korevaar, *Mathematical methods. Vol. 1: Linear algebra, normed spaces, distributions, integration*, Academic Press, New York-London, 1968 MR **37**#1209.
6. F. Riesz and B. Sz-Nagy, *Functional analysis*, Translated by Leo F. Boron, Frederick Ungar Publishing Co., New York, 1955 MR **17**,175i.
7. W. W. Rogosinski, *Volume and integral*, Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1952 MR **14**,964b.

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