NONWANDERING POINTS OF A MAP ON THE CIRCLE

JONG SOOK BAE\textsuperscript{1}, SEONG HOON CHO\textsuperscript{2},
KYUNG JIN MIN AND SEUNG KAB YANG

ABSTRACT. In this paper, we will show that for any continuous map \( f \) of the circle, if the set of periodic points of \( f \) is empty, then the set of recurrent point of \( f \) equals the set of nonwandering points of \( f \).

\section{Introduction}

In study of the dynamics of a map \( f \) from a topological space \( X \) to itself, a central role is played by the various recursive properties of the points of \( X \). One such property is periodicity. A weaker property is that of being nonwandering. Intermediate recursive properties include almost periodicity and recurrence.

Let \( C^0(X, X) \) denote the set of continuous maps from \( X \) into itself. And for any \( f \in C^0(X, X) \), let \( P(f), R(f), \Lambda(f), \Gamma(f) \) and \( \Omega(f) \) denote the set of periodic points, recurrent points, \( \omega \)-limit points, \( \gamma \)-limit points and nonwandering points of \( f \), respectively.

In 1988, J.C.Xiong \[4\] proved the following sequence of the sets and inclusion relation hold;

\[ P(f) \subset R(f) \subset \Gamma(f) \subset \overline{P(f)} \subset \Lambda(f) \subset \Omega(f) \]

for any continuous map \( f \) of the interval \( I \). But the equalities need not hold.

For a continuous map \( f \) of the circle \( S^1 \), J.S.Bae, S.H.Cho and S.K.Yang \[2\] obtained the similar result;

\[ P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f). \]
Also, in this case, the equalities need not hold.

On the other hand, in 1983, L. Block, E. Coven, I. Mulvey and Z. Nitecki [1] showed that for any continuous map $f$ of the circle, if $P(f)$ is closed and non-empty, then $P(f) = \Omega(f)$, and hence

$$ (P(f) =) R(f) = \cdots = \Omega(f). $$

In this paper, we will show that the above equalities hold unless $P(f)$ is non-empty. Consequently, we obtain the following result.

**Theorem A.** For any $f \in C^0(S^1, S^1)$, if $P(f)$ is empty, then

$$ R(f) = \Gamma(f) = \overline{K(f)} = \Lambda(f) = \Omega(f). $$

§ 2. Preliminaries and definitions

Let $(X, d)$ be a metric space and $f \in C^0(X, X)$. And let $f^{n+1} = f \circ f^n$, for $n = 1, 2, 3, \cdots$

A point $x \in X$ is called a *recurrent point* of $f$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of $f$ by $R(f)$.

A point $x \in X$ is called a *nonwandering point* of $f$ if for every neighborhood $U$ of $x$, there exists a positive integer $m$ such that $f^m(U) \cap U \neq \emptyset$. We denote the set of nonwandering points of $f$ by $\Omega(f)$.

A point $y \in X$ is called an *ω-limit point* of $x \in X$ if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of $x$ by $\omega(x)$. Define $\Lambda(f) = \bigcup_{x \in X} \omega(x)$.

A point $y \in X$ is called an *α-limit point* of $x \in X$ if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The symbol $\alpha(x)$ denotes the set of α-limit points of $x \in X$.

A point $y \in X$ is called a *γ-limit point* of $x$ if $y \in \omega(x) \cap \alpha(x)$. The symbol $\gamma(x)$ denotes the set of γ-limit points of $x$ and $\Gamma(f) = \bigcup_{x \in X} \gamma(x)$.

Let $x \in S^1$ and $f \in C^0(S^1, S^1)$ be given. Then we will use the symbol $\omega_+(x)$ (resp. $\omega_-(x)$) to denote the set of all points $y \in S^1$ such that there
exists a sequence \( \{n_i\} \) of positive integers with \( n_i \to \infty \) such that \( f^{n_i}(x) \to y \) and

\[
y < \cdots < f^{n_1}(x) < \cdots < f^{n_2}(x) < f^{n_1}(x)
\]

( resp. \( f^{n_1}(x) < f^{n_2}(x) < \cdots < f^{n_1}(x) < \cdots < y \)).

A set \( E \subset X \) is said to be \textit{invariant} under \( f \) if \( f(E) \subset E \). It is clear that \( P(f) \), \( R(f) \), \( \Lambda(f) \), \( \Gamma(f) \) and \( \Omega(f) \) are invariant under \( f \).

Let \( R \) be the set of reals and \( Z \) be the set of integers. Formally, we will think of the circle \( S^1 \) as \( R/Z \) and use \( \pi : R \to R/Z \) to denote the canonical projection. In fact, the map \( \pi : R \to S^1 \) is an example of a covering map, since it wraps \( R \) around \( S^1 \) without doubling back (i.e., without critical points). To study the dynamics of the circle map, it is helpful to using a lifting.

Let \( f \) be a continuous map on the circle. We say that a continuous map \( F \) from \( R \) to itself is a lifting of \( f \) if \( f \circ \pi = \pi \circ F \).

We will use the following notations throughout this paper.

Let \( a, b \in S^1 \) with \( a \neq b \), and let \( A \in \pi^{-1}(a) \), \( B \in \pi^{-1}(b) \) with \( |A - B| < 1 \) and \( A < B \). Then we write \( \pi((A, B)), \pi([A, B]), \pi([A, B)) \) and \( \pi((A, B]) \) to denote the open, closed and half-open arcs from \( a \) counterclockwise to \( b \), respectively, and we denote it by \( (a, b) \), \( [a, b] \), \( [a, b) \) and \( (a, b] \).

For \( x, y \in [a, b] \) with \( a \neq b \), let \( X \in \pi^{-1}(x) \), \( Y \in \pi^{-1}(y) \) with \( X, Y \in [A, B] \), then we define for \( x, y \in [a, b] \), \( x > y \) if and only if \( X > Y \). Let \( C \) be a subset of a closed arc \( [a, b] \), then we define \( \sup C = \pi(\sup(\pi^{-1}(C) \cap [A, B])) \) and \( \inf C = \pi(\inf(\pi^{-1}(C) \cap [A, B])) \). In particular, for \( a, b, c \in S^1 \), \( a < b < c \) means that \( b \) lies in the open arc \( (a, c) \), that is, \( b \in (a, c) \).

Now we consider the notation of an \( f \)-\textit{covering}. The important property of an \( f \)-covering lies in the fact that if \( J \) \( f^n \)-covers itself for some \( n \), then \( f \) has a period point in \( J \).

**Definition 2.1.** Let \( X \) be \( I \) or \( S^1 \) and \( f \in C^0(X, X) \). Let \( J \) and \( K \) be two closed intervals in \( X \). We say that \( J \) \( f \)-covers \( K \) if there is a closed subinterval \( L \subset J \) such that \( f(L) = K \).

The following two lemmas appear in [3].
LEMMA 2.2. [3, Lemma 2] Let $X$ be $S^1$ or $I$ and $f \in C^0(X, X)$. Let $J$ and $K$ be proper closed intervals in $X$ such that $J$ $f$-covers $K$. If $L$ is a closed interval with $L \subset K$, then $J$ $f$-covers $L$.

LEMMA 2.3. [3, Lemma 3] Let $f \in C^0(I, I)$. Suppose that $J$ is a proper closed interval in $X$ such that $J$ $f$-covers $K$ or $f(J) \subset J$. Then $f$ has a fixed point in $J$.

§ 3. Main Result

The following lemma appears in [2].

LEMMA 3.1. [2] For any $f \in C^0(S^1, S^1)$, $x \in \Omega(f)$ if and only if $x \in \alpha(x)$.

LEMMA 3.2. Let $f \in C^0(S^1, S^1)$ and $I = [a, b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $I \cap P(f) = \emptyset$.

(a) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x < f(x)$.

1. if $y \in I$, $x < y$ and $f(y) \notin [y, b]$, then $[x, y] f$-covers $[f(x), b]$,
2. if $y \in I$, $x > y$ and $f(y) \notin [y, b]$, then $[y, x] f$-covers $[f(x), b]$.

(b) Suppose that there exists $x \in I$ such that $f(x) \in I$ and $x > f(x)$.

1. if $y \in I$, $x < y$ and $f(y) \notin [a, y]$, then $[x, y] f$-covers $[a, f(x)]$,
2. if $y \in I$, $y < x$ and $f(y) \notin [a, y]$, then $[y, x] f$-covers $[a, f(x)]$.

Proof. We prove only part (a) because of the symmetry. Let $A, B \in R$ with $A < B$ such that $\pi((A, B)) = (a, b)$, and let $X \in (A, B) \cap \pi^{-1}(x)$. Then we can take a lifting $F$ of $f$ with $F(X) \in (A, B)$ by assumption, we know that $A < X < F(X) < B$.

1. Let $Y \in (A, B) \cap \pi^{-1}(y)$. Then $F(Y) \notin (Y + N, B + N)$ for any integer $N$. If $y > x$, then $Y > X$, and hence $F(Y) > Y$ because also $F$ has no periodic points in $[A, B]$. Since $F(Y) \notin (Y + N, B + N)$ for any integer $N$, $F(Y) > B > F(X)$. Hence $[X, Y] F$-covers $[F(X), B]$, so that $[x, y] f$-covers $[f(x), b]$.

2. Let $Y \in (A, B) \cap \pi^{-1}(y)$. Then $F(Y) \notin (Y + N, B + N)$ for any integer $N$. If $y < x$, then $Y < X$, and hence $F(Y) > Y$. Therefore we
have \( F(Y) > B > F(X) \), so that \([Y, X]\) \( F \)-covers \([F(X), B]\), and hence \([y, x]\) \( f \)-covers \([f(x), b]\).

**Lemma 3.3.** Let \( f \in C^0(S^1, S^1) \) and \( P(f) = \phi \). Then
\[
\overline{R(f)} \subset \Gamma(f).
\]

**Proof.** Without loss of generality, we assume that \( x \in \overline{R(f)} \setminus R(f) \). Then there exists an open arc \((a, b)\) in \( S^1 \) containing \( x \) such that \( f^n(x) \notin (a, b) \) for any positive integer \( n \), and hence we may assume that there exists a sequence \( \{x_i\} \) of points with \( x_i \in R(f) \) such that \( a < x_1 < x_2 < \cdots < x_i < \cdots < x < b \) and \( x_i \to x \). For each \( i = 1, 2, \ldots \), there exist \( y_i, z_i \in (x_{i-1}, x_{i+1}) \) and \( n_i, m_i \) with \( n_i < m_i \) such that
\[
x_{i-1} < f^{n_i}(y_i) < y_i < x_{i+1} < x
\]
and
\[
x_{i-1} < z_i < f^{m_i}(z_i) < x_{i+1} < x.
\]
By Lemma 3.2,
\[
[y_i, x] \ f^{n_i} \text{-covers } [a, f^{n_i}(y_i)]
\]
and
\[
[z_i, x] \ f^{m_i} \text{-covers } [f^{m_i}(z_i), b].
\]
Consequently,
\[
(*) \quad [x_{i-1}, x] \ f^{n_i} \text{-covers } [x_1, x_{i-1}] \quad \text{for each } i,
\]
and
\[
(**) \quad [x_{i-1}, x] \ f^{m_i} \text{-covers } [x_{i+1}, x] \quad \text{for each } i.
\]
Now, let \( K_i = [x_i, x] \) for all positive integer \( i \), Then \( K_i \ f^{m_i} \text{-covers } K_{i+1} \). Hence we may choose a closed arc \( L_1 \) in \( K_1 \) such that \( f^{m_1}(L_1) = K_3 \). Also, we can take a closed arc \( L_2 \) in \( L_1 \) such that \( f^{m_1+m_3}(L_2) = K_5 \). Continuing this process, we may take a closed arc \( L_i \subset K_1 \) such that \( L_1 \supset L_2 \supset \cdots \) and \( \sum_{i=1}^{\infty} m_{2i-1} (L_i) = K_{2k+1} \) for each \( k = 1, 2, \ldots \). Let \( y \in \bigcap_{i=1}^{\infty} L_i \). Then \( x \in \omega(y) \) and \( y \in [x_1, x] \). Now, take \( N \) such that \( x_{N-1} > y \). By \((*)\), for all \( i \geq N \), there exists \( y_i \in [x_{i-1}, x] \) such that \( f^{n_i}(y_i) = y \). Since \( x_i \to x \), we have \( y_i \to x \), and hence \( x \in \alpha(y) \).

Thus \( x \in \omega(y) \cap \alpha(y) \subset \Gamma(f) \).

The following lemma appears in [2].
Lemma 3.4. [2] Let \( f \in C^0(S^1, S^1) \). Then we have

\[
\Gamma(f) \subset R(f) \cup \overline{P(f)}.
\]

By using Lemma 3.3 and Lemma 3.4, we have the following proposition.

Proposition 3.5. Let \( f \in C^0(S^1, S^1) \) and \( P(f) = \phi \). Then we have \( R(f) = \Gamma(f) = \overline{R(f)} \), and hence \( R(f) \) is closed.

Proof. Suppose that \( P(f) = \phi \). Then by Lemma 3.3, we have \( \overline{R(f)} \subset \Gamma(f) \), and by Lemma 3.4, we know that \( \Gamma(f) \subset R(f) \). Therefore, we conclude \( R(f) = \Gamma(f) = \overline{R(f)} \).

Theorem A. For any \( f \in C^0(S^1, S^1) \), if \( P(f) \) is empty, then

\[
R(f) = \Gamma(f) = \overline{R(f)} = \Lambda(f) = \Omega(f).
\]

Proof. Let \( x \in \Omega(f) \setminus R(f) \) and \( D \) be a connected component of \( S^1 \setminus R(f) \) containing \( x \). By Proposition 3.5, \( R(f) \) is closed, and hence \( D = (a, b) \) for some \( a, b \in S^1 \) with \( a \neq b \). Then we know that \( a, b \in R(f) \). Since \( R(f) \) is invariant under \( f \), \( a \in \omega_-(a) \) and \( b \in \omega_+(b) \). And since \( x \in \Omega(f) \cap D \), there exists \( k > 0 \) such that \( f^k(D) \cap D \neq \phi \). Therefore, there exists a point \( y \in D \) with \( f^k(y) \in D \). Without loss of generality, we may assume that \( a < y < f^k(y) < b \). Then we know that \([y, b] \) \( f^k \)-covers \([f^k(y), f^k(b)]\) by Lemma 3.2, and \( b \in (f^k(y), f^k(b)) \) since \( f^k(b) \in R(f) \subset [b, a] \) and \( f^k(b) \neq b \). Since \( b \in \omega_+(b) \), there exist positive integers \( m, n \) such that \( b < f^{m+n}(b) < f^m(b) < f^k(b) \). Especially,

\[
[y, b] \quad f^k \text{-covers} \quad [b, f^m(b)].
\]

On the other hand, since \( f^n(b) \notin (a, b) = D \), by Lemma 3.2,

\[
[b, f^m(b)] \quad f^n \text{-covers} \quad [a, f^{m+n}(b)].
\]

In particular,

\[
[b, f^m(b)] \quad f^n \text{-covers} \quad [y, b]
\]

By Lemma 2.2, \([y, b] \) \( f^{n+k} \)-covers itself, and hence \( f \) has a periodic point in \([a, b] \) by Lemma 2.3, which is a contradiction.

The proof of Theorem A is complete.
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References


Jong Sook Bae, Kyung Jin Min and Seung Kab Yang
Deparment of Mathematics
MyongJi University
Yongin, 449-728, Korea
E-mail: 1 jsbae@wh.myongji.ac.kr.

Seong Hoon Cho
Deparment of Mathematics
Hanseo University
Chungnam 356-820, Korea