OPTIMAL BIVARIATE BONFERRONI-TYPE INEQUALITIES VIA TAKING AVERAGE

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ABSTRACT

1. Introduction

Let $A_1, A_2, \cdots, A_m$ be a sequence of events on a given probability space and let $X_m(A)$ be the number of those $A$'s which occur. Put $S_0 = 1$ and

$$S_k = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}), \quad (1 \leq k)$$

where the summation is over all subscripts satisfying $1 \leq i_1 < i_2 < \cdots < i_k \leq m$.

For convenience in some formulae we adopt the convention $S_k = 0$ if $k > m$. By turing to indicator variables we immediately find that

$$S_k = E\left(\binom{X_m(A)}{k}\right), \quad (0 \leq k \leq m).$$

Inequalities of the form $\sum_{k=0}^{m} c_k S_k \leq P(X_m(A) \geq r) \leq \sum_{k=0}^{m} d_k S_k$, where $c_k = c_k(r, m)$ and $d_k = d_k(r, m)$ are constant, possible zero, are called Bonferroni-type inequalities.

In this univariate case, we have proved that

$$P(X_m(A) \leq 1) \leq S_1 - \sum_{i < j \leq i+2} P(A_i \cap A_j) + \sum_{i=1}^{m-2} P(A_i \cap A_{i+1} \cap A_{i+2}).$$

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Taking the averages of the above upper bounds over \(i = 1, 2, \ldots, m\) of (1), we get the following Bonferroni-type inequality (B-T-I).

\[
P(X_m(A) \geq 1) \leq S_1 - \frac{(2m - 3)}{(m)}S_2 + \frac{(m - 2)}{(m - 3)}S_3
\]

This inequality is known that it is the best possible upper bound in terms of \(S_1, S_2\) and \(S_3\) [see Kwerel(1975)].

Now we can extend the univariate B-T-I into the bivariate one. Let \(A_1, A_2, \ldots, A_m\) and \(B_1, B_2, \ldots, B_n\) be two sequences of events on the same probability space. Let \(X = X_m(A)\) and \(Y = Y_n(B)\), respectively, denote the numbers of those \(A_i\) and \(B_j\) which occur. Put \(S_{0,0} = 1\) and, for integers \(r\) and \(t\), set

\[
S_{r,t} = \sum \sum P(A_{i_1}A_{i_2}\ldots A_{i_r}B_{j_1}B_{j_2}\ldots B_{j_t})
\]

where the summation is over all subscripts satisfying \(1 \leq i_1 < i_2 < \cdots < i_r \leq m\) and \(1 \leq j_1 < j_2 < \cdots < j_t \leq n\), \(0 \leq r \leq m\) and \(0 \leq t \leq n\) (We abbreviate \(A \cap B\) as \(AB\) and an empty intersection is the sample space). We can easily prove (2) by using the method of indicators. The number \(S_{r,t} , 1 \leq r , 1 \leq t\), is called the binomial moment of the vector \((X,Y)\) because

\[
S_{r,t} = E \left[ \binom{X}{r} \binom{Y}{t} \right].
\]

Although the univariate case has been studied by many authors, in the bivariate case little information is know. In fact, Eva, Galambos [1965] and Meyer [1969] introduced the classical bivariate inequalities and Lee [1992], Galambos and Lee [1992] presented its extensions by using the method of indicators and combinations, and examples of optimal inequalities are shown by Galambos and Xu [1993].

We are interested in bivariate B-T-I which mean bounds by linear combinations of the binomial moments \(S_{r,t}\). In particular, we want to establish upper bound of \(y_{1,1} = P(X \geq 1, Y \geq 1)\) which appears in many problems in statatics (See for example Galambos and Lee [1994]). Galambos and Xu have proved that

\[
P(X \geq 1, Y \geq 1) = y_{1,1} \leq S_{1,1} - \frac{2}{m}S_{2,1} - \frac{2}{n}S_{1,2} - \frac{4}{mn}S_{2,2}
\]
which insists the best upper bound among all upper bounds of the form
\( d_1 S_{1,1} + d_2 S_{2,1} + d_3 S_{1,2} + d_4 S_{2,2} \).

When we compare (4) with the classical lower bound

\[
S_{1,1} - S_{1,2} - S_{2,1} \leq P\left( (\bigcup_{i=1}^m A_i) \cap (\bigcup_{j=1}^n B_j) \right) = P(X \geq 1, Y \geq 1) = y_{1,1}
\]

We see that if we subtract from \( S_{1,1} \) all intersections of triples (i.e., \( S_{1,2} + S_{2,1} \)) we get a lower bound but if only a percentage of these triples are subtracted we get an upper bound (note that \( S_{2,2} \) is negative in (4)). This suggests that if, instead of \( S_{1,2} \) and \( S_{2,1} \), we subtract a restricted number of terms of the form \( P(A_i B_j B_k) \) and \( P(A_i A_j B_k) \) we shall have an upper bound. The question is that what kind of sums of \( P(A_i B_j B_k) \) and \( P(A_i A_j B_k) \) will guarantee to have a universal upper bound when these sums are subtracted from \( S_{1,1} \). An answer to this question is contained in Theorem 1 of the next section.

2. The Inequalities

We establish some new inequalities in which our ideas are to prove upper bounds using some of terms instead of all terms \( S_{1,2} , S_{2,1} \) and \( S_{2,2} \). Also, we prove optimal upper bounds on \( y_{1,1} \) by taking average of new ones.

**Theorem 1.** Let \( i \) and \( j \) be integers with \( 1 \leq i \leq m, 1 \leq j \leq n \). Then

\[
y_{1,1} \leq S_{1,1} - \max \left\{ \sum_{i=1}^{m-1} \sum_{j=1}^{n} P(A_i A_{i+1} B_j) + \sum_{j=1}^{n-1} P(A_k B_j B_{j+1}),
\sum_{i=1}^{m} \sum_{j=1}^{n-1} P(A_i B_j B_{j+1}) + \sum_{i=1}^{m-1} P(A_i A_{i+1} B_k) \right\}
\]

where \( A_k \) and \( B_k \) are arbitrary fixed events.

**Theorem 2.** For positive integers \( m, n \),

\[
y_{1,1} \leq \min \left( S_{1,1} - \frac{2}{m} S_{2,1} - \frac{2}{mn} S_{1,2} , S_{1,1} - \frac{2}{n} S_{1,2} - \frac{2}{mn} S_{2,1} \right)
\]
This inequality is a new one and better than the inequality in Lee [1992] in terms of the binomial moments $S_{1,1}$, $S_{1,2}$ and $S_{2,1}$; that is, for integers $m$ and $n$, $2 \leq m$, $2 \leq n$,

$$y_{1,1} \leq \min \left( S_{1,1} - \frac{2}{m} S_{2,1}, S_{1,1} - \frac{2}{n} S_{1,2} \right).$$

3. Proofs

Proof of Theorem 1. We use the method of indicators. Let

$$I(X \geq 1, Y \geq 1) = \begin{cases} 1 & \text{if } X \geq 1 \text{ and } Y \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

and by using binomial moment of (3) and indicators, the right hand side of (6) becomes

$$E \left[ \binom{X}{1} \binom{Y}{1} \right] - \max \left\{ \sum_{i=1}^{m-1} \sum_{j=1}^{n} I(A_i)I(A_{i+1})I(B_j)I(B_{j+1}) + \sum_{j=1}^{n-1} I(A_k)I(B_j)I(B_{j+1}), \right.$$  

$$\sum_{i=1}^{m} \sum_{j=1}^{m-1} I(A_i)I(B_j)I(B_{j+1}) + \sum_{j=1}^{m-1} I(A_i)I(A_{i+1})I(B_k) \right\}. \tag{8}$$

Then $E[I(X \geq 1, Y \geq 1)] = P(X \geq 1, Y \geq 1)$, and thus in order to prove (8), it suffices to show that

$$I(X \geq 1)I(Y \geq 1) \leq XY - \max \left\{ \sum_{i=1}^{m-1} \sum_{j=1}^{n} I(A_i)I(A_{i+1})I(B_j)I(B_{j+1}) + \sum_{j=1}^{n-1} I(A_k)I(B_j)I(B_{j+1}), \right.$$  

$$\sum_{i=1}^{m} \sum_{j=1}^{m-1} I(A_i)I(B_j)I(B_{j+1}) + \sum_{j=1}^{m-1} I(A_i)I(A_{i+1})I(B_k) \right\}. \tag{9}$$

Note that both sides of (9) are zero if either $X$ or $Y$ equals zero, hence, in proving (9) we may assume that $X \geq 1$ and $Y \geq 1$, in which the left hand side of (9) is identically one. Thus, we have to prove that

$$U(X, Y) = \text{ the right hand side of (9)} \geq 1 \text{ for } 1 \leq X \leq m, 1 \leq Y \leq n. \tag{10}$$
We distinguish three cases.

(i) First case. There are integers \( X \) and \( Y \) with \( X = 1, Y = 1 \); that is, there are only two events \( A_i \) and \( B_j \) occur because the events \( A_i \)'s and \( B_j \)'s occur numerical order. Then this case is evident, having one on both sides of (10).

(ii) Second case. For integers \( p, q \) with \( 2 \leq p \leq m, \ 2 \leq q \leq n \), \( X = p \) and \( Y = q \) or \( X = p \) and \( Y = 1 \); that is, there are the events that exactly one \( A_i \) and at least two more \( B_j \)'s and \( B_0 \) occur. Then

\[
 u(1, q) = 1 \cdot q - \max\{0, (q - 1) + 0\} = 1 \quad \text{and} \quad u(p, 1) = p \cdot 1 - \max\{(p - 1) + 0, 0\} = 1. 
\]

Hence, we get (10).

(iii) Third case. For integers \( p, q \) with \( 2 \leq p \leq m, \ 2 \leq q \leq n \), \( X = p \) and \( Y = q \); that is, there are the events that at least two more \( A_i \) and \( B_j \)'s occur. Then

\[
 u(p, q) = p \cdot q - \max\{(p - 1)q + (q - 1), p(q - 1) + (p - 1)\} = 1
\]

Hence, We get (10). This completes the proof.

**Proof of Theorem 2.** We turn to indicators. Hence, to prove (7) it suffices to show that

\[
(11) \quad 1 \leq XY - \frac{X(X - 1)Y}{m} - \frac{XY(Y - 1)}{mn} \quad \text{if} \quad 1 \leq X \leq m, \ 1 \leq Y \leq n \quad \text{and}
\]

\[
(12) \quad 1 \leq XY - \frac{XY(Y - 1)}{n} - \frac{X(X - 1)Y}{mn} \quad \text{if} \quad 1 \leq X \leq m, \ 1 \leq Y \leq n
\]

Let \( f(X, Y) = \) the right hand side of

\[
(11) = \frac{XY(mn - n(X - 1) - (Y - 1))}{mn}. 
\]

Then the function \( f(X, Y) \) is parabola whose minimum point, for integers \( X, Y \) with \( 1 \leq X \leq m, \ 1 \leq Y \leq n \), are at

\[
C = \left\{ (X, Y) = (1, 1), (1, n), (m, 1), (m, n), ([\frac{m+1}{2}], 1), ([\frac{m+1}{2}]+1, 1) \right\}
\]
where the ordered pair \( \left( \frac{m+1}{2}, 1 \right) \) come from \( \frac{\partial f(X, Y)}{\partial X} = 0 \) and \( \left[ \frac{m+1}{2} \right] \) means the integer part of \( \frac{m+1}{2} \). We know that \( f((X, Y) \in C) \geq 1 \) if \( 1 \leq X \leq m, \ 1 \leq Y \leq n \) which completes the proof of (11).

Let \( g(X, Y) \) = the right hand side of 
\[
12 = \frac{XY (mn - m(Y - 1) - (X - 1))}{D}.
\]
By the same way, \( g((X, Y) \in D) \geq 1 \) if \( 1 \leq X \leq m, \ \frac{mn}{2} \leq Y \leq n \) where the set
\[
D = \left\{ (X, Y) = (1, 1), (1, n), (m, 1), (m, n), (1, \left[ \frac{n+1}{2} \right]), (1, \left[ \frac{n+1}{2} \right] + 1) \right\}
\]
which the ordered pair \( \left( 1, \frac{n+1}{2} \right) \) come from \( \frac{\partial g(X, Y)}{\partial Y} = 0 \) and \( \left[ \frac{n+1}{2} \right] \) means the integer part of \( \frac{n+1}{2} \). This completes the proof of (12).

References


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