A NEW EQUILIBRIUM EXISTENCE VIA CONNECTEDNESS

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ABSTRACT. The purpose of this note is to prove a new existence of maximal element by using the connectness property, and next an existence theorem of equilibrium for 1-person game is established as an application.

In 1950, Nash [5] first proved the existence of equilibrium for games where the player’s preferences are representable by continuous quasi-concave utilities and the strategy sets are simplexes. Next Debreu [3] proved the existence of equilibrium for abstract economies. Recently, the existence of Nash equilibrium can be further generalized in more general settings by several authors, e.g. Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8]. In the above results, the convexity assumption is very essential and the main proving tools are the continuous selection technique and the existence of maximal elements. Still there have been a number of generalizations and applications of equilibrium existence theorem in generalized games.

In this note, we first give a new maximal element existence theorem using the connectedness and next we shall prove a new equilibrium existence theorem for non-compact non-convex 1-person game. We also give an example that the previous results due to Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8], Tian [7] do not work; however our result can be applicable.

We first recall the following notations and definitions. Let $A$ be a non-empty set. We shall denote by $2^A$ the family of all subsets of $A$. Let $X, Y$ be non-empty topological spaces and $T : X \to 2^Y$ be a correspondence. Then $T$ is said to be open or have open graph (respectively, closed or closed

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graph)} if the graph of $T$ (Gr\(T = \{(x, y) \in X \times Y \mid y \in T(x)\}\)) is open (respectively, closed) in $X \times Y$. We may call $T(x)$ the upper section of $T$, and $T^{-1}(y)(= \{x \in X \mid y \in T(x)\})$ the lower section of $T$. It is easy to check that if $T$ has open graph, then the upper and lower sections of $T$ are open; however the converse is not true in general. A multimap $T : X \to 2^Y$ is said to be closed at $x$ if for each net $(x_\alpha) \to x$, $y_\alpha \in T(x_\alpha)$ and $(y_\alpha) \to y$, then $y \in T(x)$. And $T$ is closed on $X$ if it is closed at every point of $X$. Note that if $T$ is single-valued, then the closedness is equivalent to continuity as a function. A correspondence $T : X \to 2^Y$ is said to be upper semicontinuous if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, then there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \subset V$ for each $y \in U$. It is easy to see that when $X$ and $Y$ are regular topological spaces and $T$ is upper semicontinuous and each $T(x)$ is non-empty closed, then $T$ has closed graph; so $T$ is closed (for the proof, see Proposition 11.9 of Border [1]).

Let $T : X \to 2^Y$ be a correspondence; then $x \in X$ is called a maximal element for $T$ if $T(x) = \emptyset$. Indeed, in real applications, the maximal element may be interpreted as the set of those objects in $X$ that are the “best” or “largest” choices.

Let $I$ be a (possibly uncountable) set of agents. For each $i \in I$, let $X_i$ be a non-empty set of actions. A generalized game (or an abstract economy) $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples $(X_i, A_i, P_i)$ where $X_i$ is a non-empty topological space (a choice set), $A_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a constraint correspondence and $P_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An equilibrium for $\Gamma$ is a point $\hat{x} \in X = \Pi_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call $\Gamma$ an $N$-person game.

We begin with the following:

**Lemma.** Let $X$ be a non-empty connected subset of a Hausdorff topological space $E$ and $T : X \to 2^X$ be closed at every $x$, where $T(x) \neq \emptyset$, such that

1. $T^{-1}(y_o)$ is non-empty open in $X$ for some $y_o \in X$,
2. $x \notin T(x)$ for each $x \in X$.

Then $T$ has a maximal element $\hat{x} \in X$, i.e., $T(\hat{x}) = \emptyset$.

**Proof.** Suppose the assertion were false. Then $T(x)$ is non-empty for each $x \in X$ and so $T$ is closed at every $x \in X$. Since $T$ is closed, the lower
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Since \( T^{-1}(y_o) \) is closed. In fact, for every net \((x_\alpha)_{\alpha \in \Gamma} \subset T^{-1}(y_o)\) with \((x_\alpha) \to x\), we have \( y_o \in T(x_\alpha) \) for each \( \alpha \in \Gamma \) and \((x_\alpha) \to x\), so by the closedness of \( T \) at \( x \), \( y_o \in T(x) \). Hence \( x \in T^{-1}(y_o) \), so \( T^{-1}(y_o) \) is closed. By the assumption (1), \( T^{-1}(y_o) \) is also non-empty open. Therefore, by the connectedness of \( X \), \( T^{-1}(y_o) = X \). Hence we have \( y_o \in T(x) \) for each \( x \in X \) and hence \( y_o \in T(y_o) \), which contradicts the assumption (2). Therefore \( T \) has a maximal element \( \hat{x} \in X \), i.e. \( T(\hat{x}) = \emptyset \). This completes the proof.

It should be noted that in the above Lemma, we do not need the compact convex assumption on \( X \) and also do not need the closed convex assumption on \( T \). We shall need the non-empty open lower section at some special point.

The following simple example is suitable for our Lemma:

**Example 1.** Let \( X = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, 0 \leq y \leq \frac{1}{2}\} \) be a connected set in \( \mathbb{R}^2 \) and a correspondence \( T : X \to 2^X \) be defined as follows:

\[
T(x, y) := \begin{cases} 
\text{line segment from (0,0) to } \frac{1}{2} (x, y), & \text{if } (x, y) \neq (0, 0), \\
\emptyset, & \text{if } (x, y) = (0, 0).
\end{cases}
\]

Then it is easy to show that the correspondence \( T \) is closed at every \( (x, y) \neq (0, 0) \) and \( (x, y) \notin T(x, y) \) for each \( (x, y) \in X \). And note that \( T^{-1}(0, 0) = X \setminus (0, 0) \) is open in \( X \). Therefore, by Lemma, \( T \) has a maximal element \((0, 0)\) in \( X \).

Using Lemma, we shall prove a basic new equilibrium existence theorem for a connected 1-person game.

**Theorem.** Let \( \Gamma = (X, A, P) \) be an 1-person game such that

1. \( X \) is a non-empty connected subset of a regular topological space,
2. the correspondence \( A : X \to 2^X \) is upper semicontinuous such that for each \( x \in X \), \( A(x) \) is non-empty closed in \( X \),
3. the correspondence \( P : X \to 2^X \) is upper semicontinuous such that \( P(x) \) is closed in \( X \) for each \( x \in X \), and \( P(x) \) is non-empty for each \( x \notin \mathcal{F} := \{x \in X : x \in A(x)\} \),
4. for some \( y_o \in X \), \( A^{-1}(y_o) \) and \( A^{-1}(y_o) \cap P^{-1}(y_o) \) are non-empty open in \( X \).
Since $X$ is non-empty open by Proposition 11.9 of Border [1], $x$ is upper semicontinuous at every $x$ continuous at every $x$

Then, by the assumption (5), we have a contradiction. Therefore, we have $O$.

We now define a correspondence $\phi : X \rightarrow 2^X$ by

$$\phi(x) = \begin{cases} P(x), & \text{if } x \notin F, \\ A(x) \cap P(x), & \text{if } x \in F. \end{cases}$$

Then, by the assumption (5), we have $x \notin \phi(x)$ for each $x \in X$. We shall show that $\phi$ is upper semicontinuous. Let $V$ be any open subset of $X$ containing $\phi(x)$. Then we let

$$U := \{x \in X : \phi(x) \subset V\}$$

$$= \{x \in F : \phi(x) \subset V\} \cup \{x \in X \setminus F : \phi(x) \subset V\}$$

$$= \{x \in F : (A \cap P)(x) \subset V\} \cup \{x \in X \setminus F : P(x) \subset V\}$$

$$= \{x \in X : (A \cap P)(x) \subset V\} \cup \{x \in X \setminus F : P(x) \subset V\}.$$ 

Since $X \setminus F$ is open, $P$ is upper semicontinuous and $A \cap P$ is upper semicontinuous at every $x$ with $(A \cap P)(x) \neq \emptyset$, $U$ is open and hence $\phi$ is also upper semicontinuous at every $x$ with $\phi(x) \neq \emptyset$. Since each $\phi(x)$ is closed, by Proposition 11.9 of Border [1], $\phi$ is closed at every $x \in X$ with $\phi(x) \neq \emptyset$.

Next we shall show that $\phi^{-1}(y_o)$ is an open subset of $X$. In fact, by the assumption (4), we have that

$$\phi^{-1}(y_o) = \{x \in X : y_o \in \phi(x)\}$$

$$= \{x \in F : y_o \in \phi(x)\} \cup \{x \in X \setminus F : y \in \phi(x)\}$$

$$= \{\phi \cap (A \cap P)^{-1}(y_o)\} \cup \{(X \setminus F) \cap P^{-1}(y_o)\}$$

$$= P^{-1}(y_o) \cap (A^{-1}(y_o) \cap (X \setminus F) \cap P^{-1}(y_o))$$

is non-empty open in $X$. Therefore, by Lemma, there exists a point $\hat{x} \in X$ such that $\phi(\hat{x}) = \emptyset$. If $\hat{x} \notin F$, then $\phi(\hat{x}) = P(\hat{x}) = \emptyset$, which is a contradiction. Therefore, we have $\hat{x} \in F$ and $\phi(\hat{x}) = A(\hat{x}) \cap P(\hat{x}) = \emptyset$, i.e., $\hat{x} \in A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. This completes the proof.
REMbek. Our Theorem is quite different from the previous many equilibrium existence theorems (e.g. Shafer-Sonnenschein [6], Berglin-Keiding [2], Yannelis-Prabhakar [8], Kim [4]). In these results, the compactness and convexity assumptions are very essential. But we do not need any compact convex assumption on the choice set $X$, but we only need the connectedness assumption. Also we do not need the convexity assumptions on the values $A(x)$ and $P(x)$ and strong open lower section assumptions; but we need the weaker open lower section property at some special point.

Next we give an example of a connected 1-person game where our Theorem can be applicable but the previous known results can not be applicable:

EXAMPLE 2. Let $X = \{(x, y) \in R^2 \mid 0 \leq x, 0 \leq y \leq \frac{1}{4}\}$ be a connected choice set and the correspondences $A, P : X \rightarrow 2^X$ be defined as follows:

$$A(x, y) := \{(s, t) \mid s = y, 0 \leq t \leq \frac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\},$$

for each $(x, y) \in X$,

$$P(x, y) := \begin{cases} \emptyset, & \text{for each } (x, x) \in X \text{ with } 0 \leq x \leq 1, \\ \{(s, t) \mid s = y, 0 \leq t \leq \frac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\}, & \text{otherwise.} \end{cases}$$

Here, we shall use $1/0$ as the infinity for simplicity of the formula. Then it is easy to show that the correspondence $A$ is upper semicontinuous and each $A(x, y)$ is non-empty closed and the fixed point set $\mathcal{F}$ of $A$ is exactly the diagonals of $X$, i.e., $\mathcal{F} = \{(x, x) \mid 0 \leq x \leq 1\}$. Also we have that $P$ is upper semicontinuous on $X \setminus \mathcal{F}$ and $P(x, y)$ is non-empty closed at every point except on the diagonals. And note that $A^{-1}(0, 0) = X$ is open and $P^{-1}(0, 0) = X \setminus \mathcal{F}$ is also open. Therefore all assumptions of Theorem are satisfied, so that we can obtain an equilibrium point $(0, 0) \in X$ such that $(0, 0) \in A(0, 0) \text{ and } A(0, 0) \cap P(0, 0) = \emptyset$.

Finally, it should be noted that by modifying the methods in Berglin-Keiding [2] or Kim [4], we can show that the case of $N$-agents can be reduced to the 1-person game.
References


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