ON THE ESTIMATE OF THE FIRST EIGENVALUE OF THE
LAPLACIAN ON A COMPACT RIEMANNIAN MANIFOLD

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§0. Introduction

Throughout this paper, let \( M \) be an \( n \)-dimensional compact Riemannian manifold with or without boundary \( \partial M \) and be assumed that \( K \) and \( H \) are non-negative constants and \( d \) denotes the diameter of \( M \).

We shall consider the solution of the equation

\[
\Delta u = -\lambda u
\]

defined on \( M \). In case \( M \) is a manifold with boundary \( \partial M \), we impose the following boundary condition:

\[
\frac{\partial u}{\partial n}\big|_{\partial M} = 0,
\]

where \( n \) is the unit normal vector to \( \partial M \).

Our purpose is largely to show that results of a lower bound of the first non-zero eigenvalue obtained for the Laplacian on a compact Riemannian manifold by P.Li, S.T.Yau, and R.Chen can be improved by a simple but sharper inequality. More precisely, this article has triple purposes: Firstly, we generalize the first non-zero eigenvalue estimate of (0.1) obtained in P.Li-S.T.Yau [4,5] with \( \text{Ricc}(M) \geq 0 \) to the case \( \text{Ricc}(M) \geq -(n - 1)K \) ([Theorem 0.1]). Secondly, we improve the first non-zero eigenvalue estimate of (0.1) given by P.Li-S.T.Yau [4,5] via the inequality \((a - b)^2 \geq \xi a^2 \frac{\xi}{(1 - \xi)} b^2 (0 < \xi < 1)\) without using a less sharp one \((a - b)^2 \geq \frac{1}{2} a^2 - b^2 \) ([Theorem 0.2]). And finally, we introduce (probably improve) a lower

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bound of the non-zero Neumann eigenvalue of (0.1) with (0.2) [2] which is a generalization of [4, Theorem 9] to a compact Riemannian manifold with possibly non-convex ∂M ([Theorem 0.3]).

Using a method similar to those of P.Li–S.T.Yau and R.Chen, we have the followings:

THEOREM 0.1. Let $M$ be a compact Riemannian manifold without boundary. Let $\lambda_1$ be the first non-zero eigenvalue of (0.1). Suppose $\text{Ricc}(M) \geq -(n-1)K$. Then

$$\lambda_1 \geq \left[ \frac{2\xi(n)}{(n-1)d^2} B^2 - (n-1)K \right] \exp(-B),$$

where

$$\xi(n) = (n-1)\{\sqrt{(n-1)^2 + 2} - (n-1)\}$$

and

$$B = 1 + \sqrt{1 + \frac{2(n-1)^2}{\xi(n)} d^2 K}$$

REMARK. Note that $\frac{1}{2} < \xi(n) < 1$. Hence this eigenvalue estimate is better than

$$\frac{1}{(n-1)d^2} \exp[-(1 + \sqrt{1 + 4(n-1)^2 d^2 K})]$$

obtained by Li-Yau [5, p.129].

THEOREM 0.2. Let $M$ be a compact Riemannian manifold with $\partial M$. Let $\eta_1$ be the first nonzero eigenvalue of (0.1) and (0.2). Let $\partial M$ satisfy the “interior rolling $\epsilon$-ball” condition. Suppose $\text{Ricc}(M) \geq -(n-1)K$ and the second fundamental form elements of $\partial M \geq -H$. By choosing $\epsilon$ small, we have

$$\eta_1 \geq \frac{1}{(1 + H)^2} \left[ \frac{(1 - \alpha^2)\xi(n, \alpha)}{2(n-1)d^2} B_1^2 - C \right] \exp(-B_1)$$

where $\alpha$ and $\epsilon < 1$,

$$B_1 = 1 + \left( 1 + \frac{2(n-1)d^2 C}{(1 - \alpha^2)\xi(n, \alpha)} \right)^{1/2}.$$
On the estimate of the first eigenvalue of the Laplacian

\[ C = (1 + H)C_1 + (1 + H)^2(n - 1)K + \frac{H^2(2n - 2 - 2\xi(n, \alpha))^2 + (8n - 8 - 4\xi(n, \alpha))\xi(n, \alpha)\alpha^2}{\xi(n, \alpha)\alpha^2\varepsilon^2(n - 1)}, \]

\[ C_1 = \frac{2(n - 1)H(3H + 1)(H + 1)}{\varepsilon} + \frac{H + H^2}{\varepsilon^2}, \]

and

\[ \xi(n, \alpha) = \frac{(n - 1)}{1 - \alpha^2} \left\{ \sqrt{(n - 1)^2 + 2(1 - \alpha^2)} - (n - 1) \right\}. \]

**Remark.** When the boundary is convex, our estimate implies the eigenvalue estimate obtained in [Theorem 0.2] (See Remark after the [Lemma 2.2] below). In contrast to the result of R. Chen [2], \( \xi(n, \alpha) \) has been inserted \((\frac{1}{2} < \xi(n, \alpha) < 1)\). But we still don’t know whether or not this result is sharper than that of R. Chen [2].

**Definition 0.3.** Let \( \partial M \) be the boundary of a compact Riemannian manifold \( M \). Then \( \partial M \) satisfies the interior rolling \( \varepsilon \)-ball condition if for each point \( p \in \partial M \), there exists a geodesic ball \( B_q(\varepsilon/2) \) s.t.

\[ p = B_q(\varepsilon/2) \cap \partial M \]

and

\[ B_q(\varepsilon/2) \subset M. \]

In §1., we shall give gradient estimates which are essential in proofs of the main results. In §2., we shall give brief proofs of main theorems.

### §1. Gradient Estimates

We have the followings:

**Lemma 1.1.** Let \( M \) be as above with \( \text{Ricc}(M) \geq -(n - 1)K \) and let \( u \) be a non-constant first eigenfunction of (0.1) with \( \lambda_1 \) s.t.

\[ 1 = \sup u > \inf u = -k \geq -1 \quad (0 < k \leq 1). \]

Then we get

\[ |\nabla u|^2 \leq [\lambda_1 \frac{2}{1 + k} + (n - 1)K](1 - u)(k + u). \]
Proof. This result follows from the proof of P.Li-S.T.Yau [5, p.121] with handling the condition $\text{Ricc}(M) \geq -(n-1)K$ carefully.

**Lemma 1.2.** If $u$ is a non-constant first eigenfunction of (0.1) with $\sup u = 1$ and $\lambda_1$, then

$$|\nabla u| \leq \left[ \frac{2(n-1)}{\xi(n)} \left( \frac{\beta}{\beta - 1} \lambda_1 + (n-1)K \right) \right]^{1/2} (\beta - u),$$

where $\beta > 1$.

**Remark.** If $M$ is a compact Riemannian manifold with $\partial M$ being convex, then we have the same gradient estimate with the same $G(x)$ defined amid the proof of [Lemma 1.2] under the Newmann condition by the maximum principle. Thus in that case the same first eigenvalue estimate as in [Theorem 0.2] can be obtained.

Proof. The proof is a slight modification of the proof given by P.Li-S.T.Yau [5]. As in [5], consider the function defined by

$$G(x) = \frac{|\nabla u|^2}{(\beta - u)^2}$$

By applying the maximum principle and the Bochner identity in line with the proof in [5], it is easy to get

$$\sum u_{ij}^2 - (\lambda_1 + (n-1)K)|\nabla u|^2 - G[\lambda_1 u(\beta - u) + |\nabla u|^2] \leq 0$$

at $x_0$, at which $G$ attains its maximum.

If we choose an orthonormal frame at $x_0$ s.t. $u_1 = |\nabla u|$ and $u_i = 0$ for all $i \neq 1$, then, from $\nabla G(x_0) = 0$,

$$u_{11} = -\frac{|\nabla u|^2}{(\beta - u)}$$

and

$$u_{ii} = 0, \; i \neq 1$$
Also, at $x_0$,

\[(1.4)\]

\[
\sum u_{ij}^2 \geq \sum u_{ii}^2 = u_{11}^2 + \sum_{i \neq 1} u_{ii}^2
\]

\[
\geq u_{11}^2 + \frac{(\sum_{i \neq 1} u_{ii})^2}{(n-1)}
\]

\[
= u_{11}^2 + \frac{(\Delta u - u_{11})^2}{(n-1)}
\]

\[
\geq u_{11}^2 + \frac{\xi}{(n-1)} u_{11}^2 - \frac{\xi}{(n-1)(1-\xi)} (\Delta u)^2
\]

\[
= \frac{(n-1+\xi)}{(n-1)} u_{11}^2 - \frac{\xi}{(n-1)(1-\xi)} (\Delta u)^2
\]

for $0 < \xi < 1$. Substituting (1.2),(1.3) and (1.4) into (1.1), we get

\[
\frac{(n-1+\xi)}{(n-1)} |\nabla u|^4 - \frac{\xi \lambda_1^2 u^2}{(n-1)(1-\xi)} - (\lambda_1 + (n-1)K)|\nabla u|^2
\]

\[
- \lambda_1 u |\nabla u|^2 - \frac{\xi}{(n-1)(1-\xi)} (\Delta u)^2 \leq 0
\]

Since \(\frac{u}{(\beta-u)} \leq \frac{1}{(\beta-1)}\), simple computations show

\[
G \leq \max \left\{ \frac{2(n-1)}{\xi} [\lambda_1 + (n-1)K + \lambda_1 \frac{1}{(\beta-1)}], \left( \frac{2}{1-\xi} \right)^{1/2} \lambda_1 \frac{1}{(\beta-1)} \right\},
\]

so that

\[
|\nabla u|^2 \leq \frac{2(n-1)}{\xi(n)} \left[ \frac{\beta}{(\beta-1)} [\lambda_1 + (n-1)K]](\beta-u)^2, \right.
\]

for $\xi(n)$ defined previously.

**Lemma 1.3.** Let $M$ be the same as [Theorem 0.3]. Let $u$ be a non-costant solution of (0.1) and (0.2) with $\sup u = 1$ and $\eta_1$. If $\beta > 1$ and $\epsilon$ is “small”, then

\[
|\nabla u|^2 \leq \frac{2(n-1)}{(1-\alpha^2)\xi(n, \alpha)} \left( C + (1 + H)^2 \eta_1 \frac{\beta}{(\beta-1)} \right)(\beta-u)^2,
\]

where

\[ C, \xi(n, \alpha) = the\ same\ as\ [Theorem\ 0.3]. \]
Proof. Let $\psi(r), r, \phi, G(x)$ be the same completely as in the proof of [2]. The same kind of reasoning used in [2] gives

\[
(1.5) \quad 0 \geq -\frac{C_1}{1 + \phi} - \frac{2(\psi')^2}{\epsilon^2(1 + \phi)^2} - \frac{2\psi'u_1r_1}{\epsilon(1 + \phi)(\beta - u)} + \sum_{i \neq 1} \frac{u_{ii}^2}{u_1^2} + R_{i1} - \frac{\eta_1 \beta}{\beta - u},
\]

at the interior point $x_0$, where $G$ attains its maximum and we choose an orthonormal frame $\{e_i\}$ ($1 \leq i < n$) s.t. $u_1(x_0) = |\nabla u|(x_0)$. Then, at $x_0$, $\nabla G(x_0) = 0$ gives

\[
(1.6) \quad u_{1j} = -\frac{\psi'u_1r_j}{\epsilon(1 + \phi)} - \frac{u_1u_j}{\beta - u}
\]

It is obvious with (1.6) that

\[
(1.7) \quad \sum_{i \neq 1} u_{ii}^2 \geq \frac{1}{n - 1} \left( \sum_{i > 1} u_{ii} \right)^2
\]

\[
= \frac{1}{n - 1} (\Delta u - u_{11})^2
\]

\[
\geq \frac{1}{n - 1} \left( \frac{\xi(u_{11})^2}{\xi} - \frac{\xi(\Delta u)^2}{\xi} \right) \quad (0 < \xi < 1)
\]

\[
= \frac{\xi u_1^4}{(n - 1)(\beta - u)} + \frac{2\xi u_1^3 \psi'r_1}{\epsilon(n - 1)(1 + \phi)(\beta - u)} + \frac{\xi(\psi')^2 u_{11}^2 r_1^2}{\epsilon^2(n - 1)(1 + \phi)^2}
\]

\[
= \frac{\xi}{(1 - \xi)(n - 1)}
\]

Combining (1.5) with (1.7),

\[
(1.8) \quad 0 \geq -\frac{\xi u_1^2}{(n - 1)(\beta - u)} - \frac{2(n - 1 - \xi)\psi'u_1r_1}{\epsilon(n - 1)(1 + \phi)(\beta - u)}
\]

\[
- \frac{C_1}{1 + \phi} - \frac{2(n - 1)(\psi')^2}{\epsilon^2(n - 1)(1 + \phi)^2} - \frac{\eta_1 \beta}{\beta - u} - \frac{\eta_1^2 u_1^2}{(n - 1)u_1^2 - \xi}.\]
It is clear that
\begin{equation}
\frac{1}{n-1} \left( \frac{\alpha^2 u_1^2}{(\beta - u)^2} - \frac{2(2n - 2 - 2\xi)u_1 \psi r_1}{\xi \epsilon (1 + \phi)(\beta - u)} \right) \geq - \frac{(2n - 2 - 2\xi)^2 (\psi')^2}{\xi^2 \epsilon^2 \alpha^2 (n - 1)(1 + \phi)^2}.
\end{equation}
Hence, with (1.9), (1.8) gives
\[
0 \geq \left[ 1 - \frac{\alpha^2}{\xi} \right] \left( \frac{u_1}{n-1} \right)^2 - \frac{[(2n - 2 - 2\xi)^2 - 4\alpha^2 \xi^2] (\psi')^2}{4\xi \alpha^2 \epsilon^2 (n - 1)(1 + \phi)^2} - \frac{2(\psi')^2}{\epsilon^2 (1 + \phi)^2} - \frac{C_1}{1 + \phi} - (n - 1)K - \frac{\eta_1 \beta}{\beta - u} - \frac{\eta_1^2 \xi u^2}{(n - 1)(1 - \xi)u_1^2}.
\]
Multiplying through by \((1 + \phi)^4 \frac{u_1^2}{(\beta - u)^2}\), it is easy to show
\[
\frac{(1 - \alpha^2)\xi}{n-1} G^2 \leq \left[ \frac{(2n - 2 - 2\xi)^2 + (8n \xi - 8 \xi^2 - 4 \xi^2 \alpha^2)}{4\xi \alpha^2 \epsilon^2 (n - 1)} (\psi')^2 + (1 + \phi)C_1 + (1 + \phi)^2 (n - 1)K + \frac{\eta_1 \beta}{\beta - u} (1 + \phi)^2 \right] G + \frac{\eta_1^2 \xi}{(n - 1)(1 - \xi)} (1 + \phi)^4 \frac{u^2}{(\beta - u)^2}
\]
i.e.
\[
G \leq \max \left\{ \frac{2(n - 1)}{(1 - \alpha^2)\xi} \left[ C + (1 + H)^2 \frac{\beta \eta_1}{\beta - 1} \right], \frac{\eta_1 (1 + H)^2}{\sqrt{1 - \alpha^2}} \left( \frac{2}{1 - \xi} \right)^{1/2} \frac{1}{\beta - 1} \right\}
\]
Hence, if \(\xi(n, \alpha)\) is the same as above, then our result comes out.

§3. Proofs of Main Theorems 0.1-3

A similar argument to that of [2,5] with the newly-made gradient estimates in [Lemma 1.1-3] gives the conclusions immediately.
References


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