MINIMUM PERMANENT ON THE POLYTOPES DETERMINED BY A VECTOR MAJORIZATION*

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1. Introduction

Let \( \Omega_n \) denote the set of all \( n \times n \) doubly stochastic matrices. Then it is well known that \( \Omega_n \) forms a convex polytope of dimension \((n - 1)^2 \) with \( n! \) extreme points in the \( n^2 \)-dimensional Euclidean space.

For an \( n \times n \) (0,1)-matrix \( U = [u_{ij}] \), let

\[
F(U) = \{ X = [x_{ij}] \in \Omega_n | x_{ij} = 0 \text{ whenever } u_{ij} = 0 \}.
\]

Then \( F(U) \) is a face of the polytope \( \Omega_n \).

One of the most interesting problems concerning the polytope \( F(U) \) is that of determining the minimum value of the permanent function and the set of all minimizing matrices on it.

For integers \( k, n \) with \( 1 \leq k \leq n \), let \( V_{k,n} \) denote the set of all \( n \times 1 \) (0,1)-matrices whose entries have sum \( k \), and let \( \mathbb{R}^n \) denote the set of all real \( n \times 1 \) matrices. For \( x, y \in \mathbb{R}^n \), \( y \) is said to be majorized by \( x \), written as \( y \prec x \), if

\[
\max \{ v^T y | v \in V_{k,n} \} \leq \max \{ v^T x | v \in V_{k,n} \}
\]

for all \( k = 1, \ldots, n \) and the equality holds in (1.1) when \( k = n \).

It is well known that \( y \prec x \) for \( x, y \in \mathbb{R}^n \) if and only if there exists \( S \in \Omega_n \) with \( y = Sx \). Let

\[
\Omega_n(y \prec x) = \{ S \in \Omega_n | y = Sx \}.
\]

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Then $\Omega_n(y < x)$ forms a convex subpolytope of $\Omega_n$. We call it the *polytope of the majorization* $y < x$.

In [1], R.A. Brualdi determined the dimensions of majorization polytopes and the support matrix of the majorization. But as remarked by Marshall and Olkin [5], very little is known about this polytope.

Let $\mathcal{F}(U)$ be the face determined by the support matrix $U$ of the majorization. Then $\mathcal{F}(U)$ is the largest face of $\Omega_n$ whose interior has a nonempty intersection with the majorization polytope $\Omega_n(y < x)$.

The purpose of this paper is to determine the minimum permanent and the set of all minimizing matrices on the polytope $\mathcal{F}(U)$.

### 2. The support matrix of the majorization

Throughout this paper, let $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$ be in $\mathbb{R}^n$ with $y < x$, and without loss of generality, we may assume that $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$.

For integers $k$, $n$ with $1 \leq k \leq n$, we say that $y < x$ has a coincidence at $k$ if $y_1 + \cdots + y_k = x_1 + \cdots + x_k$, and is $k$-decomposable if it has a coincidence at $k$ ($< n$) and $x_k > x_{k+1}$.

Suppose that $y < x$ has a coincidence at $k$ ($< n$), but is not $k$-decomposable. Then there exist integers $j_i, k_i, k'_i, l_i$ ($i = 1, \ldots, p$), such that

\[
\begin{align*}
& (i) \quad \text{the only coincidences of } y < x \text{ occur at } k_i, k_i + 1, \ldots, k'_i, \\
& (ii) \quad k'_{i-1} < l_{i-1} < j_i \leq k_i \leq k'_i, \\
& (iii) \quad x_{j_i-1} = \cdots = x_{k_i} = \cdots = x_{k'_i-1} = \cdots = x_{l_i} > x_{l_i+1}.
\end{align*}
\]

Note that $k'_p = l_p = n$, and we shall also use $k'_0 = l_0 = 0$ and $j_0 = 1$.

For $A = [a_{ij}] \in \Omega_n$, we define the support matrix of $A$ to be the $n \times n$ $(0,1)$-matrix $S_A = [s_{ij}]$ by

\[
s_{ij} = \begin{cases} 
1 & \text{if } a_{ij} > 0 \\
0 & \text{otherwise}. 
\end{cases}
\]

Let $x$, $y \in \mathbb{R}^n$ with $y < x$. We define the *support matrix of the majorization* $y < x$ to be the $n \times n$ $(0,1)$-matrix $U = [u_{ij}]$ having the following two properties:

(i) $A \in \Omega_n(y < x)$ implies $A \leq U$

(ii) there is a matrix $B = [b_{ij}] \in \Omega_n(y < x)$ such that $S_B = U$. 


THEOREM 2.1. (R.A. Brualdi [1]) Let \( x, y \in \mathbb{R}^n \) and suppose \( y < x \) is not \( k \)-decomposable for each \( k = 1, \ldots, n - 1 \). Then the support matrix of \( y < x \) is the \( n \times n \) \((0, 1)\)-matrix \( U = [u_{rs}] \) such that for \( 1 \leq r, s \leq n \), \( u_{rs} = 1 \) if and only if for some \( i = 1, \ldots, p \),

\[
k'_{i-1} + 1 \leq r \leq k_i, \ j_{i-1} \leq s \leq l_i \text{ or } k_i + 1 \leq r \leq k'_i, \ j_i \leq s \leq l_i.
\]

In the following we represent the support matrix of the majorization \( y < x \) in a different manner.

With \( j_i, k_i, k'_i \) and \( l_i \), \( (i = 1, \ldots, p) \), defined by (2.1) we define frames \( \sigma \) and \( \sigma' \) of \( U \) by

\[
\sigma = (k_1, \ldots, k_p, n : j_1 - 1, \ldots, j_{p-1} - 1, n) \quad \text{and} \quad \sigma' = (k'_1, \ldots, k'_p : l_1, \ldots, l_p).
\]

In particular, if \( k_p = k'_p \) then we denote \( \sigma \) by

\[
\sigma = (k_1, \ldots, k_p : j_1 - 1, \ldots, j_{p-1} - 1, n).
\]

Throughout this paper, \( K_{p \times q} \) will denote the \( p \times q \) matrix all of whose entries are 1 and \( K \) the matrix of 1’s of suitable sizes.

For \( \sigma \) and \( \sigma' \) defined by (2.2), let \( U_{\sigma \sigma'} = [U_{ij}] \) be \( n \times n \) \((0,1)\)-matrix with

\[
U_{ij} = \begin{cases} 
K_{g(t) \times h(t)} & \text{if } 2t - 1 \leq i \leq j \leq 2t \text{ and } i = 2t - 1, j = 2t - 2, (t = 1, \ldots, p) \\
0 & \text{otherwise}
\end{cases}
\]

where, for some \( t = 1, \ldots, p \),

\[
g(t) = \begin{cases} 
k_i - k'_{t-1} & \text{if } i = 2t - 1 \\
k'_i - k_t & \text{if } i = 2t
\end{cases} \quad \text{and} \quad h(t) = \begin{cases} 
j_i - l_{t-1} - 1 & \text{if } j = 2t - 1 \\
l_t - j_i + 1 & \text{if } j = 2t
\end{cases}
\]

Note that there is no \( U_{2t} \) for \( t \) with \( k'_i = k_i \) and \( U_{(2t-1)} \) for \( t \) with \( j_i = l_{t-1} + 1 \). Thus those submatrices do not appear in the matrix \( U_{\sigma \sigma'} \).
Then the (0,1)-matrix $U_{0}$ is the support matrix of $y < x$, and $U_{\sigma \sigma'}$ has the following block representation

$$U_{\sigma \sigma'} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ \end{bmatrix}$$

Figure 1

**Example 2.1.**

Let

$$y = (8, 7, 6, 5, 5, 4, 2, 2)^T \quad \text{and} \quad x = (9, 7, 6, 6, 5, 4, 3, 3, 1)^T.$$ 

Then $y < x$ but is not $k$-decomposable. It has 4 coincidences, namely at $k_1 = 3, \quad 4 = k', \quad k_2 = 8 = k_2', \quad k_3 = 10 = k_3'$. Moreover, $j_1 = 3, \quad l_1 = 5, \quad j_2 = 8, \quad l_2 = 9, \quad j_3 = l_3 = 10$.

From (2.3) and (2.2), we get the frames

$$\sigma = (3, 8, 10 : 2, 7, 10), \quad \sigma' = (4, 8, 10 : 5, 9, 10)$$

and the support matrix $U_{\sigma \sigma'}$ of $y < x$ is

$$U_{\sigma \sigma'} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ \end{bmatrix}$$
3. The barycenter of the polytope $\mathcal{F}(U_{\sigma'})$ and its permanent

An $n \times n$ matrix is called *partly decomposable* if it contains a $k \times (n-k)$ zero submatrix for some $k, 1 \leq k \leq n-1$, and an $n$-square matrix which is not partly decomposable is called *fully indecomposable*. A matrix $A \in \mathcal{F}(U)$ is called a *minimizing matrix* on $\mathcal{F}(U)$ if

$$\text{per } A = \min \{ \text{per } X | X \in \mathcal{F}(U) \}.$$ 

For a matrix $A$, $A(i_1, \ldots, i_s|j_1, \ldots, j_t)$ will denote the $(n-s) \times (n-t)$ matrix obtained from $A$ by deleting the rows $i_1, \ldots, i_s$ and the columns $j_1, \ldots, j_t$, and $A[i_1, \ldots, i_s|j_1, \ldots, j_t]$ the $s \times t$ submatrix of $A$ complementary to $A(i_1, \ldots, i_s|j_1, \ldots, j_t)$. In particular, if we delete the rows $i_1, \ldots, i_i$ only, the resulting matrix is denoted by $A(i_1, \ldots, i_s|\cdot)$. Similarly, the matrices $A(\cdot|j_1, \ldots, j_t)$, $A[i_1, \ldots, i_s|\cdot]$ and $A(\cdot|j_1, \ldots, j_t)$ are defined.

For $x, y \in \mathbb{R}^n$, suppose that $y < x$ has a coincidence at $k (< n)$ but is not $k$-decomposable. With frames $\sigma = (k_1, \ldots, k_p, n : j_1 - 1, \ldots, j_p - 1, n)$ and $\sigma' = (k'_1, \ldots, k'_p : l_1, \ldots, l_p)$ defined by (2.2), we define some quantities $\alpha_i$’s, $\beta_i$’s, $\gamma_i$’s and $\delta_i$’s, $i = 1, \ldots, p$, by

$$\alpha_i = k_i - j_i + 1, \quad \beta_i = l_i - k'_i, \quad \gamma_i = l_i - j_i + 1, \quad \text{and} \quad \delta_i = k_i - k'_{i-1},$$

and let $U_{\sigma', \sigma}$ be the support matrix of the majorization $y < x$. Then by Theorem 2.5 of [2], we get

$$\dim \mathcal{F}(U_{\sigma', \sigma}) = (n-1)^2 - \sum_{i=1}^{p} (j_i - j_{i-1})(n-k_i) - \sum_{i=1}^{p-1} (k'_i - k'_{i-1})(n-l_i)$$

where

$$U_\sigma = \begin{bmatrix} \vdots \\ \end{bmatrix} \quad \text{and} \quad U_{\sigma'} = \begin{bmatrix} \vdots \\ \end{bmatrix}.$$ 

Note that S.G. Hwang [4] determined the set of all minimizing matrices and its minimum permanent on $\mathcal{F}(U_\sigma)$ or $\mathcal{F}(U_{\sigma'})$. 
It readily follows by induction that the number of vertices of $\mathcal{F}(U_{\sigma'})$, per $U_{\sigma'}$, is

\begin{equation}
\prod_{i=1}^{p} \frac{\gamma_i! \delta_i!}{\alpha_i! \beta_i!},
\end{equation}

where $\alpha_i$, $\beta_i$, $\gamma_i$ and $\delta_i$ are defined by (3.1).

Now, we consider the barycenter $B(U_{\sigma'})$ of $\mathcal{F}(U_{\sigma'})$ given by

\begin{equation}
B(U_{\sigma'}) = \frac{1}{\text{per } U_{\sigma'}} \sum_{P \leq U_{\sigma'}} P,
\end{equation}

where the summation extends over the set of all permutation matrices $P$ with $P \leq U_{\sigma'}$.

Finally, we define an $n \times n$ matrix $M_{\sigma'}$ with the same block representation as $U_{\sigma'}$ in (2.4) by

\begin{equation}
M_{\sigma'} = 
\begin{bmatrix}
M_{11} & \cdots & M_{1p2} \\
\vdots & & \vdots \\
M_{2p1} & \cdots & M_{2p2p}
\end{bmatrix}
\end{equation}

where, for each $t = 1, \cdots, p$,

\begin{equation}
M_{ij} = \begin{cases}
  r_t K & \text{if } i = j = 2t - 1, \\
  s_t K & \text{if } i = 2t - 1, j = 2t, \\
  t_t K & \text{if } i = j = 2t, \\
  q_t K & \text{if } i = 2t - 1, j = 2t - 2, \\
  0 & \text{otherwise}
\end{cases}
\end{equation}

where

\begin{equation}
r_t = \frac{1}{\delta_t}, \quad s_t = \frac{\alpha_t}{\gamma_t \delta_t}, \quad t_t = \frac{1}{\gamma_t} \quad \text{and} \quad q_t = \frac{\beta_{t-1}}{\gamma_{t-1} \delta_t}.\end{equation}

**Theorem 3.1.** Let $U_{\sigma'}$ be the support matrix of $y < x$ with $\sigma$ and $\sigma'$ given by (2.2). Then $B(U_{\sigma'}) = M_{\sigma'}$. 

Proof. Let $B(U_{\sigma'}) = [b_{rs}]_{n \times n}$ be the barycenter of $\mathcal{F}(U_{\sigma'})$. Then since

$$b_{rs} = \frac{\text{per } U_{\sigma'}(r|s)}{\text{per } U_{\sigma'}}, \quad (r, s = 1, \ldots, n)$$

$B(U_{\sigma'}) = [B_{ij}]_{n \times n}$ has the same zero-one pattern as $U_{\sigma'}$ and the $(i, j)$-block $B_{ij}$ has the same entries. Thus we have

$$b_{rs} = \begin{cases} 
\text{per } U_{\sigma'}(k_i|j_i - 1) & \text{if } b_{rs} \in B_{2t-1}2t-1, \\
\text{per } U_{\sigma'}(k_i|l_i) & \text{if } b_{rs} \in B_{2t-1}2t, \\
\text{per } U_{\sigma'}(k_i'|l_i) & \text{if } b_{rs} \in B_{2t+1}2t, \\
\text{per } U_{\sigma'}(k_i|l_i-1) & \text{if } b_{rs} \in B_{2t-1}2t-2
\end{cases}$$

(3.7)

for each $t = 1, \ldots, p$.

If we notice that $k'_{t-1} < l_{t-1} < j_t \leq k_t < k'_t < l_t$ for each $t = 1, \ldots, p$, then from (3.1), (3.2) and (3.7) it follows that $B(U_{\sigma'}) = M_{\sigma'}$.

For $\alpha_i, \beta_i, \gamma_i$ and $\delta_i$ ($i = 1, \ldots, p$) given by (3.1), we define some ordered pair $(\pi_t, \theta_t)$ by

$$\pi_t = \begin{cases} 
(\delta_t - \alpha_t - \beta_{t-1}, \delta_t - \beta_{t-1}) & \text{if } t = 4i - 3, \\
(\alpha_t, \gamma_t) & \text{if } t = 4i - 2, \\
(\gamma_{i-1} - \alpha_i, \gamma_{i-1} - \alpha_i) & \text{if } t = 4i - 1, \\
(\beta_{i-1} - 1, \delta_i) & \text{if } t = 4i - 4.
\end{cases}$$

(3.8)

Notice that, for each $t = 1, \ldots, 2p - 1$,

$$\begin{align*}
\theta_{2t-1} - \pi_{2t-1} &= \pi_{2t} \\
\theta_{2t} - \pi_{2t} &= \theta_{2t+1}.
\end{align*}$$

(3.9)

**Theorem 3.2.** Let $\alpha_i, \beta_i, \gamma_i$ and $\delta_i$ be as defined in (3.1). Then

$$\text{per } M_{\sigma'} = \prod_{i=1}^{p} \frac{\gamma_i! \delta_i! \alpha_i^{\alpha_i} \beta_i^{\beta_i}}{\alpha_i! \beta_i! \gamma_i! \delta_i!}.$$
Proof. First, we let
\[ \phi_{\sigma \sigma'} = \prod_{i=1}^{p} \Delta(i), \quad \Delta(i) = \frac{\gamma_i! \delta_i! \alpha_i^\alpha \beta_i^\beta}{\alpha_i! \beta_i! \gamma_i^\gamma \delta_i^\delta}, \]
and for an \( n \times n \) matrix \( A \), let
\[ \text{per } A(1, \cdots, \hat{\pi}_i|1, \cdots, \hat{\pi}_i) = P_{\hat{\pi}_i}(A) \]
where \( \hat{\pi}_i = \sum_{j=1}^{i} \pi_j \).

Then we get the following recursion formula for \( \text{per } M_{\sigma \sigma'} \) using the Laplace expansion for permanent:
\[ P_{\hat{\pi}_{i-1}}(M_{\sigma \sigma'}) = \frac{\theta_i!}{(\theta_i - \pi_i)!} x_{f(i)}^{\pi_i} P_{\hat{\pi}_i}(M_{\sigma \sigma'}), \quad (i = 1, \cdots, 4p - 1) \quad (3.11) \]
where, for each \( i = 1, \cdots, p \),
\[ x_{f(i)} = \begin{cases} r_i & \text{if } t = 4i - 3, \\ s_i & \text{if } t = 4i - 2, \\ t_i & \text{if } t = 4i - 1, \\ q_i & \text{if } t = 4i - 4 \end{cases} \quad (3.12) \]
where \( r_i, s_i, t_i \) and \( q_i \) are defined by (3.6). Note that
\[ P_{\hat{\pi}_0}(M_{\sigma \sigma'}) = \text{per } M_{\sigma \sigma'} \quad \text{and} \quad P_{\hat{\pi}_{4p-1}}(M_{\sigma \sigma'}) = 1. \]

Thus from (3.11) we have
\[ \text{per } M_{\sigma \sigma'} = \prod_{i=1}^{4p-1} \frac{\theta_i!}{(\theta_i - \pi_i)!} \prod_{i=1}^{4p-1} x_{f(i)}^{\pi_i} \quad (3.13) \]

On the other hand, from (3.8) and (3.9), it is easily seen that
\[ \prod_{i=1}^{4p-1} \frac{\theta_i!}{(\theta_i - \pi_i)!} = \prod_{i=1}^{p} \frac{\gamma_i! \delta_i!}{\alpha_i! \beta_i!} \quad (3.14) \]
and also from (3.8), (3.9) and (3.12), we get

\[
\prod_{t=1}^{4p-1} x_{f(t)}^{\pi_t} = \prod_{i=1}^{p} \alpha_i^{\alpha_i} \beta_i^{\beta_i} \gamma_i^{\gamma_i} \delta_i^{\delta_i}.
\]

(We used the convention \(0^0 = 1\)).

Therefore from (3.13), (3.14) and (3.15), we have

\[
\text{per } M_{\sigma\sigma'} = \phi_{\sigma\sigma'}
\]

which completes the proof.

For the Example 2.1, we get \(\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 1; \beta_1 = 1, \beta_2 = 1, \beta_3 = 0; \gamma_1 = 3, \gamma_2 = 2, \gamma_3 = 1; \delta_1 = 3, \delta_2 = 4, \delta_3 = 2 \) and \(r_1 = \frac{1}{3}, r_2 = \frac{1}{4}; s_1 = \frac{1}{9}, s_2 = \frac{1}{8}, s_3 = \frac{1}{2}; t_1 = \frac{1}{3}; q_2 = \frac{1}{112}, q_3 = \frac{1}{4}\). (Note that \(q_1, t_2, t_3 \) and \(r_3 \) do not appear.)

Thus we have

\[
M_{\sigma\sigma'} = \begin{bmatrix}
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\
0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\
0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\
0 & 0 & 1/12 & 1/12 & 1/12 & 1/4 & 1/4 & 1/8 & 1/8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/2
\end{bmatrix}
\]

and

\[
\text{per } M_{\sigma\sigma'} = \frac{3!3!2!4!2!}{3^3 3^2 2^2 4^2 2^2} = \frac{1}{864}.
\]

### 4. The minimum permanent and the minimizing matrices

The following Lemma is due to Foregger [3].
LEMMA 4.1. Let $U = [u_{ij}]$ be an $n \times n$ fully indecomposable $(0, 1)$-matrix, and let $A = [a_{ij}]$ be a minimizing matrix on the face $F(U)$. Then $A$ is fully indecomposable, and more, for $(i, j)$ such that $u_{ij} = 1$

$$\per A(i|j) \begin{cases} = \per A & \text{if } a_{ij} > 0 \\ \geq \per A & \text{if } a_{ij} = 0. \end{cases}$$

And we need the following Lemma due to Minc [6].

LEMMA 4.2. Let $A = [a_{ij}]$ be a minimizing matrix on $F(U)$ where $U = [u_1, \ldots, u_n]$ is an $n \times n$ $(0, 1)$-matrix. If, for some $k \leq n$, $d_1 = \cdots = d_k$, then for any $p \leq k$, $A(J_p \oplus I_{n-p}) \in F(U)$ and $\per A(J_p \oplus I_{n-p}) = \per A$. From now on in the sequel let $J_p$ denote the $p$-square matrix all of whose entries are $1/p$ and $I$ the square identity matrix, i.e., the matrix obtained from $A$ by replacing each of its first $p$ columns by their average remains a minimizing matrix on $F(U)$. A similar statement holds for rows.

Let $U_{\sigma \sigma'}$ be the support matrix of the majorization $y < x$ with $\sigma$ and $\sigma'$ given by (2.2), and let $M_{\sigma \sigma'}$ be the matrix defined by (3.4). Since $U_{\sigma \sigma'}$ has a block representation, by Lemma 4.2, it follows that $M_{\sigma \sigma'}$ is a minimizing matrix on $F(U_{\sigma \sigma'})$. Hence, by Theorem 3.2, we get the following.

THEOREM 4.1. Let $U_{\sigma \sigma'}$ be the support matrix of the majorization $y < x$ with $\sigma$ and $\sigma'$ given by (2.2). Then

$$\min_{X \in F(U_{\sigma \sigma'})} \per X = \phi_{\sigma \sigma'}.$$  

With $j_i, k_i, k'_i$ and $l_i$ defined in (2.1), let

$$C_{\sigma \sigma'} = \{t|k_t = j_t, \quad t = 1, \ldots, p\}$$

and

$$\widehat{C}_{\sigma \sigma'} = \{s|k'_s = l_s - 1, \quad s = 1, \ldots, p-1\}.$$  

Suppose that $C_{\sigma \sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma \sigma'} \neq \emptyset$. Let $C_{\sigma \sigma'} = \{t_1, \ldots, t_k\}$ and $\widehat{C}_{\sigma \sigma'} = \{s_1, \ldots, s_l\}$, and let $m_1, \ldots, m_{k+l}$ be positive integers obtained from $2t_1 - 1, \ldots, 2t_k - 1, 2s_1, \ldots, 2s_l$ by rearranging in increasing order. Then we get a sequence

$$m_1, \ldots, m_{k+l}.$$
Let \( \mathcal{I} \) be the set of all indices of the sequence with nonconsecutive index in (4.4) and let \( t_o \) and \( t_e \) be an odd and an even number in \( \mathcal{I} \) respectively. Notice that

\[
\left\{
\begin{array}{ll}
t \in C_{\sigma'} & \text{if } t_o = 2t - 1, \ t = 1, \cdots, p \\
s \in \widehat{C}_{\sigma'} & \text{if } t_e = 2s, \ s = 1, \cdots, p - 1.
\end{array}
\right.
\]

Finally, we define a matrix \( M^*_{\sigma\sigma'} = [M^*_{ij}] \) on \( \mathcal{F}(U_{\sigma\sigma'}) \) obtained from \( M_{\sigma\sigma'} = [M_{ij}] \) by

\[
M^*_{ij} = \begin{cases} 
S_i & \text{if } i = t_o, \ j = t_o + 1 \text{ for } t_o \in \mathcal{I} \\
S_j & \text{if } i = t_e + 1, \ j = t_e \text{ for } t_e \in \mathcal{I} \\
M_{ij} & \text{otherwise.}
\end{cases}
\]  

(4.5)

where the entries of \( S_i \) and \( S_j \) can be chosen freely as long as \( M^*_{\sigma\sigma'} \) remains doubly stochastic.

Let \( \tau(S) \) be the sum of all the entries in the matrix \( S \). Then \( \tau(S_{2t-1}) = k_i - j_i + 1 \) for \( t = 1, \cdots, p \) and \( \tau(S_{2s}) = l_s - k_s' \) for \( s = 1, \cdots, p - 1 \). Thus we have the following.

**Lemma 4.3.** Let \( C_{\sigma'} \) and \( \widehat{C}_{\sigma'} \) be defined in (4.2) and (4.3). Then \( \tau(S_{2t-1}) = 1 \) for \( t \in C_{\sigma'} \), and \( \tau(S_{2s}) = 1 \) for \( s \in \widehat{C}_{\sigma'} \).

**Lemma 4.4.** For each \( t \) and \( s \) such that \( 2t - 1 = t_o \) and \( 2s = t_e \) for \( t_o, \ t_e \in \mathcal{I} \),

\[
P_{\widehat{h}(k)}(M^*_{\sigma\sigma'}) = \left\{ \prod_{i=h(k)+1}^{\widehat{h}(k)+3} \Delta(i) \right\} P_{\widehat{h}(k)+3}(M_{\sigma\sigma'})
\]

where

\[
(x_k, y_k) = \begin{cases} 
(q_t, r_t) & \text{if } h(t) = 4t - 5 \text{ for } k = t, \ t = 2, \cdots, p \\
(s_s, t_s) & \text{if } h(s) = 4s - 3 \text{ for } k = s, \ s = 1, \cdots, p - 1.
\end{cases}
\]

**Proof.** First, let \( h(t) = 4t - 5 \) for \( k = t, \ t = 2, \cdots, p \). For a matrix \( M^*_{\sigma\sigma'} = [M^*_{ij}] \) with \( t \in C_{\sigma'} \) we get

\[
M^*_{ij} = \begin{cases} 
S_i & \text{if } i = 2t - 1, \ j = 2t \\
M_{ij} & \text{otherwise.}
\end{cases}
\]
Thus if we notice that \( \pi_{4t-2} = 1 \) and \( s_t = \frac{1}{\alpha_{u-2} \alpha_{u-4}} \), then, by (3.8), (3.9), (3.11) and Lemma 4.3, we get

\[
P_{\hat{\pi}_{h(t)}}(M_{\sigma \sigma'}) = (\theta_{h(t)+1} - 1)! q_{t}^{\hat{\pi}_{h(t)+2}} \tau(S_{2l-1}) P_{\hat{\pi}_{h(t)+3}}(M_{\sigma \sigma'})
\]

\[
= \left\{ \prod_{i=h(t)+1}^{\hat{\pi}_{h(t)+3}} \Delta(i) \right\} P_{\hat{\pi}_{h(t)+3}}(M_{\sigma \sigma'}).
\]

Similarly, the case \( k = s \) can also be easily proved. Thus the proof is complete.

**Theorem 4.2.** If \( C_{\sigma \sigma'} = \emptyset \) and \( \hat{C}_{\sigma \sigma'} = \emptyset \), then \( M_{\sigma \sigma'} \) is the unique minimizing matrix on \( F(U_{\sigma \sigma'}) \).

**Proof.** First, we prove the theorem for \( k_p \neq k_p' \). Note that if \( n \leq 3 \) then \( C_{\sigma \sigma'} \neq \emptyset \) or \( \hat{C}_{\sigma \sigma'} \neq \emptyset \). Hence we may assume that \( n \geq 4 \). Let \( A \) be a minimizing matrix on \( F(U_{\sigma \sigma'}) \). Since \( U_{\sigma \sigma'} \) has a block representation (2.4), we have \( A = M_{\sigma \sigma'} \) by averaging method of Lemma 4.2.

Now, we suppose that \( C_{\sigma \sigma'} = \emptyset \) and \( \hat{C}_{\sigma \sigma'} = \emptyset \), and let \( B \) be an another minimizing matrix on \( F(U_{\sigma \sigma'}) \), and we will show that \( B = M_{\sigma \sigma'} \). Let

\[
\tilde{B} = (I_1 \oplus J_{k_1-1} \oplus J_{k_1'} \oplus \cdots \oplus I_1 \oplus J_{k_p-1} \oplus I_1 \oplus J_{k_p'}) B
\]

\[
(I_1 \oplus J_{j_1-2} \oplus I_1 \oplus J_{l_1-j_1} \oplus \cdots \oplus I_1 \oplus J_{j_p-l_p} \oplus I_1 \oplus J_{l_p})
\]

Then \( \tilde{B} \) is also a minimizing matrix on \( F(U_{\sigma \sigma'}) \) by Lemma 4.2 applied to the rows \( 2, \ldots, k_1, k_1 + 2, \ldots, k_1', \ldots, k_p - 1 + 2, \ldots, k_p, k_p + 2, \ldots, k_p' \) and columns \( 2, \ldots, j_1 - 1, j_1 + 1, \ldots, l_1, \ldots, l_p - 1 + 2, \ldots, j_p - 1, j_p + 1, \ldots, l_p \).

Note that \( I_1 \oplus J_{k_1'-k_1-1} \) for \( t \) such that \( k_t = k_1' \) and \( I_1 \oplus J_{j_1-l_1-2} \) for \( s \) such that \( j_s = l_{s-1} + 1 \) do not appear.

Thus, we may assume, without loss of generality, that \( B = \tilde{B} \). Applying Lemma 4.1 to \( \tilde{B} \), then it is easily seen that \( \tilde{B} = M_{\sigma \sigma'} \).

It remains the case \( k_p = k_p' \), which can be handled similarly. Hence the proof is completed.

**Theorem 4.3.** If \( C_{\sigma \sigma'} \neq \emptyset \) or \( \hat{C}_{\sigma \sigma'} \neq \emptyset \), then \( M_{\sigma \sigma'}^{*} \) is a minimizing matrix on \( F(U_{\sigma \sigma'}) \).
Proof. Since the submatrices $S_i$ and $S_j$ in $M_{\sigma\sigma'}^{*}$ for all $i, j \in \mathcal{I}$ do not have common rows and columns, it suffices to show that $M_{\sigma\sigma'}^{*}$ determined by only one submatrix $S_i$ is a minimizing matrix on $\mathcal{F}(U_{\sigma\sigma'})$.

Let $i$ be an odd number in $\mathcal{I}$. Then for $t \in C_{\sigma\sigma'}$ with $2t - 1 = i$, $t = 1, \ldots, p$, we have

$$M_{\sigma\sigma'}^{*}[1, \ldots, (\pi_{4t-4}, \ldots, \pi_{4t-2})^\wedge, \ldots, n| \cdot]$$

$$= M_{\sigma\sigma'}[1, \ldots, (\pi_{4t-4}, \ldots, \pi_{4t-2})^\wedge, \ldots, n| \cdot]$$

and

$$M_{\sigma\sigma'}^{*}[\cdot | 1, \ldots, (\pi_{4t-2}, \ldots, \pi_{4t})^\wedge, \ldots, n]$$

$$= M_{\sigma\sigma'}[\cdot | 1, \ldots, (\pi_{4t-2}, \ldots, \pi_{4t})^\wedge, \ldots, n]$$

where $\wedge$ stands for the deletion under it.

Thus, by Lemma 4.4 and (3.11), we get

$$\text{per } M_{\sigma\sigma'}^{*} = \begin{Bmatrix} \prod_{i=1}^{4t-5} \Delta(i) \\ \prod_{i=1}^{4t-2} \Delta(i) \\ \prod_{i=1}^{p} \Delta(i) \end{Bmatrix} P_{\pi_{4t-5}}(M_{\sigma\sigma'})$$

$$= \begin{Bmatrix} \prod_{i=1}^{4t-2} \Delta(i) \\ \prod_{i=1}^{4t-2} \Delta(i) \\ \prod_{i=1}^{4t-2} \Delta(i) \end{Bmatrix} P_{\pi_{4t-2}}(M_{\sigma\sigma'})$$

$$= \begin{Bmatrix} \prod_{i=1}^{p} \Delta(i) \end{Bmatrix}$$

$$= \phi_{\sigma\sigma'}.$$

Also, we can easily seen with similar argument for even number $i$ in $\mathcal{I}$. Hence the proof is complete.

Theorem 4.3 tells us that, if $C_{\sigma\sigma'} \neq \emptyset$ or $\widehat{C}_{\sigma\sigma'} \neq \emptyset$, then there are infinitely many minimizing matrices on $\mathcal{F}(U_{\sigma\sigma'})$.

Example 4.1. Let

$$y = (14, 13, 12, 11, 10, 8, 6, 6, 3, 3)^T$$

$$x = (16, 12, 12, 12, 12, 9, 7, 7, 7, 3, 1)^T.$$
Then \( y < x \) and we get the frames \( \sigma = (3, 9, 13 : 1, 7, 13) \) and \( \sigma' = (4, 9, 13 : 6, 11, 13) \). Note that \( C_{\sigma \sigma'} = \emptyset \) and \( \widehat{C}_{\sigma \sigma'} = \emptyset \). Thus

\[
\begin{bmatrix}
1/3 & 2/15 & 2/15 & 2/15 & 2/15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 2/15 & 2/15 & 2/15 & 2/15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 2/15 & 2/15 & 2/15 & 2/15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/5 & 1/5 & 1/5 & 1/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/10 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2/25 & 2/25 & 2/25 & 2/25 & 2/5 & 1/10 & 1/10 & 1/10 & 1/10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

is the unique minimizing matrix on \( \mathcal{F}(U_{\sigma \sigma'}) \). But, for

\[
y = (12, 12, 11, 10, 9, 8, 7, 6, 5, 5, 3, 3)^T \\
x = (13, 12, 10, 10, 10, 8, 6, 6, 6, 5, 2, 2)^T
\]

with \( y < x \), we get \( \sigma = (3, 7, 13 : 2, 6, 13) \) and \( \sigma' = (4, 8, 13 : 5, 10, 13) \). Thus \( C_{\sigma \sigma'} = \{1, 2\} \) and \( \widehat{C}_{\sigma \sigma'} = \{1\} \). Therefore any doubly stochastic matrix of the form

\[
\begin{bmatrix}
1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/9 & 1/9 & 1/9 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/9 & 1/9 & 1/9 & 1/3 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/9 & 1/9 & 1/9 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\
\end{bmatrix}
\]
or

\[
\begin{bmatrix}
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/3 & 1/3 & 1/9 & 1/9 & 1/9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 1/12 & 0 & 0 & 0 & 0 \\
0 & 0 & * & 1/3 & 1/12 & 1/12 & 1/12 & 1/12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/10 & 1/10 & 1/10 & 1/5 & 1/5 & 1/5
\end{bmatrix}
\]

is a minimizing matrix on \(\mathcal{F}(U_{\sigma^*})\).

We close our discussion here by giving an answer to the problem of minimizing the permanent on some special class of majorization polytopes.

For \(x = (x_1, \ldots, x_n)^T\) and \(y = (y_1, \ldots, y_n)^T\) with \(y \prec x\), let \(y^{(i)} = (y_{k_{i-1}}, \ldots, y_k)^T, i = 1, \ldots, p\), where \(k_i\) and \(k'_i\) are defined in (2.1). If \(z_1 = \cdots = z_n\) then a vector \(z = (z_1, \cdots, z_n)^T\) is called a scalar vector.

**Theorem 4.5.** Let \(x = (x_1, \cdots, x_n)^T\) and \(y = (y_1, \cdots, y_n)^T\) in \(\mathbb{R}^n\). Suppose that \(y \prec x\) is not \(k\)-decomposable for each \(k = 1, \cdots, n - 1\). Then \(M_{\sigma^*}\) is a minimizing matrix over \(\Omega_n(y \prec x)\) iff \(y^{(i)}\) is the scalar vector for each \(i = 1, \cdots, p\). Moreover, if \(C_{\sigma^*} \neq \emptyset\) or \(\hat{C}_{\sigma^*} \neq \emptyset\) then \(M^*_{\sigma^*}\) holds above conclusion.

**Proof.** Since \(\Omega_n(y \prec x) \subset \mathcal{F}(U_{\sigma^*})\) and \(M_{\sigma^*}\) is a minimizing matrix on \(\mathcal{F}(U_{\sigma^*})\), it is sufficient to show that \(M_{\sigma^*} \in \Omega_n(y \prec x)\). It is readily proved from \(y = M_{\sigma^*}x\).

Also, we can easily seen with similar argument for \(C_{\sigma^*} \neq \emptyset\) or \(\hat{C}_{\sigma^*} \neq \emptyset\).

**Problem A.** Determine the permanental minimizing matrix on the majorization polytope \(\Omega_n(y \prec x)\).

**References**


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