NORMS FOR SCHUR PRODUCTS

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ABSTRACT. We first show that if \( \psi : M_n(B(H)) \to M_n(B(H)) \) is a \( D_n \otimes F(H) \)-bimodule map, then there is a matrix \( A \in M_n \) such that \( \psi = S_A \). Secondly, we show that for an operator space \( \mathcal{E} \), \( A \in M_n \), the Schur product map \( S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E}) \) and \( \phi_A : M_n(\mathcal{E}) \to \mathcal{E} \), defined by \( \phi_A([x_{ij}]) = \sum_{i,j=1}^n a_{ij}x_{ij} \), we have \( \|S_A\| = \|S_A\|_{cb} = \|A\|_S \), \( \|\phi_A\| = \|\phi_A\|_{cb} = \|A\|_1 \) and obtain some characterizations of \( A \) for which \( S_A \) is contractive.

1. Introduction

Schur products on \( M_n \) have been studied in several areas. In particular, Paulsen, Power and Smith [4] proves that for \( A \in M_n \), a Hilbert space \( H \) and the Schur product map \( S_A : M_n \to M_n(B(H)) \), \( \|S_A\| = \|S_A\|_{cb} \) and obtains a characterization of \( A \) for which \( S_A \) is contractive.

In this paper, we first show that if \( \psi : M_n(B(H)) \to M_n(B(H)) \) is a \( D_n \otimes F(H) \)-bimodule map, then there is a matrix \( A \in M_n \) such that \( \psi = S_A \). Secondly, we show that for an operator space \( \mathcal{E} \), \( A \in M_n \), the Schur product map \( S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E}) \) and \( \phi_A : M_n(\mathcal{E}) \to \mathcal{E} \), we have \( \|S_A\| = \|S_A\|_{cb} = \|A\|_S \), \( \|\phi_A\| = \|\phi_A\|_{cb} = \|A\|_1 \), where \( \|A\|_1 \) is the trace of \( |A| = (A^*A)^{1/2} \), and obtain some characterizations of \( A \) for which \( S_A \) is contractive.

2. Main Results

An operator space is a subspace of \( B(H) \) for some Hilbert space and an operator system is a self-adjoint subspace of \( B(H) \) containing the identity.

For an operator space \( \mathcal{E} \subseteq B(H) \) we identify \( M_n \otimes \mathcal{E} \) with \( M_n(\mathcal{E}) \) which is a subspace of \( B(H^n) \) where \( H^n \) is the \( n \)-fold direct sum of copies of \( H \).
If \( A = [a_{ij}] \), \( B = [b_{ij}] \) are elements of \( M_n \) or \( M_n(B(H)) \), then we denote the Schur product by \( A \circ B = [a_{ij}b_{ij}] \). For \( A = [a_{ij}] \in M_n \) and an operator space \( \mathcal{E} \), let \( S^\mathcal{E}_A : M_n(\mathcal{E}) \to M_n(\mathcal{E}) \) be the Schur product map defined by \( S^\mathcal{E}_A(x) = A \circ x \), for \( \phi^\mathcal{E}_A : M_n(\mathcal{E}) \to \mathcal{E} \) be the map defined by \( \phi^\mathcal{E}_A([x_{ij}]) = \sum_{i,j=1}^n a_{ij}x_{ij} \), and let \( \|A\|_S \) denote the norm of the operators on \( M_n \) corresponding to Schur multiplication by \( A \). When there is no danger of confusion we let \( \phi_A \) and \( S_A \) denote \( \phi^\mathcal{E}_A \) and \( S^\mathcal{E}_A \) respectively.

If \( \mathcal{E}, \mathcal{F} \) are operator spaces and \( \varphi : \mathcal{E} \to \mathcal{F} \) is a linear map, then we can define the linear maps

\[
\varphi_n : M_n(\mathcal{E}) \to M_n(\mathcal{F}) \quad \text{via} \quad \varphi_n([x_{ij}]) = [\varphi(x_{ij})].
\]

The map \( \varphi \) is called contractive if \( \|\varphi\| \leq 1 \), completely bounded if \( \|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in N\} \) is finite and completely contractive if \( \|\varphi\|_{cb} \leq 1 \).

In the case that \( \mathcal{E}, \mathcal{F} \) are operator systems, the map \( \varphi \) is called positive if \( \varphi(x) \) is positive for every positive \( x \) in \( \mathcal{E} \), and completely positive if \( \varphi_n \) is positive for every \( n \).

Let \( \{e_{ij}\}_{i,j=1}^n \) be the canonical matrix units for \( M_n \), let \( D(x_1, \cdots, x_n) \) be the diagonal operator matrix in \( M_n(B(H)) \), and let \( A = [a_{ij}], B = [b_{ij}] \), \( C \) in \( M_n(B(H)) \) be operator matrices with mutually commuting entries. Then by elementary calculations, we get the following Lemma.

**Lemma 1.** \((AB) \circ C = \sum_{k=1}^n D(a_{1k}, \cdots, a_{nk})CD(b_{k1}, \cdots, b_{kn})\).

It is a well-known theorem that the Schur product of two positive matrices is positive. Using Lemma 1, we give a new elementary proof of a generalization of the above well-known theorem.

**Proposition 2.** Let \( \mathcal{E} \) be an operator system. If \( A \in M_n \), \( B \in M_n(\mathcal{E}) \) are positive, then \( A \circ B \) is positive.

**Proof.** Let \( A^{\frac{1}{2}} = [a_{ij}], D_k = D(a_{1k}, \cdots, a_{nk}). \) By Lemma 1, \( A \circ B = (A^{\frac{1}{2}}A^{\frac{1}{2}}) \circ B = \sum_{k=1}^n D_kBD_k^* \). Hence \( A \circ B \) is positive. \( \square \)

Let \( A \in M_n \) be a positive matrix and \( \mathcal{E} = C \). Then \( S_A \) and \( \phi_A \) are completely positive. The following shows that it holds for any operator system \( \mathcal{E} \).

**Proposition 3.** Let \( \mathcal{E} \) be an operator system and let \( A = [a_{ij}] \in M_n \) be a matrix. For \( \phi_A : M_n(\mathcal{E}) \to \mathcal{E}, S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E}) \), the following are equivalent:

1. \( A \) is positive,
(2) $\phi_A$ is completely positive,
(3) $\phi_A$ is completely positive,
(4) $\sum_{i,j=1}^n a_{ij} \bar{\alpha}_i \alpha_j \geq 0$ for any $\alpha_1, \cdots, \alpha_n \in C$,
(5) $S_A$ is positive,
(6) $S_A$ is completely positive.

**Proof.** (1) $\Rightarrow$ (6) Let $A_k = [A_{ij}] \in M_k(M_n)$ with $A_{ij} = A$. Then $(S_A)_k = S_{(A_k)}$. Since $A$ is positive, $A_k$ is positive. Hence by Proposition 2 $(S_A)_k$ is positive and $S_A$ is completely positive.

(6) $\Rightarrow$ (3) Since $(\phi_A)_k(x) = V(S_A)_k(x)V^*$ for some $V \in M_{k, kn}$, $\phi_A$ is completely positive.

(2) $\Rightarrow$ (4) Let $x = (\alpha_1 I, \cdots, \alpha_n I) \in M_{1,n}(\mathcal{E})$ with $\alpha_i \in C$. Then $\phi_A(x^*x) = (\sum_{i,j=1}^n \bar{\alpha}_i \alpha_j a_{ij})I$ is positive. Hence $\sum_{i,j=1}^n a_{ij} \bar{\alpha}_i \alpha_j \geq 0$ for any $\alpha_1, \cdots, \alpha_n \in C$.

(3) $\Rightarrow$ (2), (4) $\Rightarrow$ (1), (6) $\Rightarrow$ (5) $\Rightarrow$ (1) Clear. □

Let $\mathcal{E}$ be an operator space. For $A, B, C \in M_n$, $x \in M_n(\mathcal{E})$, let $A^t$ be the transpose of $A$, $L_A(x) = Ax$, $R_A(x) = xA$. Then by elementary calculations, we get the following Lemma.

**Lemma 4.** $\phi_{BAC} = \phi_A L_B R_{C^t}$. In particular, if $U, V \in M_n$ are unitaries, then $\|(\phi_{UAV})_k\| = \|(\phi_A)_k\|$ for each $k \in N$.

For an operator space $\mathcal{E}$ and a positive matrix $A \in M_n$, $S_A$ is completely bounded and $\|S_A\|_{cb} = \max\{a_{ii}\}_{i=1}^n$. When $A$ is not positive, it is more difficult to calculate $\|S_A\|$. But, using Lemma 4, we can easily calculate $\|\phi_A\|_{cb}$. Let $\|A\|_1$ denote the trace norm of the matrix $A$, $i, e, \|A\|_1$ is the trace of $|A| = (A^*A)^{\frac{1}{2}}$.

**Theorem 5.** Let $\mathcal{E}$ be an operator space and let $A \in M_n$ be a matrix. Then we have $\|\phi_A\| = \|\phi_A\|_{cb} = \|A\|_1$.

**Proof.** Note that there is a unitary matrix $U \in M_n$ such that $A = U|A|$. Then by Lemma 4, $\|(\phi_A)_k\| = \|(\phi_{|A|})_k\|$ for each $k \in N$. Clearly $\|(\phi_{|A|})\| \geq \|A\|_1$. Let $\mathcal{E} \subseteq B(H)$. Since $\phi_{|A|}^{B(H)}$ is completely positive, $\|\phi_{|A|}^{B(H)}\| = \|\phi_{|A|}^{B(H)}\|_{cb} = \|\phi_{|A|}^{B(H)}(I)\| = \|A\|_1$. Hence $\|\phi_A^{\mathcal{E}}\| = \|\phi_A^{\mathcal{E}}\|_{cb} = \|A\|_1$. □

Let $A$ be a subalgebra of $B(H)$. A linear map $\psi : B(H) \rightarrow B(H)$ is called a $A$-bimodule map if $\psi(axb) = a\psi(x)b$ for all $a, b \in A$ and $x \in B(H)$. Let
Let $F(H)$ be the set of all finite rank operators on $H$, let $D_n$ be the set of all $n \times n$ diagonal matrices, and let $\{e_{ij}\}_{i,j=1}^n$ be the canonical matrix units for $M_n$.

**THEOREM 6.** If $\psi : M_n(B(H)) \to M_n(B(H))$ is a $D_n \otimes F(H)$-bimodule map, then there is a matrix $A \in M_n$ such that $\psi = S_A$.

**PROOF.** For any projection $p \in F(H)$ and fixed $i, j$

\[
\psi(e_{ij} \otimes p) = \psi(((e_{ii} \otimes p)(e_{ij} \otimes p)(e_{jj} \otimes p)) = (e_{ii} \otimes p)\psi(e_{ij} \otimes p)(e_{jj} \otimes p)
\]

Hence $\psi(e_{ij} \otimes I) = e_{ij} \otimes x_{ij}$ for some $x_{ij} \in B(H)$. Since $(e_{ii} \otimes y)(e_{ij} \otimes I) = (e_{ij} \otimes I)(e_{jj} \otimes y)$ for $y \in F(H)$ and $\psi$ is a $D_n \otimes F(H)$-bimodule map

\[
e_{ij} \otimes y x_{ij} = (e_{ii} \otimes y)(e_{ij} \otimes x_{ij}) = \psi((e_{ii} \otimes y)(e_{ij} \otimes I)) = \psi((e_{ij} \otimes I)(e_{jj} \otimes y)) = (e_{ij} \otimes x_{ij})(e_{jj} \otimes y) = e_{ij} \otimes x_{ij} y
\]

for any $y \in F(H)$. Hence $x_{ij} \in F(H)' = CI$ and we can put $x_{ij} = a_{ij} I$ for some $a_{ij} \in C$. Put $A = [a_{ij}] \in M_n$. Then clearly $\psi = S_A$. □

**REMARK 7.** Let $B(H) = M_2$, $C = M_n \otimes I \subseteq M_n(M_2)$ or $C = D_n \otimes I \subseteq M_n(M_2)$ and let $\psi : M_n(M_2) \to M_n(M_2)$ be defined by $\psi([x_{ij}]) = [x_{ij}']$. Then $\psi$ is a $C$-bimodule map but there is no $A \in M_n$ such that $\psi = S_A$.

**COROLLARY 8.** If $\psi : M_n(B(H)) \to M_n(B(H))$ is a $D_n \otimes F(H)$-bimodule map, then $\psi$ is also a $D_n \otimes B(H)$-bimodule map.

**PROOF.** By Theorem 6, $\psi = S_A$ for some $A \in M_n$. It is trivial that $S_A^{B(H)}$ is a $D_n \otimes B(H)$-bimodule map. □

Let $E \subseteq B(H)$ be an operator space and let

\[
V = \left\{ \begin{bmatrix} P & S \\ T^* & Q \end{bmatrix} : P, Q \in D_n \otimes B(H), S, T \in M_n(E) \right\},
\]

\[
W = \left\{ \begin{bmatrix} P & S \\ T^* & Q \end{bmatrix} : P, Q \in D_n \otimes I, S, T \in M_n(E) \right\}
\]

and let $P_A = \begin{bmatrix} I & A \\ A^* & I \end{bmatrix} \in M_{2n}$ for $A \in M_n$. 

PROPOSITION 9. Let $\mathcal{E}$ be an operator space and let $A = [a_{ij}] \in M_n$ be a matrix. For $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$, the following are equivalent:

1. $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ is contractive,
2. $S_A : M_n \to M_n$ is contractive,
3. There exist vectors $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ in $\mathbb{C}^n$ of the norm less than or equal to 1 with $a_{ij} = (w_j v_i)$,
4. $S_A : M_n(\mathcal{E}) \to M_n(\mathcal{E})$ is completely contractive,
5. $S_A : M_n \to M_n$ is completely contractive,
6. $S_{P_A} : V \to V$ is positive,
7. $S_{P_A} : V \to V$ is completely positive,
8. $S_{P_A} : W \to W$ is completely positive,
9. $S_{P_A} : W \to W$ is positive.

PROOF. (1) $\Rightarrow$ (2), (4) $\Rightarrow$ (5) By [5, Proposition 2.2], $\|B \otimes x\| = \|B\| \|x\|$ for $B \in M_m$, $x \in \mathcal{E}$. Hence they are trivial.

(2) $\Rightarrow$ (3) [4, Theorem 3.2].

(3) $\Rightarrow$ (4) Let $P_1 = [(v_j v_i)]$, $P_2 = [(w_j w_i)] \in M_n$. Then $A = \begin{bmatrix} P_1 & A \\ A^* & P_2 \end{bmatrix} \in M_{2n}$ is positive and by Proposition 3, the map $S_{\alpha} : M_{2n}(\mathcal{E}) \to M_{2n}(\mathcal{E})$ is completely positive. Hence $\|S_{\alpha}\|_{cb} = \|S_{\alpha}(I)\| \leq 1$ and $\|S_{\alpha}\|_{cb} \leq \|S_{\alpha}\|_{cb}$.

(2) $\Rightarrow$ (6), (5) $\Rightarrow$ (7), (9) $\Rightarrow$ (1) Similar to the proof of [4, Lemma 3.1].

(6) $\Rightarrow$ (9), (7) $\Rightarrow$ (8) $\Rightarrow$ (9) Trivial. □

REMARK 10. If $A = [a_{ij}] \in M_n$ with $a_{ij} = 1$, then $S_A : M_n(B(H)) \to M_n(B(H))$ is contractive, so $S_{P_A} : V \to V$ is positive, but $S_{P_A} : M_2(M_n(B(H))) \to M_2(M_n(B(H)))$ is not positive since $P_A$ is not positive.

From Proposition 9, we get the following theorem.

THEOREM 11. If $\mathcal{E} \subseteq B(H)$ is an operator space and $A \in M_n$, then we have $\|S^\mathcal{E}_A\| = \|S^\mathcal{E}_{cb}\| = \|A\|_S$.

PROOF. Clearly $\|A\|_S \leq \|S^\mathcal{E}_A\|$ and $\|S^\mathcal{E}_{cb}\| \leq \|S^{B(H)}_A\|_{cb}$. By Proposition 9, $\|S^{B(H)}_A\| = \|S^{B(H)}_{cb}\|_{cb} = \|A\|_S$. Hence $\|S^\mathcal{E}_A\| = \|S^\mathcal{E}_{cb}\| = \|A\|_S$. □

Let $\{e_{ij}\}_{i,j=1}^n$ be the canonical matrix units for $M_n$, let $c_{ij} = I - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, $d_{i}(\lambda) = I + (\lambda - 1) e_{ii}$ for $|\lambda| = 1$, let $G$ be the multiplicative group generated by $\{c_{ij}, d_{i}(\lambda) : 1 \leq i, j \leq n, |\lambda| = 1\}$ and let

$$R = \{B \in M_n : \|A\|_S = \|AB\|_S \text{ for all } A \in M_n\}$$
\[ L = \{ B \in M_n : \| A \|_S = \| BA \|_S \text{ for all } A \in M_n \} \]

\[ LR = \{ B \in M_n : \| A \|_S = \| B^* AB \|_S \text{ for all } A \in M_n \} \]

By elementary calculations we get \[ (Ac_{ij}) \circ (xc_{ij})c_{ij} = c_{ij}(Ac_{ij}) \circ (c_{ij}x) \]
\[ = [Ad_i(\lambda) \circ x]d_i(\bar{\lambda}) = d_i(\bar{\lambda})(Ad_i(\lambda) \circ x) = A \circ x \text{ for } A \in M_n, x \in M_n(E) \]
where \( E \) is an operator space and |\( \lambda \)| = 1. Hence \( \| Ac_{ij} \|_S = \| Ac_{ij} \|_S = \| A \|_S \) and \( \| Ad_i(\lambda) \|_S = \| d_i(\lambda)A \|_S = \| A \|_S \) for \( A \in M_n, |\lambda| = 1 \). Clearly \( L \cdot L = L, R \cdot R = R, (LR) \cdot (LR) = LR. \) So \( G \subseteq L \cap R \cap LR. \)

For \( B = [b_{ij}] \in R, \| e_{kk} \|_S = \| e_{kk}B \|_S = \max\{|b_{k1}|, \ldots, |b_{kn}|\} \) by Lemma 1. That is, for \( 1 \leq k \leq n \)

\[ \max\{|b_{k1}|, \ldots, |b_{kn}|\} = 1 \]
Choose \( \lambda_{ij} \) with \( |\lambda_{ij}| = 1, \lambda_{ij}b_{ij} = |b_{ij}| \) and put \( A_k = \sum_{i=1}^{n} \lambda_{ki}e_{ki} \). Then \( \| A_kB \|_S = \| A_k \|_S = 1 \) and the \((k, k)\) entry of \( A_kB \) is \( \sum_{i=1}^{n} |b_{ik}| \). Hence for \( 1 \leq k \leq n \)

\[ \sum_{i=1}^{n} |b_{ik}| \leq 1 \]

By (1),(2), each column and each row of \( B \) have exactly one entry whose absolute value is 1 and the others are 0. Therefore \( B \in G \) and \( R = G \).

Similarly \( L = G \).

For \( B = [b_{ij}] \in LR, \| B^* e_{kk} B \|_S = \max\{|b_{k1}^2|, \ldots, |b_{kn}^2|\} \) by Lemma 1. Hence \( \max\{|b_{k1}^2|, \ldots, |b_{kn}^2|\} = 1 \), that is, \( \max\{|b_{k1}|, \ldots, |b_{kn}|\} = 1 \).

Since \( \| B^* B \|_S = 1 \) and the \((k, k)\) entry of \( B^* B \) is \( \sum_{i=1}^{n} |b_{ik}^2|, \sum_{i=1}^{n} |b_{ik}| \leq 1 \). Hence each column and each row of \( B \) have exactly one entry whose absolute value is 1 and the others are 0. Therefore \( B \in G \) and \( LR = G \).

By the above, we get the following Proposition.

**Proposition 12.** \( L = R = LR = G \).

**References**


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