STABILITY IN DISTRIBUTION FOR A CLASS OF DIFFUSIONS WITH JUMP

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1. Introduction

We consider a diffusion \( \{X^x(t); t \geq 0\} \) on \( \mathbb{R}^1 \) satisfying the following stochastic differential equation.

\[
X^x(t) = x + \int_0^t \sigma(X^x(s))dB(s) + \int_0^t b(X^x(s))ds + \int_0^t \int \tilde{c}(X^x(s), u) \tilde{v}(du, ds)
\]

where \( \sigma \) and \( b \) are Lipschitz continuous functions on \( \mathbb{R}^1 \), \( c \) is a measurable function on \( \mathbb{R}^2 \), \( \{B(t); t \geq 0\} \) is a standard 1-dimensional Brownian motion and \( \tilde{v} \) is a compensated Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R} \). That is, there is a \( \sigma \)-finite measure \( \pi \) on \( \mathbb{R}^1 \setminus \{0\} \) such that \( \tilde{v}([0, t] \times A) = v([0, t] \times A) - t\pi(A) \) where \( v \) is a Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R} \) with \( E[v([0, t] \times A)] = t\pi(A) \) for any Borel set \( A \) of \( \mathbb{R}^1 \). Let \( p(t, x, dy) \) denote the transition probability of the diffusion. First, we introduce the following definitions applying to general diffusions.

DEFINITION 1. A diffusion is stable in distribution if its transition probability \( p(t, x, dy) \) converges weakly to some probability measure \( \pi(dy) \) as \( t \to \infty \), for every \( x \).

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DEFINITION 2. The stochastic flow \( \{X^x(t); t \geq 0, x \in R\} \) is asymptotically flat (in probability) uniformly on compacts if

\[
(2) \lim_{t \to \infty} \sup_{x,y \in K} P(|X^x(t) - X^y(t)| > \epsilon) = 0
\]
as \( t \to \infty \) for every \( \epsilon > 0 \) and every compact set \( K \).

It is simple to check that stability in distribution follows from the following (ref. [3]):

(i) tightness of \( \{p(t, x, dy); 0 \leq t < \infty\} \) and

(ii) asymptotic flatness (definition 2).

Now (2) can be derived by the following property.

DEFINITION 3. The stochastic flow \( \{X^x(t); t \geq 0, x \in R\} \) is asymptotically flat in the second mean if every compact set \( K \) in \( R \),

\[
(3) \lim_{t \to \infty} \sup_{x,y \in K} E[|X^x(t) - X^y(t)|^2] = 0.
\]

REMARK 1. The exponent 2 in (3) can be changed to any \( \delta > 0 \) to imply the Definition 2.

In this paper, we consider the question: under what conditions on \( \sigma, b, c \) and \( \nu \), is the diffusion tight or stable in distribution? In the next section, we have some sufficient conditions for this even though it is very special. If \( c \equiv 0 \), then \( X^x(t) \) is continuous a.s. and in that case, there are lots of literature including the above question ([1], [5], [2]). But with nonzero \( c(x, u) \), \( X^x(t) \) is right continuous with left limit. Sufficient conditions that \( X^x(t) \) exists uniquely are well known ([4]) and those are given in the following section. We primarily follow the idea in [1] to have some conditions that \( x^2 \) is a Liapunov function for the generator of the process (1).

2. Main results

Consider the following conditions for \( \sigma, b, \) and \( c \).

There is a constant \( M \) such that for all \( x \in R \),

\[
(4) \sigma^2(x) + b^2(x) + \int c(x, u)^2 \pi(du) \leq L(1 + x^2).
\]
Stability in distribution

Lipschitz conditions; there exist positive constants \( \lambda_0, \lambda_1 \) and \( \lambda_2 \) such that for all \( x, y \)

\[
|\sigma(x) - \sigma(y)| \leq \lambda_0 |x - y|
\]

\[
|b(x) - b(y)| \leq \lambda_1 |x - y|, \quad \frac{d}{dx} b \leq -\lambda_1 < 0,
\]

and

\[
\int |c(x, u) - c(y, u)|^2 \pi(du) \leq \lambda_2 |x - y|^2.
\]

With above conditions (4)-(7), the diffusion exists uniquely and \( E|X_t|^2 \) < \( \infty \) for all \( t \) and \( x \). (cf. [4] part II, ch.2). And without (5) and (6), but if \( \sigma, b \) are continuous, \( X_t \) in (1) exists. (cf: [6])

**THEOREM 1.**

(1) Assume conditions (4)-(7). If there exists a constant \( \beta > 0 \) such that

\[-2\lambda_1 + \frac{\sigma^2(x)}{x^2} + \lambda_2 \leq -\beta\]

for all sufficiently large \( |x| \), there exists an invariant probability.

(2) Assume the conditions (4)-(7). If there exists a constant \( \alpha > 0 \) such that

\[-2\lambda_1 + \lambda_0^2 + \lambda_2 < -\alpha, \]

then the diffusion (1) is stable in distribution.

**REMARK 1.** Note that

\[
\sigma^2(x) = (\sigma(x) - \sigma(0) + \sigma(0))^2
\]

\[
= (\sigma(x) - \sigma(0))^2 + \sigma(0)^2 + 2\sigma(0)(\sigma(x) - \sigma(0))
\]

\[
\leq \lambda_0^2 x^2 + \sigma(0)^2 + 2|\sigma(0)|\lambda_0 |x|.
\]

Hence

\[-2\lambda_1 + \frac{\sigma^2(x)}{x^2} + \lambda_2 \leq (-2\lambda_1 + \lambda_0^2 + \lambda_2) + \frac{\sigma(0)^2}{x^2} + 2|\sigma(0)|\frac{\lambda_0 |x|}{x^2}
\]

\[
= (-2\lambda_1 + \lambda_0^2 + \lambda_2) + O(|x|)
\]

as \( |x| \to \infty \) and we see that (2) in Theorem 1 implies (1).
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Proof of Theorem 1. (1) Since \( p(t, x, dy) \) is Feller continuous (ref: Theorem 1 p.276 and Lemma 2 p.284 of [4]), it is sufficient to show that \( \sup_{t \geq 0} E|X_t^x|^2 < \infty \) and for this, by the following Lemma 1, it is enough to show that for some constant \( \beta' > 0 \), \( L\phi(y) \leq -\beta'y^2 \) for large enough \( |y| \) where \( \phi(y) = y^2 \) and \( L\phi(y) = 2yb(y) + \sigma^2(y) + \int c^2(y, u)\pi(du) \). By Ito’s formula for \( X_t^x \) in (1),

\[
(X_t^x)^2 = x^2 + \int_0^t 2X_s\sigma(X_s^x)dB_s
\]

\[
+ \int_0^t \int ((X_s^x + c(X_s, u))^2 - (X_s^x)^2)\tilde{\nu}(du, ds)
\]

\[
+ \int_0^t \int (2X_s^x b(X_s^x) + \sigma^2(X_s^x))ds
\]

\[
+ \int_0^t \int ((X_s^x + c(X_s^x, u))^2 - (X_s^x)^2 - 2c(X_s^x, u)X_s^x)\pi(du)ds
\]

\[
= x^2 + \int_0^t 2X_s\sigma(X_s^x)dB_s
\]

\[
+ \int_0^t \int (2c(X_s^x, u)X_s^x + \sigma^2(X_s^x, u))\tilde{\nu}(du, ds)
\]

\[
+ \int_0^t (2X_s^x b(X_s^x) + \sigma^2(X_s^x)) + \int c^2(X_s^x, u)\pi(du)ds.
\]

Then by the conditions (4), (6) and (7),

\[
2yb(y) + \sigma^2(y) + \int c(y, u)^2\pi(du)
\]

\[
= 2y(b(y) - b(0)) + 2yb(0) + \sigma^2(y) + \int (c(y, u) - c(0, u) + c(0, u))^2\pi(du)
\]

\[
\leq -2\lambda_1 y^2 + 2yb(0) + \sigma^2(y) + \lambda_2 y^2 + M + 2 \int (c(y, u) - c(0, u)c(0, u)\pi(du)
\]

\[
\leq (-2\lambda_1 + \frac{\sigma^2(y)}{y^2} + \lambda_2)y^2 + 2yb(0) + M + 2(M\lambda_2 y^2)^{1/2}
\]

\[
\leq (-2\lambda_1 + \frac{\sigma^2(y)}{y^2} + \lambda_2)y^2 + O(|y|) \leq -\beta'y^2
\]

for large enough \( |y| \). Hence by the following Lemma 1, \( \sup_{t \geq 0} E|X_t^x|^2 < \infty \). Therefore there exists an invariant probability.

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(2) Define for a given pair \((x, y)\) with \(x \neq y\),
\[ Z_t^{x,y} = X_t^x - X_t^y \]
\[ = x - y + \int_0^t (b(X_s^x) - b(X_s^y))ds + \int_0^t (\sigma(X_s^x) - \sigma(X_s^y))dB_s + \int_0^t \int_c^d c(X_s^x, u) - c(X_s^y, u))\tilde{\nu}(du, ds) \]
\[ = (x - y) + \int_0^t f(s)ds + \int_0^t \gamma(s)dB_s + \int_0^t \int g(s, u)\tilde{\nu}(du, ds). \]

Let \(\tau_0 := \inf\{t \geq 0 : Z_t^{x,y}(t) = 0\}\). By Ito formular applied to \(\phi(x) = x^2\),
\[ (Z_t^{x,y})^2 = (x - y)^2 + \int_0^t 2Z_s^{x,y} \gamma_s dB_s + \int_0^t \int ((Z_s^{x,y} + g(s, u))^2 - (Z_s^{x,y})^2)\tilde{\nu}(du, ds) + \int_0^t \bar{L}(\phi)(X_s^x, X_s^y)ds \]
where \(\bar{L}(\phi)(X_s^x, X_s^y) = 2Z_s^{x,y} f(s) + \gamma(s)^2 + \int g(s, u)^2\pi(du)\). Now
\[ (x - y)(b(x) - b(y)) = \int_0^1 (x - y)^2b'(y + \theta (x - y))d\theta \leq -\lambda_1(x - y)^2 \]
by (6). Hence
\[ \bar{L}(\phi)(X_s^x, X_s^y) \leq (-2\lambda_1 + \lambda_0^2 + \lambda_2)(Z_s^{x,y})^2 \leq -\alpha(Z_s^{x,y})^2. \]

Hence we have
\[ E(Z_t^{x,y}(t \wedge \tau_0))^2 \leq (x - y)^2 - \alpha E(\int_0^{t \wedge \tau_0} (Z_s^{x,y})^2)ds. \]

Notice that \(Z_t^{x,y} = 0\) a.s. for all \(t \geq \tau_0\). Therefore \(E(Z_t^{x,y})^2 \leq e^{-\alpha t}(x - y)^2\) for all \(t \geq 0\). Hence \(E(Z_t^{x,y})^2 \to 0\) as \(t \to \infty\) for any \(x, y\) in compact set and it implies the asymptotic flatness of the second mean. Hence by Remark 1, the process of (1) is stable in distribution.
LEMMA 1. If $\phi(y) = y^2$ and for some $\beta > 0$, $L\phi(y) = 2yb(y) + \sigma^2(y) + \int c(y, u)^2 \pi(du) \leq -\beta y^2$ for all large $|y|$, then for the process $X^x_t$ of (1), we have $\sup_{t \geq 0} E(X^x_t)^2 < \infty$ for any $x$.

Proof. Take $N$ large enough so that $L\phi(y) \leq -\beta y^2$ if $|y| \geq N$. Then take expectation on (8) using usual truncation (p.275 of [4]), we have

$$E(X^x_t)^2 = x^2 + \int_0^t \mathbb{E} L\phi(X^x_s) ds$$

since with the conditions (4)-(7), we can take constant $C$ such that $L\phi(y) \leq C(1 + y^2)$ for all $y$, hence $\int_0^t \mathbb{E} L\phi(X^x_s) ds < \infty$ and stochastic integrals with respect to $B$ and $\tilde{\nu}$ are martingales. Therefore we have $\frac{d}{dt} E(X^x_t)^2 = E L\phi(X^x_t)$ and we can take positive constants $M_1, M_2$ such that

$$\frac{d}{dt} E(X^x_t)^2 = E L\phi(X^x_t) 1_{|X^x_t| < N} + E L\phi(X^x_t) 1_{|X^x_t| \geq N}$$

$$\leq M_1 - \beta E((X^x_t)^2) 1_{|X^x_t| \geq N}$$

$$= M_1 - \beta E(X^x_t)^2 + \beta E((X^x_t)^2) 1_{|X^x_t| < N}$$

$$\leq M_1 + M_2 - \beta E(X^x_t)^2.$$

Hence by routine calculation, $\sup_{t} E(X^x_t)^2 < \infty$. □

REMARK 2. Without any difficulty, we can extend Theorem 1 to multidimensional case under the same condition of the components of coefficients and $|x|^2$ instead of $x^2$.

References