THE INDEX OF THE CORESTRICTION
OF A VALUED DIVISION ALGEBRA

Yoon Sung Hwang

ABSTRACT. Let $L/F$ be a finite separable extension of Henselian valued fields with same residue fields $\bar{L} = \bar{F}$. Let $D$ be an inertially split division algebra over $L$, and let $\mathcal{C}D$ be the underlying division algebra of the corestriction $\text{cor}_{L/F}(D)$ of $D$. We show that the index $\text{ind}(\mathcal{C}D)$ of $\mathcal{C}D$ divides $[Z(\bar{D}) : Z(\mathcal{C}D)] \cdot \text{ind}(D)$, where $Z(\bar{D})$ is the center of the residue division ring $\bar{D}$.

For any finite separable extension $L/F$ of fields and any central simple algebra $A$ over $L$, the corestriction of $A$ is a central simple $F$-algebra obtained as the fixed point algebra under a Galois group action (cf. [Ri]). This induces the map from the Brauer group $\text{Br}(L)$ to $\text{Br}(F)$ corresponding to the homological corestriction. Though this algebraic corestriction is an important tool in the theory of division algebras, it is actually very hard to work with. To gain a better insight into the behavior of the corestriction, we analyze here the corestriction for valued division algebras over Henselian valued fields, for which there is a well-developed structure theory.

For any ring $R$ we write $Z(R)$ and $R^*$ for the center of $R$ and the group of units of $R$, respectively. We will consider only central simple algebras $A$ finite-dimensional over a field $F$. By Wedderburn’s theorem, $A \cong M_n(D)$, a matrix ring over a division algebra $D$, which is called the underlying division algebra of $A$.

A valued field $(F, v)$ is called Henselian if $v$ extends uniquely to each field algebraic over $F$. For a nice account for several other characterizations of Henselian valuations, see Ribenboim’s paper [Rb]. Recall (e.g. from [W]) that if $D$ is a central division algebra over a Henselian valued field $(F, v)$, there exists one and only one valuation on $D$ extending $v$ on $F$.
For a central $L$-division algebra $D$, $^c D$ denote the underlying division algebra of the corestriction $\text{cor}_{L/F}(D)$ of $D$. The index $\text{ind}(^c D)$ of $^c D$ divides $\text{ind}(D)^{[L:F]}$. ([D, Lemma 7, p. 54]) We will show that when $D$ is inertially split over $L$ and $L/F$ is a finite separable extension of Henselian valued fields with same residue fields $\overline{L}=\overline{F}$, the index $\text{ind}(^c D)$ of $^c D$ divides $[Z(D): Z(^c D)] \cdot \text{ind}(D)$, where $Z(D)$ is the center of the residue division ring $\overline{D}$. (See below for terminology.)

We now fix most of the basic terminology and notation that we will employ throughout this paper.

Let $(L, v) \supseteq (F, v)$ be a finite separable extension of Henselian fields. We say that $L$ is inertial (or unramified) over $F$ if $[\overline{L}: \overline{F}] = [L: F]$ and $\overline{L}$ is separable over $\overline{F}$.

Let $(F, v)$ be a Henselian valued field. Let $D$ be a central division $F$-algebra (with a unique valuation extending $v$ on $F$). We say $D$ is tame and totally ramified (or $D$ is tame) over $F$ if $\text{char}(\overline{F}) \mid [D: F]$ and $|\Gamma_D : \Gamma_F| = [D: F]$. $D$ is said to be inertially split over $F$ if $D$ is split by $F_{nr}$ where $F_{nr}$ is the maximal unramified extension in some algebraic closure of $F$. Also, $D$ is said to be tame if $\text{char}(\overline{F}) = 0$ or $\text{char}(\overline{F}) = q \neq 0$ and the $q$-primary component of $D$ is split by $F_{nr}$. (See [JW, Lemma 5.1] and [JW, Lemma 6.1] for other characterizations of inertially split and tame division algebras.) Recall also that $D$ is said to be inertial over $F$ if $[\overline{D}: \overline{F}] = [D: F]$ and $Z(D) = \overline{F}$. $D$ is said to be nicely semiramified over $F$ if $D$ has a maximal subfield $L$ which is inertial over $F$, and another maximal subfield $K$ which is totally ramified of radical type over $F$. (Then, $\overline{D} = \overline{L}, \Gamma_D = \Gamma_K$ and $[\overline{D}: \overline{F}] = |\Gamma_D : \Gamma_F| = \text{ind}(D)$.) (See [JW, Sec. 4].) Let

$$\mathcal{D}(F) = \{D \mid D \text{ is a central division } F\text{-algebra with } [D: F] < \infty\}$$

$$\mathcal{D}_{ttr} = \{D \in \mathcal{D}(F) \mid D \text{ is tame and totally ramified over } F\}$$

$$\mathcal{D}_i(F) = \{D \in \mathcal{D}(F) \mid D \text{ is inertial over } F\}$$

$$\mathcal{D}_{is}(F) = \{D \in \mathcal{D}(F) \mid D \text{ is inertially split over } F\} \text{ and}$$

$$\mathcal{D}_t(F) = \{D \in \mathcal{D}(F) \mid D \text{ is tame over } F\}.$$  

It is clear that $\mathcal{D}_t(F) \subseteq \mathcal{D}_{is}(F) \subseteq \mathcal{D}_i(F)$ and $\mathcal{D}_{ttr}(F) \subseteq \mathcal{D}_i(F)$.

$(K/F, \sigma, a)_y$ is the cyclic $F$-algebra generated over $K$ by a single element $x$ with defining relations $x\sigma x^{-1} = \sigma(c)$ for all $c \in K$ and $x^n = a \in F^*$, where $K$ is a Galois extension of $F$ with cyclic Galois group generated by $\sigma$ and $n = [K: F]$. 
Now, we give a lemma to compute the corestriction $\text{cor}_{L/F}(D)$ of $D$ of a NSR cyclic division algebra $D$ over $L$ when $L/F$ is a finite separable extension of Henselian valued fields with $\overline{L} = \overline{F}$.

**Lemma 1.** Let $L/F$ be a finite separable extension of Henselian valued fields with same residue fields $\overline{L} = \overline{F}$. Let $D = (M'/L, \sigma, \alpha)'$ be a NSR cyclic division algebra over $L$. (So, $M'/L$ is inertial with cyclic Galois group generated by $\sigma$.) Let $M$ be the inertial lift of $M'$ over $F$. Then,

$$\text{cor}_{L/F}(D) \sim (M/F, \sigma, N_{L/F}(\alpha))'',$$

where $N_{L/F}$ is the norm map from $L$ to $F$.

**Proof.** Let $L_{\text{sep}} = F_{\text{sep}}$ be the separable closure of $L$ and $F$. Let $G = \text{Gal}(F_{\text{sep}}/F)$ and $H = \text{Gal}(L_{\text{sep}}/L)$

Since $M/F$ is Galois and $L \cap M = F$, $L$ and $M$ are linearly disjoint over $F$ and $L \otimes_F M$ is the field $L \cdot M = M'$. Let $N = \text{Gal}(F_{\text{sep}}/M)$. Then since $M/F$ is Galois and $L \cap M = F$, $N$ is normal in $G$ and $G = HN$. Also, $\text{Gal}(M'/F) \cong G/N \cong \langle \sigma \rangle$ and $\text{Gal}(M'/L) \cong H/(H \cap N) \cong \langle \sigma \rangle$. Since $L \otimes_F M$ is the field $M'$, by [D, p. 56, Ex. 1] $\text{cor}_{L/F}(D) \otimes_F M \sim \text{cor}_{M'/M}(D \otimes_L M') \sim \text{cor}_{M'/M}(M') \sim M$ in $\text{Br}(M)$.

Since $D = (M'/L, \sigma, \alpha)' \in \text{Br}(M'/L) \cong H^2(H/(H \cap N), M^*)$, $D$ is represented by $\text{inf}^H_{H/(H \cap N)}(f)$ where $f \in H^2(H/(H \cap N), M^*)$ is given by $(\sigma^i, \sigma^j) \mapsto 1$ if $0 \leq i + j \leq n - 1$ and $(\sigma^i, \sigma^j) \mapsto \alpha$ if $i + j \geq n$. Since the algebraic corestriction corresponds to the homological corestriction, in $\text{Br}(F)$, $\text{cor}_{L/F}(D)$ is represented by $\text{cor}_{H}^{G}(\text{inf}^H_{H/(H \cap N)}(f))$. But, by [H, Th. 5] $\text{cor}_{H}^{G}(\text{inf}^H_{H/(H \cap N)}(f)) = \text{inf}^{G}_{G/N}(N_{G/N}^*(f))$, where $N_{G/N}^*: H^2(H/(H \cap N), M^*) \rightarrow H^2(G/N, M^*)$ is induced by the norm map from $M^*$ to $M$. Hence, $\text{cor}_{L/F}(D) \sim (M/F, \sigma, N_{L/F}(\alpha))'$ in $\text{Br}(F)$. □

We can now prove our theorem.

**Theorem 2.** Let $L/F$ be a finite separable extension of Henselian valued fields with same residue fields $\overline{L} = \overline{F}$. If $D$ is inertially split over $L$, then the index $\text{ind}(D)$ of $D$ divides $|Z(D) : Z(\overline{D})| \cdot \text{ind}(D) = |\Gamma_D : \Gamma_{\overline{D}}| \cdot \text{ind}(D)$, where $Z(\overline{D})$ is the center of the residue division ring $\overline{D}$, and $\Gamma_D$ is the value group of $D$. 

Proof. Since $D$ is inertially split over $L$, by [JW, Lemma 5.14] there exist $I', N' \in \mathcal{D}(L)$ with $I'$ inertial over $L$ and $N'$ NSR over $L$, such that $D \sim I' \otimes_L N'$ in $Br(L)$. Then by [JW, Th. 4.4], $N' = \bigotimes_{i=1}^{k} (M'_i / L, \sigma_i, \alpha_i)_{t_i}$ where $M'_i / L$ is inertial cyclic Galois with $Gal(M'_i / L) = \langle \sigma_i \rangle$.

By Lemma 1 above, $\text{cor}_{L/F}(N') \sim \bigotimes_{i=1}^{k} (M_i / F, \sigma_i, N_{L/F}(\alpha_i))_{t_i}$ where $M_i$ is the inertial lift of $M'_i$ over $F$.

Let $a_i = N_{L/F}(\alpha_i)$ and let $v(a_i)$ map to an element of $\Gamma_F / t_i' \Gamma_F$ of order $t_i$. So, $t_i v(a_i) = t_i' v(p_i)$ for some $p_i \in F^*$, and $a_i = u_i p_i^{t_i}$ where $s_i = t_i' / t_i$ and $u_i$ is a $\alpha$-unit of $F$. Let $K_i$ be an extension of $F$ of degree $t_i$ with $F \subseteq K_i \subseteq M_i$ and $\text{Gal}(K_i / F) = \langle \sigma_i \rangle$ where $\sigma_i$ is the restriction of $\sigma_i$ to $K_i$. Then by [R, Th. 30. 10, p. 262] $\text{cor}_{L/F}(N') \sim \bigotimes_{i=1}^{k} (M_i / F, \sigma_i, u_i)_{t_i} \otimes_F \bigotimes_{i=1}^{k} (K_i / F, \sigma_i, p_i)_{t_i}$. Also, by [H, Lemma 4] $cI' \in \mathcal{D}_t(F)$ and $\frac{cI'}{L} \sim \frac{cI'}{L} \otimes_L F$ in $Br(F)$.

Let $I$ be the underlying division algebra of $cI' \otimes_F (\bigotimes_{i=1}^{k} (M_i / F, \sigma_i, u_i))_{t_i}$ and let $N = \bigotimes_{i=1}^{k} (K_i / F, \sigma_i, p_i)_{t_i}$. Then $cD \sim I \otimes_F N$ in $Br(F)$ with $I$ inertial over $F$ and $N$ NSR over $F$. So, by [JW, Th. 5.15 (a)] $\text{ind}(D) = \text{ind}(\frac{cI'}{L}) \cdot |\Gamma' / \Gamma_L| = \text{ind}(\frac{cI'}{L}) \cdot \prod_{i=1}^{k} t_i$, and $\text{ind}(cD) = \text{ind}(\frac{cI'}{L}) \cdot |\Gamma' / \Gamma_F| = \text{ind}(\frac{cI'}{L}) \cdot \prod_{i=1}^{k} t_i$. But $\text{ind}(\frac{cI'}{L})$ divides $\text{ind}(\frac{cI'}{L} \otimes_L F) \cdot \prod_{i=1}^{k} \text{ind}((\frac{M_i}{F}, \sigma_i, u_i))_{t_i} \otimes_F N$.

Since $\frac{cI'}{L} \sim \frac{cI'}{L} \otimes_L F$, by [P, Prop. 13. 4] and [D, Th. 12, p. 67] $\text{ind}(\frac{cI'}{L}) | \text{ind}(\frac{cI'}{L})$ and $\text{ind}(\frac{cI'}{L}) | \text{ind}(\frac{cI'}{L}) \otimes_L F$.

Note that $\text{Gal}(M_i / N) \mathcal{N} / \mathcal{N} \cong \text{Gal}(M_i / N) \mathcal{N} / \mathcal{N} = \mathcal{N} / \mathcal{N}$. So, by [R, Th. 30. 8, p. 261] $(M_i / F, \sigma_i, \sigma_i)_{t_i} \otimes_F N \sim (M_i / N, \sigma_i, \sigma_i)_{t_i}$, whence $\text{ind}((\frac{M_i}{F}, \sigma_i, \sigma_i)_{t_i} \otimes_F N)$ divides $s_i$. Therefore, $\text{ind}(cD)$ divides $(\mathcal{N} / \mathcal{N}) \cdot \text{ind}(\frac{cI'}{L}) \cdot \prod_{i=1}^{k} s_i t_i = (\mathcal{N} / \mathcal{N}) \cdot \text{ind}(D)$. Since $Z(D) = \mathcal{N}$ and $Z(\frac{cD}{L}) = \mathcal{N}$ by [JW, Th. 5.15 (a)], $\text{ind}(cD)$ divides $\text{ind}(D)$.

Note that $\text{ind}(\frac{Z(\mathcal{D})}{\mathcal{D}}) = |\Gamma_D / \Gamma_D|$, since $\Gamma_D = \Gamma' / \mathcal{N}$ and $\Gamma_D = \Gamma' / \mathcal{N}$ by [JW, Th. 5.15 (a)] and $\mathcal{N} / \mathcal{N} = |\Gamma' / \mathcal{N} / \mathcal{N}|$ as $N'$ is NSR over $L$ and $N$ is NSR over $F$. □
This theorem gives us a best relation between \( \text{ind}(D) \) and \( \text{ind}^{(c)}(D) \) when \( D \) is inertially split over \( L \) and \( L/F \) is a finite separable extension of Henselian valued fields with same residue fields \( \overline{F} = \overline{F} \), as the following examples illustrate.

**Example 3.** Let \( L/F \) be as above and let \( D \) be inertial over \( L \). Then by [H, Lemma 4] \( ^{c}D \) is inertial over \( F \) and \( \overline{D} \sim D \otimes [L:F] \) in \( \text{Br}(\overline{F}) \). So \( Z(\overline{D}) = \overline{L} = \overline{F} = Z(\overline{cD}) \), and \( \text{ind}^{(c)}(D) = \text{ind}(\overline{cD}) = \text{ind}(D) \otimes [L:F] \) by [JW, Th. 2.8 (b)]. So, by [P, Prop. 13.4] \( \text{ind}^{(c)}(D) \mid \text{ind}(D) \).

**Example 4.** Let \( (F, v) \) be a Henselian field with \( \Gamma_F = \mathbb{Z} \) and \( \pi \in F \) with \( v(\pi) = 1 \). Let \( L = F(\sqrt[n]{\pi}) \). (So \( \overline{L} = \overline{F} \) and \( \Gamma_F = \frac{1}{n} \mathbb{Z} \)). Let \( t \geq 1 \) with \( \gcd(n,t) = 1 \) and \( D = (M'/L, \sigma, \pi_t) \), be a NSR division algebra over \( L \), where \( M'/L \) is inertial with \( \text{Gal}(M/L) = \langle \sigma \rangle \) and \( [M':L] = t \). Then by Lemma 1, \( ^{c}D \sim \text{cor}_{L/F}(D) \sim (M/F, \sigma, \pi^n)_t \) in \( \text{Br}(F) \), where \( M \) is the inertial lift of \( M' \) over \( F \). But since \( (M/F, \sigma, \pi^n) \), is a NSR division algebra over \( F \) as shown in [JW, Ex. 4.3], \( \text{ind}^{(c)}(D) = t = \text{ind}(D) = [Z(\overline{D} : \overline{Z(\overline{D})})] \cdot \text{ind}(D) \).

**References**


Department of Mathematics
Korea University
Anam-Dong, Sungbuk-ku
Seoul 136-701, Korea
E-mail: yhwang@semi.korea.ac.kr