STABLE CLASS OF EQUIVARIANT ALGEBRAIC VECTOR BUNDLES OVER REPRESENTATIONS

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Dedicated to Professor Fuichi Uchida on his 60th birthday

Abstract. Let $G$ be a reductive algebraic group and let $B, F$ be $G$-modules. We denote by $\text{VEC}_G(B, F)$ the set of isomorphism classes in algebraic $G$-vector bundles over $B$ with $F$ as the fiber over the origin of $B$. Schwarz (or Kraft-Schwarz) shows that $\text{VEC}_G(B, F)$ admits an abelian group structure when $\dim B/G = 1$. In this paper, we introduce a stable functor $\text{VEC}_G(B, F^\infty)$ and prove that it is an abelian group for any $G$-module $B$. We also show that this stable functor will have nice properties.

1. Introduction

Throughout this paper, we will work in the algebraic category over the field of complex numbers $\mathbb{C}$ and $G$ will denote a reductive group unless otherwise stated. Finite groups, $\mathbb{C}^*$-tori (i.e., products of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$) and semisimple groups are examples of reductive groups, and it is known that any reductive group is obtained as a group extension by these three types of groups (see [2] for example). One may also think of a reductive group as “complexification” of a compact Lie group (see [20] for example), e.g. the complexification of the circle group $S^1$ is $\mathbb{C}^*$.

The research of this paper is motivated by the following problem.

Equivariant Serre Problem. Is any $G$-vector bundle over a $G$-module $B$ (= a $G$-representation space) trivial, i.e., isomorphic to a product bundle $F := B \times F \to B$ for some $G$-module $F$?

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One can ask the same question in other categories. It is a classical result that the problem has an affirmative solution in the smooth category because the base space $B$ is equivariantly contractible. Recently it has affirmatively been answered in the holomorphic category ([5]).

However, the situation is not so simple in the algebraic category. When $G$ is trivial, the Equivariant Serre Problem is nothing but the famous Serre conjecture which was solved affirmatively by D. Quillen [18] and A. Suslin [21]. This result is extended to the case when $G$ is abelian by Masuda-Moser-Petrie [14]. Another type of partial affirmative solution to the problem is as follows. The affine variety $B//G$, whose coordinate ring is the ring $\mathcal{O}(B)^G$ of $G$-invariant polynomials on $B$, is called the algebraic quotient of $B$ by the $G$-action. When $\dim B//G = 0$, it follows from Luna slice theorem [10] that the Equivariant Serre Problem has an affirmative solution. G. Schwarz [19] (see also [8]) attacked the next case where $\dim B//G = 1$, and surprisingly found counterexamples to the problem for many non-abelian groups $G$. After his breakthrough, more counterexamples have been found ([6], [13, 15], [16, 17]), where $\dim B//G$ is not necessarily one. On the other hand, Bass and Haboush ([4]) proved (before the breakthrough by Schwarz) that every $G$-vector bundle over a $G$-module is stably trivial, i.e., it becomes trivial when added to a suitable trivial $G$-vector bundle, for any $G$. See [12] for more information on our subject.

For $G$-modules $B$ and $F$ we denote by $\text{VEC}_G(B, F)$ the set of isomorphism classes in $G$-vector bundles over $B$ whose fiber over the origin is isomorphic to $F$. We often abbreviate a $G$-vector bundle $\pi: E \rightarrow B$ as $E$, and denote its isomorphism class by $[E]$. Needless to say, $\text{VEC}_G(B, F)$ contains the isomorphism class of the product bundle $F$, and if $\text{VEC}_G(B, F)$ contains an element different from $[F]$, then it provides a counterexample to the Equivariant Serre Problem. Following [16, 17] we also consider a subset

$$\text{VEC}_G(B, F; S) := \{ [E] \in \text{VEC}_G(B, F) \mid [E \oplus S] = [F \oplus S] \}$$

for a $G$-module $S$. The result of Bass and Haboush mentioned above says that the union of $\text{VEC}_G(B, F; S)$ over all $G$-modules $S$ agrees with $\text{VEC}_G(B, F)$.

Schwarz [19] (and Kraft-Schwarz [8]) proved that if $\dim B//G = 1$, then $\text{VEC}_G(B, F)$ admits an abelian group structure and is isomorphic to $\mathbb{C}^p$ for some non-negative integer $p$ depending on $B$ and $F$. They also established a formula to compute the dimension $p$ in terms of invariant theory and found that $p$ could be positive for many $G$, $B$ and $F$. 
The group structure on $\text{VEC}_G(B, F)$ is as follows. When $\dim B/G = 1$, they showed that the Whitney sum with $F$ induces a bijective correspondence
\[(*) \quad \text{VEC}_G(B, F) \overset{\oplus F}{\cong} \text{VEC}_G(B, F \oplus F).\]

Therefore, given $[E_1]$ and $[E_2]$ in $\text{VEC}_G(B, F)$, there is a unique element $[E_3]$ in $\text{VEC}_G(B, F)$ such that $[E_1 \oplus E_2] = [E_3 \oplus F]$, and the sum of $[E_1]$ and $[E_2]$ is defined to be $[E_3]$, giving the abelian group structure on $\text{VEC}_G(B, F)$. The map $(*)$ above also induces a bijection between $\text{VEC}_G(B, F; S)$ and $\text{VEC}_G(B, F \oplus F; S)$ for any $S$, so that $\text{VEC}_G(B, F; S)$ becomes a subgroup of $\text{VEC}_G(B, F)$ when $\dim B/G = 1$.

However, when $\dim B/G \geq 2$, the map $(*)$ above is not known to be bijective, so we do not know whether $\text{VEC}_G(B, F)$ admits an abelian group structure under Whitney sum. To get around this, we consider the following direct system
\[
\begin{align*}
\oplus F & \quad \text{VEC}_G(B, F^n) \overset{\oplus F}{\to} \text{VEC}_G(B, F^{n+1}) \overset{\oplus F}{\to} \\
\end{align*}
\]
where $F^n$ denotes the direct sum of $n$ copies of $F$, and define
\[\text{VEC}_G(B, F^{\infty}) := \lim_{\to n} \text{VEC}_G(B, F^n).\]

Similarly $\text{VEC}_G(B, F^{\infty}; S)$ can be defined. $\text{VEC}_G(B, F^{\infty})$ and $\text{VEC}_G(B, F^{\infty}; S)$ are apparently abelian monoids under Whitney sum, but it turns out

**Theorem 1.1.** $\text{VEC}_G(B, F^{\infty})$ is an abelian group and $\text{VEC}_G(B, F^{\infty}; S)$ is its subgroup under Whitney sum for any $G$-modules $B, F$ and $S$.

**Remark.** $\text{VEC}_G(B, F^{\infty})$ and $\text{VEC}_G(B, F^{\infty}; S)$ are both trivial when $\dim B/G = 0$, and isomorphic to $\text{VEC}_G(B, F)$ and $\text{VEC}_G(B, F; S)$ respectively when $\dim B/G = 1$.

In the proof of the theorem above, we define a surjective homomorphism
\[V \colon (R/I)^* \to \text{VEC}_G(B, F^{\infty}; S),\]
where $R$ is the ring of $G$-vector bundle endomorphisms of $S$, $I$ is a two sided ideal in $R$ and $(R/I)^*$ is the group of units in $R/I$. Note that when $S$ is the trivial one-dimensional module $\mathbb{C}$, $R$ is isomorphic to $\mathcal{O}(B)^G$, in particular, commutative. The homomorphism $V$ has a nontrivial kernel $\Gamma^{\infty}$ in general. When $(R/I)^*$ is commutative (e.g.
If $R/I$ is commutative (e.g. $S = \mathbb{C}$), then $\text{VEC}_G(B, F^\infty; S)$ is isomorphic to a finitely generated $O(B)^G$-module, as groups.

The author believes that the theorem above would hold without the commutativity assumption on $R/I$ and even for $\text{VEC}_G(B, F^\infty)$. In fact, when $\dim B / G = 1$, $\text{VEC}_G(B, F^\infty)$ is isomorphic to $\text{VEC}_G(B, F)$ as remarked above and $\text{VEC}_G(B, F)$ is isomorphic to a truncated polynomial ring $\mathbb{C}[t]/(t^p)$ in one variable $t$ for some non-negative integer $p$ by the result of Schwarz. The assumption that $\dim B / G = 1$ is equivalent to $O(B)^G$ being a polynomial ring in one variable, so $\mathbb{C}[t]$ can be identified with $O(B)^G$ and then $\mathbb{C}[t]/(t^p)$ is certainly a finitely generated $O(B)^G$-module in this case.

When $\dim B / G = 1$, Schwarz proved more. He showed that there is a “universal” $G$-vector bundle $\mathcal{E} \in \text{VEC}_G(B \oplus \mathbb{C}^p, F)$ such that mapping $c \in \mathbb{C}^p$ to $\mathcal{E}|_{B \times \{c\}} \in \text{VEC}_G(B, F)$ is bijective. Let $m$ be a non-negative integer. To any morphism (i.e., polynomial map) $f: \mathbb{C}^m \rightarrow \mathbb{C}^p = \text{VEC}_G(B, F)$, we assign a bundle induced from $\mathcal{E}$ by a map $1 \oplus f: B \oplus \mathbb{C}^m \rightarrow B \oplus \mathbb{C}^p$. This produces a map $\text{Mor}(\mathbb{C}^m, \text{VEC}_G(B, F)) = \text{VEC}_G(B, F) \otimes O(\mathbb{C}^m) \rightarrow \text{VEC}_G(B \oplus \mathbb{C}^m, F)$ where $\text{Mor}(X, Y)$ denotes the set of morphisms from $X$ to $Y$ and the tensor product is taken over $\mathbb{C}$. The universality of the bundle $\mathcal{E}$ implies that the above map is injective, and it is claimed in [11] that the map is actually bijective. The following result implies that there might exist the product formula above even when $\dim B / G \geq 2$.

**Theorem 1.3.** If $R/I$ is commutative (e.g. $S = \mathbb{C}$), then

\[ \text{VEC}_G(B \oplus \mathbb{C}^m, F^\infty; S) \cong \text{VEC}_G(B, F^\infty; S) \otimes O(\mathbb{C}^m) \]

as groups.

This paper is organized as follows. In Section 2 we review the method introduced in [16, 17] to produce elements in $\text{VEC}_G(B, F; S)$ and to distinguish them. It is the main tool used in this paper. We discuss its stable version in Section 3 and Theorem 1.1 is proved in Section 4. In Section 5 we consider a $\mathbb{C}^*$-action on $B$ commuting with the
G-action. In Section 6 we study \((R/I)^*/\Gamma^\infty\), which is isomorphic to \(\text{VEC}_G(B, F^\infty; S)\), using the \(C^*\)-action on \(B\) when \(R/I\) is commutative, and prove Theorem 1.2. Theorem 1.3 is proved in Section 7.

2. Subbundle method

In this section we review the method introduced in [16, 17]. Let \([E]\) be an element of \(\text{VEC}_G(B, F; S)\). Since \(E \oplus S\) is isomorphic to \(F \oplus S\), there is a \(G\)-vector bundle surjective homomorphism \(L : F \oplus S \to S\) whose kernel \(\ker L\) is isomorphic to \(E\). Let \(L' : F \oplus S \to S\) be another surjective homomorphism. Then it is not difficult to see that \(\ker L'\) is isomorphic to \(\ker L\) if and only if there is a \(G\)-vector bundle automorphism \(A\) of \(F \oplus S\) such that \(L' = LA\). Therefore, the study of \(\text{VEC}_G(B, F; S)\) splits into two steps: one is the study of \(G\)-vector bundle surjective homomorphisms from \(F \oplus S\) to \(S\) (in other words, construction of \(G\)-vector bundles) and the other is the study of \(G\)-vector bundle automorphisms of \(F \oplus S\) (in other words, distinction of \(G\)-vector bundles). One can formulate this as follows. Let \(\text{sur}(F \oplus S, S)^G\) be the set of \(G\)-vector bundle surjective homomorphisms from \(F \oplus S\) to \(S\) and let \(\text{aut}(F \oplus S)^G\) be the group of \(G\)-vector bundle automorphisms of \(F \oplus S\). The group \(\text{aut}(F \oplus S)^G\) acts on \(\text{sur}(F \oplus S, S)^G\) as above. Then the fact mentioned above can be restated as follows.

**Theorem 2.1** ([16, 17]). The map sending \(L \in \text{sur}(F \oplus S, S)^G\) to \(\ker L\) induces a bijection

\[
\text{sur}(F \oplus S, S)^G / \text{aut}(F \oplus S)^G \cong \text{VEC}_G(B, F; S).
\]

The following example will illustrate our method well.

**Example 2.2**. Let \(O_2 = C^* \rtimes \mathbb{Z}/2\). For a positive integer \(n\) we denote by \(V_n\) the 2-dimensional \(O_2\)-module with the actions of \(g \in C^*\) and of the nontrivial element in \(\mathbb{Z}/2\) respectively given by

\[
\begin{pmatrix}
g^n & 0 \\
0 & g^{-n}
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Then one easily checks that \(O(V_n)^{O_2}\) is a polynomial ring in one variable and it is proved in [19] that \(\text{VEC}_{O_2}(V_1, V_m) \cong C^{m-1}\) and \(\text{VEC}_{O_2}(V_1, V_m) = \text{VEC}_{O_2}(V_1, V_m; C)\). This provided the first counterexamples to the Equivariant Serre Problem.

Here is an explicit description of elements in \(\text{VEC}_{O_2}(V_1, V_m)\) found in [16, 17]. To a polynomial \(f(t)\) in one variable \(t\) with \(f(0) = 1\), we
associate

\[ E_f := \{(a, b, x, y, z) \in V_1 \times (V_m \oplus \mathbb{C}) \mid b^m x + a^m y + f(ab) z = 0\}, \]

where \((a, b) \in V_1, (x, y) \in V_m\) and \(z \in \mathbb{C}\). Taking the projection on \(V_1\), one sees that \(E_f\) defines an element of \(\text{VEC}_{O_2}(V_1, V_m; \mathbb{C})\). In fact, the \(1 \times 3\) matrix \(L_f := (b^m, a^m, f(ab))\) is of rank one at any point \((a, b) \in V_1\), so

\[ L_f : V_1 \times (V_m \oplus \mathbb{C}) \to V_1 \times \mathbb{C} \]

is a surjective \(O_2\)-vector bundle homomorphism and \(\ker L_f = E_f\).

On the other hand, it follows from the equivariance that an \(O_2\)-vector bundle automorphism \(A\) of the product bundle \(V_1 \times (V_m \oplus \mathbb{C})\) is a \(3 \times 3\) matrix of this form

\[
A = \begin{pmatrix}
p & a^{2m}q & a^m r \\
b^{2m}q & p & b^m r \\
b^m s & a^m s & w
\end{pmatrix},
\]

where \(p, q, r, s, w\) are polynomials in \(ab = t\). An elementary computation shows that

\[ \det A = (p - t^m q)(pw + t^m qw - 2t^m rs). \]

Since \(A\) is algebraic and invertible, \(\det A\) must be a nonzero constant and hence so are the both factors above. It follows that

\[ w \equiv \text{a nonzero constant} \pmod{t^m}. \]

Let \(h(t)\) be another polynomial with \(h(0) = 1\) and suppose that \([E_h] = [E_f]\) in \(\text{VEC}_{O_2}(V_1, V_m; \mathbb{C})\). Then \(L_h = L_f A\) for some automorphism \(A\). Comparing the last entries in \(L_h\) and \(L_f\) and using the congruence on \(w\) above, one concludes that \(h(t) \equiv f(t) \pmod{t^m}\). This shows that the correspondence \(\mathbb{C}^{m-1} \to \text{VEC}_{O_2}(V_1, V_m; \mathbb{C})\) given by \((c_1, \ldots, c_{m-1}) \to [E_c]\), where \(c(t) = 1 + c_1 t + \cdots + c_{m-1} t^{m-1}\), is injective. A more careful but elementary observation shows that this correspondence is bijective.

In this case, the universal bundle \(E\) mentioned in the introduction can be described as

\[ E = \{(a, b, c_1, \ldots, c_{m-1}, x, y, z) \in (V_1 \oplus \mathbb{C}^{m-1}) \times (V_m \oplus \mathbb{C}) \mid b^m x + a^m y + c(ab) z = 0\} \]

with the projection on \(V_1 \times \mathbb{C}^{m-1}\).

The following general argument was developed keeping the above example in mind. We review the definition of the invariants which distinguish elements in \(\text{VEC}_G(B, F; S)\).
For $G$-vector bundles $P$ and $Q$ over the same base space $B$, we denote by $\text{mor}(P, Q)^G$ the set of $G$-vector bundle homomorphisms from $P$ to $Q$. We write an element $L$ in $\text{sur}(F \oplus S, S)^G$ as $L = (L(F, S), L(S, S))$ where $L(F, S) \in \text{mor}(F, S)^G$ and $L(S, S) \in \text{mor}(S, S)^G =: R$. Since $L$ is a surjective homomorphism and $G$ is reductive, there is an element $M \in \text{mor}(S, F \oplus S)^G$ such that $LM$ is the identity map on $S$ (see [3]), i.e.,

$$L(S, S)M(S, S) + L(F, S)M(S, F) = 1,$$

where $M(S, S)$ and $M(S, F)$ are defined similarly to $L(F, S)$ and $L(S, S)$.

We denote by $I$ the ideal in $R$ generated by $G$-vector bundle endomorphisms of $S$ which factor through $F$, i.e., $I$ is generated by composition of elements in $\text{mor}(F, S)^G$ and $\text{mor}(S, F)^G$. The identity above implies that $L(S, S)$ is in $(R/I)^*$, i.e., a unit in $R/I$.

Now let $A$ be an element in $\text{aut}(F \oplus S)^G$. Then $\text{ker}(LA)$ is isomorphic to $\text{ker} L$ and we have

$$(LA)(S, S) = L(F, S)A(S, F) + L(S, S)A(S, S),$$

where $A(S, F)$ and $A(S, S)$ are defined similarly to $L(F, S)$ and $L(S, S)$. The first term at the right hand side above is an element of $I$ and it is not difficult to see that $A(S, S)$ is a unit in $R/I$. Therefore, if we denote by $\Gamma$ the subgroup of $(R/I)^*$ represented by elements $A(S, S)$ for $A \in \text{aut}(F \oplus S)^G$, then we have a well-defined map

$$\rho: \text{VEC}_G(B, F; S) \to (R/I)^*/\Gamma$$

sending $[\text{ker} L]$ to the equivalence class of $L(S, S)$. This is the invariant introduced in [16, 17] and used to distinguish elements in $\text{VEC}_G(B, F; S)$ (see also [13, 15]). In Example 2.2, one can check that $R = \mathcal{O}(V_1)^G = \mathbb{C}[t] \ (t = ab)$, $I = (t^m)$ and $\Gamma = \mathbb{C}^*$; so $(R/I)^*/\Gamma$ bijectively corresponds to the set of truncated polynomials of degree at most $m - 1$ and with constant term 1. Moreover, the map $\rho$ is bijective in this case. There are many cases where $\rho$ is bijective but it is not known whether $\rho$ is always bijective. However we will see later that the map $\rho^\infty$ induced from $\rho$ on $\text{VEC}_G(B, F^\infty; S)$ is bijective for any $G$-modules $B, F$ and $S$.

### 3. Stabilization

First we make sure that $\text{VEC}_G(B, F^\infty; S)$ is closed under Whitney sum. Suppose $[E_i] \in \text{VEC}_G(B, F^m; S)$ for $i = 1, 2$. Then, since $E_i \oplus S \simeq$
For each positive integer $n$, we have
\[ E_1 \oplus E_2 \oplus S \cong E_1 \oplus F^{n_2} \oplus S \cong F^{n_1} \oplus F^{n_2} \oplus S \cong F^{n_1+n_2} \oplus S, \]
which shows that $[E_1 \oplus E_2]$ lies in $\VEC_G(B, F^{n_1+n_2}; S)$. It follows that $\VEC_G(B, F^\infty; S)$ is closed under Whitney sum.

$\VEC_G(B, F^\infty; S)$ can be described in terms of $\sur$ and $\aut$ as in Theorem 2.1. We think of $\sur(F^n \oplus S, S)^G$ (resp. $\aut(F^n \oplus S)^G$) as a subset (resp. a subgroup) of $\sur(F^{n+1} \oplus S, S)^G$ (resp. $\aut(F^{n+1} \oplus S)^G$) by defining to be zero (resp. the identity) on the added factor $F$, and define $\sur(F^\infty \oplus S, S)^G$ (resp. $\aut(F^\infty \oplus S)^G$) to be the union of $\sur(F^n \oplus S, S)^G$ (resp. $\aut(F^n \oplus S)^G$) over all positive integers $n$. The group $\aut(F^n \oplus S)^G$ acts on $\sur(F^n \oplus S, S)^G$ and it follows from Theorem 2.1 that we have a bijection
\[ \sur(F^n \oplus S, S)^G/\aut(F^n \oplus S)^G \cong \VEC_G(B, F^n; S) \]
for each $n$. Therefore, the group $\aut(F^\infty \oplus S)^G$ acts on $\sur(F^\infty \oplus S, S)^G$ and we obtain a bijection
\[ \sur(F^\infty \oplus S, S)^G/\aut(F^\infty \oplus S)^G \cong \VEC_G(B, F^\infty; S). \]

The map $\rho$ applied to $F^n$ instead of $F$ produces a map
\[ \rho^n: \VEC_G(B, F^n; S) \rightarrow (R/I)^*/\Gamma^n \]
for each positive integer $n$. Here $\Gamma^n$ is a subgroup of $(R/I)^*$ defined for $F^n$, and since $\aut(F^n \oplus S)^G$ is a subgroup of $\aut(F^{n+1} \oplus S)^G$, $\Gamma^n$ is a subgroup of $\Gamma^{n+1}$. We define $\Gamma^\infty$ to be the union of $\Gamma^n$ over all positive integers $n$. Then the maps $\rho^n$ induce a map
\[ \rho^\infty: \VEC_G(B, F^\infty; S) \rightarrow (R/I)^*/\Gamma^\infty. \]
We do not know whether $\rho^n$ is bijective for each $n$, but we will prove the following in the next section.

**Theorem 3.1.** The map $\rho^\infty$ is bijective (in fact, a group isomorphism) for any $G$-modules $B, F$ and $S$.

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**4. Proof of Theorem 1.1**

As we did in the previous section for $\sur(F^n \oplus S, S)^G$, we think of $\mor(F^n, S)^G$ (resp. $\mor(S, F^n)^G$) as a subset of $\mor(F^{n+1}, S)^G$ (resp. $\mor(S, F^{n+1})^G$) by defining to be zero on the added factor and denote by $\mor(F^\infty, S)^G$ (resp. $\mor(S, F^\infty)^G$) the union of $\mor(F^n, S)^G$ (resp. $\mor(S, F^n)^G$) over all positive integers $n$. Let $\phi_1, \ldots, \phi_\ell$ be elements
in \( \text{mor}(F^\infty, S)^G \). Then each \( \phi_i \) lies in \( \text{mor}(F^{n_i}, S)^G \) for some positive integer \( n_i \). We define

\[
(\phi_1, \ldots, \phi_k)(v) := \sum_{i=1}^{k} \phi_i(v) \quad \text{for } v \in F,
\]

so that \((\phi_1, \ldots, \phi_k)\) is an element in \( \text{mor}(F^\sum_{i=1}^{k} n_i, S)^G \) and hence in \( \text{mor}(F^\infty, S)^G \).

Since \( \text{mor}(F, S)^G = \text{Mor}(B, \text{Hom}(F, S))^G \) and \( \text{Mor}(B, V)^G \) is finitely generated as an \( \mathcal{O}(B)^G \)-module for any \( G \)-module \( V \) as is well-known, \( \text{mor}(F, S)^G \) is a finitely generated \( \mathcal{O}(B)^G \)-module. Let \( \Phi_1, \Phi_2, \ldots, \Phi_k \) be generators of \( \text{mor}(F, S)^G \) as an \( \mathcal{O}(B)^G \)-module. We set

\[
\Phi := (\Phi_1, \Phi_2, \ldots, \Phi_k) \in \text{mor}(F^k, S)^G \subset \text{mor}(F^\infty, S)^G
\]

and think of it as an element of \( \text{mor}(F^\infty, S)^G \).

**Lemma 4.1.** Any element in the ideal \( I \) is of the form \( \Phi \Psi \) with some \( \Psi \in \text{mor}(S, F^\infty)^G \).

**Proof.** By definition, the ideal \( I \) is generated by elements in \( R = \text{mor}(S, S)^G \) which factors through \( F \). Therefore, any element \( \alpha \) in \( I \) is of the form \( \sum \phi_j \psi_i \) with some \( \phi_j \in \text{mor}(F, S)^G \) and \( \psi_i \in \text{mor}(S, F)^G \). Since \( \Phi_j \)'s are generators of \( \text{mor}(F, S)^G \) as an \( \mathcal{O}(B)^G \)-module, each \( \phi_i \) is a linear combination of \( \Phi_1, \ldots, \Phi_k \) over \( \mathcal{O}(B)^G \). Therefore, \( \alpha = \sum \phi_j \psi_i = \sum_{j=1}^{k} \Phi_j \psi_j \) with some \( \psi_j \in \text{mor}(S, F)^G \) because \( \text{mor}(S, F)^G \) is also an \( \mathcal{O}(B)^G \)-module. This means that if we set \( \Psi = (\Psi_1, \Psi_2, \ldots, \Psi_k) \in \text{mor}(S, F^\infty)^G \), then \( \alpha = \Phi \Psi \). \( \square \)

If \((\phi, T)\) is an element of \( \text{sur}(F^\infty \oplus S, S)^G \), where \( \phi \in \text{mor}(F^\infty, S)^G \) and \( T \in R = \text{mor}(S, S)^G \), then \([T] \) in \( R/I \) is a unit as is observed in Section 2. Conversely, if \( T \) is an element of \( R \) whose image \([T] \) in \( R/I \) is a unit, then there is an element \( Y \) in \( R \) such that \( TY \equiv 1 \mod I \). It follows from Lemma 4.1 that there is \( \Psi \in \text{mor}(S, F^\infty)^G \) such that \( \Phi \Psi + TY = 1 \). This means that the pair \((\Phi, T)\) is an element of \( \text{sur}(F^\infty \oplus S, S)^G \).

We denote \( \ker(\phi, T) \) by \( E_\phi(T) \), and by \( \{E\} \) the element in \( \text{VEC}_G(B, F^\infty; S) \) determined by a \( G \)-vector bundle \( E \). The argument above shows that if \( \{E_\phi(T)\} \) is an element in \( \text{VEC}_G(B, F^\infty; S) \), then so is \( \{E_\Phi(T)\} \).

With this understood we have

**Lemma 4.2.** \( \{E_\phi(T)\} = \{E_\Phi(T)\} \).
Proof. Since \((\phi, T) \in \text{sur}(\mathbf{F}^\infty \oplus \mathbf{S}, \mathbf{S})^G\), there are elements \(\psi \in \text{mor}(\mathbf{S}, \mathbf{F}^\infty)^G\) and \(Y \in R\) such that \(\phi \psi + TY = 1\). Hence we have
\[
(\phi, \Phi, T) \begin{pmatrix} 1 & -\psi \Phi & 0 \\ 0 & 1 & 0 \\ 0 & -Y \Phi & 1 \end{pmatrix} = (\phi, 0, T),
\]
where the square matrix above is in \(\text{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G\). This together with (3.1) shows that \(\{E_{\Phi \oplus \Phi}(T)\} = \{E_{\Phi \oplus 0}(T)\}\). Here \(\{E_{\phi \oplus 0}(T)\} = \{E_{\phi}(T)\}\) because \(E_{\phi \oplus 0}(T)\) is isomorphic to Whitney sum of \(E_{\phi}(T)\) and a certain number of \(\mathbf{F}\). Therefore we have \(\{E_{\phi \oplus \Phi}(T)\} = \{E_{\Phi}(T)\}\). Changing the role of \(\phi\) and \(\Phi\), we obtain \(\{E_{\Phi \oplus \phi}(T)\} = \{E_{\Phi}(T)\}\). Thus, it suffices to prove that \(\{E_{\phi \oplus \Phi}(T)\} = \{E_{\Phi \oplus \phi}(T)\}\), but this follows from the following identity and (3.1):
\[
(\phi, \Phi, T) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (\Phi, \phi, T),
\]
where the square matrix above is in \(\text{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G\). \(\square\)

As noted before Lemma 4.2, we have an element \(\{E_{\Phi}(T)\} \in \text{VEC}_G(B, F^\infty; S)\) for any \(T \in R\) such that \([T] \in (R/I)^*\).

**Lemma 4.3.** If \([T] = [T'] \in (R/I)^*\), then \(\{E_{\Phi}(T)\} = \{E_{\Phi}(T')\}\).

**Proof.** Since \(T \equiv T' \mod I\), there is \(\Psi \in \text{mor}(\mathbf{S}, \mathbf{F}^\infty)^G\) such that \(T' = T + \Phi \Psi\) by Lemma 4.1. Then
\[
(\Phi, T) \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix} = (\Phi, T')
\]
where the square matrix above is in \(\text{aut}(\mathbf{F}^\infty \oplus \mathbf{S})^G\). This together with (3.1) proves the lemma. \(\square\)

Lemma 4.3 tells us that the correspondence \([T] \rightarrow \{E_{\Phi}(T)\}\) yields a well-defined map
\[
\mathcal{V}: (R/I)^* \rightarrow \text{VEC}_G(B, F^\infty; S),
\]
and Lemma 4.2 tells us that \(\mathcal{V}\) is independent of the choice of \(\Phi\) and is surjective.

**Lemma 4.4.** (1) \(\mathcal{V}([1]) = \{\mathbf{F}\}\).
(2) \(\mathcal{V}([T'] [T]) = \mathcal{V}([T']) \oplus \mathcal{V}([T])\) for any \([T'], [T] \in (R/I)^*\).
Proof. (1) Since \((0, 1) \in \text{sur}(F^\infty \oplus S, S)^G\), \(\{E_0(1)\} = \{E_\Phi(1)\}\) by Lemma 4.2. Here \(E_0(1)\) is nothing but \(F\), so statement (1) is proved.

(2) By definition
\[
V([T'])[T]) = V([T'T]) = \{E_\Phi(T')\},
\]
\[
V([T]) = \{E_\Phi(T)\}.
\]
Since \(E_{\Phi}(1) \cong F\) by (1) above, it suffices to prove that
\[
E_\Phi(T'T) \oplus E_\Phi(1) \cong E_\Phi(T') \oplus E_\Phi(T).
\]
Here the left hand side is the kernel of
\[
L := \begin{pmatrix}
\Phi & 0 & T'T & 0 \\
0 & \Phi & 0 & 1
\end{pmatrix} \in \text{sur}(F^\infty \oplus S \oplus S \oplus S)^G
\]
while the right hand side is the kernel of
\[
L' := \begin{pmatrix}
\Phi & 0 & T' & 0 \\
0 & \Phi & 0 & T
\end{pmatrix} \in \text{sur}(F^\infty \oplus S \oplus S \oplus S)^G.
\]
Since \([T] \in (R/I)^*\) and \((R/I)^*\) is a group, there is \(Y \in R\) such that \(TY \equiv YT \equiv 1\) mod \(I\). Set \(P := 1 - YT\) and \(Q := Y(Y - 1)\). Then \(P \equiv 0\) mod \(I\) and \(TQ \equiv Y - 1\) mod \(I\). Observe that
\[
L \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & Y - PQ & P \\
0 & 0 & Y - 1 - (T + P)Q & T + P
\end{pmatrix} = \begin{pmatrix}
\Phi & 0 & T' + p_1 & p_2 \\
0 & \Phi & p_3 & T + p_4
\end{pmatrix},
\]
where \(p_i \in I\), and that
\[
\begin{pmatrix}
\Phi & 0 & T' + p_1 & p_2 \\
0 & \Phi & p_3 & T + p_4
\end{pmatrix} \begin{pmatrix}
1 & 0 & -\Psi_1 & -\Psi_2 \\
0 & 1 & -\Psi_3 & -\Psi_4 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = L',
\]
where \(\Psi_i \in \text{mor}(S, F^\infty)^G\) such that \(p_i = \Phi \Psi_i\) for each \(i\) (such \(\Psi_i\) exists by Lemma 4.1). One can check that the two square matrices above are both in \(\text{aut}(F^\infty \oplus S \oplus S)^G\) by applying elementary operations. This shows that the kernels of \(L\) and \(L'\), which are respectively \(E_\Phi(T'T) \oplus F\) and \(E_\Phi(T') \oplus E_\Phi(T)\), are isomorphic.

Proof of Theorem 1.1. The map \(V: (R/I)^* \to \text{VEC}_G(B, F^\infty; S)\) is surjective as noted before and \((R/I)^*\) is a group. So it follows from Lemma 4.4 that the abelian monoid \(\text{VEC}_G(B, F^\infty; S)\) is actually an abelian group, i.e., any element in \(\text{VEC}_G(B, F^\infty; S)\) has an inverse in it.
It follows from the result of Bass-Haboush mentioned in the introduction that the union of $\text{VEC}_G(B, F^n; S)$ over all $G$-modules $S$ agrees with $\text{VEC}_G(B, F^n)$. Therefore the union of $\text{VEC}_G(B, F^n; S)$ over all $G$-modules $S$ agrees with $\text{VEC}_G(B, F^n)$. Since $\text{VEC}_G(B, F^n; S)$ is a group under Whitney sum, so is $\text{VEC}_G(B, F^n)$.

**Proof of Theorem 3.1.** Any element in $\Gamma^\infty$ is represented by $[A(S, S)]$ for some $A \in \text{aut}(F^{\infty} \oplus S)^G$. Since $(A(F^{\infty}, S), A(S, S))A^{-1} = (0, 1)$, the element $(A(F^{\infty}, S), A(S, S))$ in $\text{sur}(F^{\infty} \oplus S, S)^G$ produces the trivial element in $\text{VEC}_G(B, F^{\infty}; S)$. This shows that $\text{ker} \nu \supset \Gamma^\infty$. On the other hand, the composition $\rho^\infty \nu: (R/I)^* \to (R/I)^*/\Gamma^\infty$ is just the projection, so $\text{ker} \nu \subset \Gamma^\infty$. Thus $\text{ker} \nu = \Gamma^\infty$ and $\nu$ induces an isomorphism $\check{\nu}: (R/I)^*/\Gamma^\infty \to \text{VEC}_G(B, F^{\infty}; S)$. Since $\rho^\infty \check{\nu}$ is the identity and $\check{\nu}$ is an isomorphism, $\rho^\infty$ is also an isomorphism.

5. $\mathbb{C}^*$-action and grading

Since $B$ is a $G$-module, scalar multiplication gives a $\mathbb{C}^*$-action on $B$ commuting with the $G$-action. Keeping this example in mind, we consider a general $\mathbb{C}^*$-action on $B$ commuting with the $G$-action. The $\mathbb{C}^*$-action induces an action on $\text{Mor}(B, V)^G$ and makes it a $\mathbb{C}^*$-module for any $G$-module $V$. In fact, we define $(cf)(x) := f(cx)$ for $c \in \mathbb{C}^*, f \in \text{Mor}(B, V)^G$ and $x \in B$. Then $\text{Mor}(B, V)^G$ decomposes into a direct sum of eigenspaces, i.e.,

$$\text{Mor}(B, V)^G = \bigoplus_{k \in \mathbb{Z}} \text{Mor}(B, V)^G_{(k)},$$

where $\mathbb{C}^*$ acts on $\text{Mor}(B, V)^G_{(k)}$ as scalar multiplication by $k$-th power. Note that

$$\text{Mor}(B, V)^G_{(0)} = \text{Mor}(B, V)^{G \times \mathbb{C}^*} = \text{Mor}(B//\mathbb{C}^*, V)^G.$$

For an element $P \in \text{Mor}(B, V)^G$, we denote by $P_{(k)}$ the degree $k$ homogeneous component of $P$. It is obvious that $\text{sur}(F \oplus S, S)^G$ and $\text{aut}(F \oplus S)^G$, which are respectively subsets of $\text{Mor}(B, \text{Hom}(F \oplus S, S))^G$ and $\text{Mor}(B, \text{Hom}(F \oplus S, F \oplus S))^G$, are invariant under the $\mathbb{C}^*$-actions, so both of them inherit gradings. Moreover, it is obvious that the map from $\text{sur}(F \oplus S, S)^G$ and $\text{aut}(F \oplus S)^G$ to $R$ defined by taking the $(S, S)$-component is $\mathbb{C}^*$-equivariant and hence so is the map $\rho: \text{VEC}_G(B, F; S) \to (R/I)^*/\Gamma$.
The $\mathbb{C}^*$-action makes $\mathcal{O}(B)$ a $\mathbb{C}^*$-module as above. We say that $\mathcal{O}(B)$ is \textit{positively graded} if $\mathcal{O}(B)_{(k)} = 0$ for all $k < 0$. The $\mathbb{C}^*$-actions we will use later are the ones obtained as scalar multiplication on $B$ or on a factor of $B$ when $B$ is a direct sum of two $G$-modules, and $\mathcal{O}(B)$ is positively graded for these actions. The following lemma can easily be checked for them.

\textbf{Lemma 5.1 ([3])}. If $\mathcal{O}(B)$ is positively graded for the $\mathbb{C}^*$-action, then the algebraic quotient map $\pi: B \to B/\mathbb{C}^*$ restricted to the $\mathbb{C}^*$-fixed point set $B_{\mathbb{C}^*}$ gives an isomorphism between $B_{\mathbb{C}^*}$ and $B/\mathbb{C}^*$.

We note that if the grading on $\mathcal{O}(B)$ is positive, then so is the grading on $\text{Mor}(B, V^G)$.

\textbf{Lemma 5.2}. If $\mathcal{O}(B)$ is positively graded by the $\mathbb{C}^*$-action, then $L(0) \in \text{sur}(F \oplus S, S)^{G \times \mathbb{C}^*}$ and $A(0) \in \text{aut}(F \oplus S)^{G \times \mathbb{C}^*}$ for $L \in \text{sur}(F \oplus S, S)^G$ and $A \in \text{aut}(F \oplus S)^G$.

\textit{Proof}. As remarked in the previous section, there is an element $M \in \text{mor}(S, F \oplus S)^G$ such that $LM$ is the identity. Since $(LM)(0) = L(0)M(0)$ (where we use the assumption that our grading is positive) and the identity is of degree zero, it follows that $L(0)M(0)$ is the identity. This shows that $L(0): F \oplus S \to S$ is also surjective. A similar argument shows that $A(0)$ is again an automorphism of $F \oplus S$. \hfill $\square$

It follows from the above lemma that sending $L$ to $L(0)$ induces a correspondence

$\text{sur}(F \oplus S, S)^G/\text{aut}(F \oplus S)^G \to \text{sur}(F \oplus S, S)^{G \times \mathbb{C}^*}/\text{aut}(F \oplus S)^{G \times \mathbb{C}^*}$.

Here the left hand side is identified with $\text{VEC}_G(B, F; S)$ while the right hand side is identified with $\text{VEC}_G(B_{\mathbb{C}^*}, F; S)$ because $\mathcal{O}(B)_{\mathbb{C}^*} = \mathcal{O}(B_{\mathbb{C}^*})$ by Lemma 5.1. Through these identifications, the above map is nothing but the restriction of $G$-vector bundles over $B$ to $B_{\mathbb{C}^*}$.

One can apply the above argument to $F^n$ for each $n$ in place of $F$, so all the statements above hold for $F^n$ in place of $F$.

\section{Analysis of $(R/I)^*/\Gamma^\infty$}

Since the map $\rho^\infty$ is bijective by Theorem 3.1, we are led to study its target group $(R/I)^*/\Gamma^\infty$. Henceforth we assume that $R/I$ is commutative. Suppose that our $\mathbb{C}^*$-action on $B$ commutes with the $G$-action and induces a positive grading on $\mathcal{O}(B)$. Then $R$ has a positive grading and
I becomes a graded ideal in $R$ because it is invariant under the induced $\\mathbb{C}^*$-action on $R$. Therefore $R/I$ inherits the grading from $R$. Since the grading on $R/I$ is positive, the degree zero term of a unit in $R/I$ is again a unit. We denote by $(R/I)^*_0$ the subgroup of $(R/I)^*$ consisting of elements of degree zero. Then we have a decomposition
\[(R/I)^* = (R/I)^*_0 \times (1 + (R/I)_1)^*,\]
where $(R/I)_1$ denotes the set of elements in $R/I$ whose degree zero terms vanish. On the other hand, $\Gamma^\infty_{(0)}$, which is the projection image of $\text{aut}(\\mathcal{F}^\infty \oplus S)^G_{(0)}$, is a subgroup of $\Gamma^\infty$ and we have a decomposition
\[\Gamma^\infty = \Gamma^\infty_{(0)} \times \Gamma^\infty_*,\]
where $\Gamma^\infty_*$ denotes a subgroup of $\Gamma^\infty$ with 1 as the degree zero term. The above two decompositions give rise to the following decomposition
\[(R/I)^*/\Gamma^\infty = (R/I)^*_0/\Gamma^\infty_{(0)} \times (1 + (R/I)_1)^*/\Gamma^\infty_*.
\]
We note that $(R/I)^*_0/\Gamma^\infty_{(0)}$ is the target of the invariant $\rho^\infty$ for $\text{VEC}_G(B/\mathbb{C}^*, F^\infty; S)$ and that $B/\mathbb{C}^*$ can be identified with $B^{\mathbb{C}^*}$ by Lemma 5.2. Therefore the restriction map $\iota^*: \text{VEC}_G(B, F^\infty; S) \to \text{VEC}_G(B^{\mathbb{C}^*}, F^\infty; S)$, where $\iota: B^{\mathbb{C}^*} \to B$ is the inclusion map, corresponds to the projection
\[(R/I)^*/\Gamma^\infty = (R/I)^*_0/\Gamma^\infty_{(0)} \times (1 + (R/I)_1)^*/\Gamma^\infty_* \to (R/I)^*_0/\Gamma^\infty_{(0)},\]
and thus we have

**Lemma 6.1.** If $\text{VEC}_G(B^{\mathbb{C}^*}, F; S)$ consists of one element, then $\text{VEC}_G(B, F^\infty; S)$ is isomorphic to $(1 + (R/I)_1)^*/\Gamma^\infty_*$.

An element $x \in (R/I)_1$ is nilpotent if and only if $1+x \in (1 + (R/I)_1)^*$, (see [1], Exercise 2 in p.11). Therefore we have a logarithmic map
\[\log: (1 + (R/I)_1)^* \to \text{Nil}(R/I)_1\]
where $\text{Nil}(R/I)_1$ denotes the set of nilpotent elements in $(R/I)_1$. $\text{Nil}(R/I)_1$ is an $\mathcal{O}(B)^G$-submodule of $(R/I)_1$ and hence of $R/I$. The map log is an isomorphism, the inverse being an exponential map.

**Lemma 6.2.** $\log \Gamma^\infty_*$ is an $\mathcal{O}(B)^G$-submodule of $\text{Nil}(R/I)_1$.

**Proof.** The groups $(1 + (R/I)_1)^*$ and $\text{Nil}(R/I)_1$ have the $\mathbb{C}^*$-actions and the map log are equivariant with respect to the actions. Therefore, $\log \Gamma^\infty_*$ is a $\mathbb{C}^*$-invariant additive subgroup of $\text{Nil}(R/I)_1$. It follows that if $x$ is an element of $\log \Gamma^\infty_*$, then all its homogeneous terms $x_{(d)}$ lie in
log $\Gamma^\infty_*$. In fact, since $x = \sum_{d=1}^{\infty} x(d)$, where $x(d) = 0$ for sufficiently large $d$, is an element of the $\mathbb{C}^*$-invariant additive subgroup log $\Gamma_*^\infty$, $\sum z^d x(d)$ lies in log $\Gamma_*^\infty$ for any $z \in \mathbb{C}^*$. Suppose that $x(d) = 0$ for all $d > m$ where $m$ is a certain positive integer. Then we take $m$ nonzero different integers for $z$. For those $m$ values of $z$, $\sum z^d x(d)$ lies in log $\Gamma_*^\infty$. Using the non-singularity of Vandermonde matrix and the fact that log $\Gamma_*^\infty$ is an additive group, one sees that $x(d)$'s lie in log $\Gamma_*^\infty$ for all $d$.

In the sequel, it suffices to show that if $x \in \log \Gamma_*^\infty$ is homogeneous, then $fx$ lies again in log $\Gamma_*^\infty$ for any $f \in \mathcal{O}(B)^G$. This can be seen as follows. Since the exponetional map $\exp: \text{Nil}(R/I)_1 \to (1 + (R/I)_1)^*$ is the inverse of log, $\exp(x)$ is an element of $\Gamma_*^\infty$. Remember that an element in $\Gamma_*^\infty$ is the $(S, S)$-component of an element of $\text{aut}(\mathcal{F}^\infty \oplus S)^G$ with 1 as the degree zero term. Suppose that $\exp(x)$ is the $(S, S)$-component of such an element $A = \sum_{d=0}^{\infty} A(d)$ where $A(0) = 1$. Then, $\sum_{d=0}^{\infty} f^d A(d)$ again lies in $\text{aut}(\mathcal{F}^\infty \oplus S)^G$ for $f \in \mathcal{O}(B)^G$. In fact, if $A' = \sum_{d=0}^{\infty} A'(d)$ is the inverse of $A$, then one checks that $\sum_{d=0}^{\infty} f^d A'(d)$ is the inverse of $\sum_{d=0}^{\infty} f^d A(d)$. Taking degrees into account, one sees that the $(S, S)$-component of $\sum_{d=0}^{\infty} f^d A(d)$ is equal to $\exp(fx)$. Therefore $fx$ lies again in log $\Gamma_*^\infty$, proving the lemma.

**Lemma 6.3.** The group $(1 + (R/I)_1)^*/\Gamma_*^\infty$ is isomorphic to a finitely generated $\mathcal{O}(B)^G$-module.

**Proof.** The group $(1 + (R/I)_1)^*/\Gamma_*^\infty$ is isomorphic to $\text{Nil}(R/I)_1/\log \Gamma_*^\infty$ through the map log. As is well known, $R = \text{Mor}(B, \text{Hom}(S, S))^G$ is finitely generated as $\mathcal{O}(B)^G$-module and hence so is the quotient $R/I$. Since the ring $\mathcal{O}(B)^G$ is Noetherian and $\text{Nil}(R/I)_1$ is an $\mathcal{O}(B)^G$-submodule of $R/I$, $\text{Nil}(R/I)_1$ is finitely generated as $\mathcal{O}(B)^G$-module, (see Propositions 6.2 and 6.5 in [1]) and hence so is the quotient $\text{Nil}(R/I)_1/\log \Gamma_*^\infty$. This proves the lemma.

**Proof of Theorem 1.2.** We take the $\mathbb{C}^*$-action on $B$ defined by scalar multiplication. Then $B^{\mathbb{C}^*}$ is a point, that is the origin, so VEC$_G(B^{\mathbb{C}^*}, F^\infty; S)$ consists of one element. Therefore the theorem follows from Lemmas 6.1 and 6.3.

7. **Product formula**

We shall prove Theorem 1.3. We use the notation $R, I$ and $\Gamma^\infty$ for the base space $B$ as before and $\bar{R}, \bar{I}$ and $\bar{\Gamma}^\infty$ for the base space $B \oplus \mathbb{C}^m$. 
Lemma 7.1. \( \bar{R} = R \otimes \mathcal{O}(\mathbb{C}^m) \) and \( \bar{I} = I \otimes \mathcal{O}(\mathbb{C}^m) \).

Proof. As is well known, (7.1) \( \text{Mor}(B, V)^G \) is canonically isomorphic to \( (V \otimes \mathcal{O}(B))^G \)
for any \( G \)-module. In fact, an element \( f \in \text{Mor}(B, V)^G \) induces an equivariant algebra homomorphism \( f^* : \mathcal{O}(V) \to \mathcal{O}(B) \). Since \( V \) is a module, \( \mathcal{O}(V) \) is a symmetric tensor algebra of \( V^* = \text{Hom}(V, \mathbb{C}) \). Therefore, \( f^* \) is determined by its restriction to \( V^* \) and hence \( f^* \) can be identified with an element of \( \text{Hom}(V^*, \mathcal{O}(B))^G = (V \otimes \mathcal{O}(B))^G \). This is the correspondence giving the isomorphism (7.1). Applying (7.1) to \( B \oplus \mathbb{C}^m \) in place of \( B \), we get

\[
\text{Mor}(B \oplus \mathbb{C}^m, V)^G = (V \otimes \mathcal{O}(B \oplus \mathbb{C}^m))^G
= (V \otimes \mathcal{O}(B) \otimes \mathcal{O}(\mathbb{C}^m))^G
= (V \otimes \mathcal{O}(B))^G \otimes \mathcal{O}(\mathbb{C}^m)
= \text{Mor}(B, V)^G \otimes \mathcal{O}(\mathbb{C}^m).
\]

(7.2) Since \( \bar{R} = \text{Mor}(B \oplus \mathbb{C}^m, \text{Hom}(S, S))^G \) and \( R = \text{Mor}(B, \text{Hom}(S, S))^G \), the isomorphism (7.2) applied with \( V = \text{Hom}(S, S) \) proves the first identity in the lemma.

As for the latter identity, we remember that \( I \) is generated by composition of elements in \( \text{mor}(F, S)^G \) and \( \text{mor}(S, F)^G \). Since \( \text{mor}(F, S)^G = \text{Mor}(B, \text{Hom}(F, S))^G \) and \( \text{mor}(S, F)^G = \text{Mor}(B, \text{Hom}(S, F))^G \), the isomorphism (7.2) applied with \( V = \text{Hom}(F, S) \) or \( \text{Hom}(S, F) \) implies the latter identity in the lemma. \( \square \)

Now we consider the \( \mathbb{C}^* \)-action on \( B \oplus \mathbb{C}^m \) defined by scalar multiplication on the factor \( B \). This action commutes with the \( G \)-action on \( B \oplus \mathbb{C}^m \), where the \( G \)-action on \( \mathbb{C}^m \) is trivial, and \( \mathcal{O}(B \oplus \mathbb{C}^m) = \mathcal{O}(B) \otimes \mathcal{O}(\mathbb{C}^m) \) is positively graded by the \( \mathbb{C}^* \)-action, so that we can apply the results in Section 6. Then, since \( (B \oplus \mathbb{C}^m)^{\mathbb{C}^*} = \{0\} \oplus \mathbb{C}^m \) and \( \text{VEC}_G(\mathbb{C}^m, F^\infty; S) \) consists of one element (because any \( G \)-vector bundle over \( \mathbb{C}^m \) is trivial, which follows from the Quillen-Suslin Theorem, see Corollary in p.113 of [7]), we have

\[
(\bar{R}/\bar{I})^*/\bar{\Gamma}^\infty = (1 + (\bar{R}/\bar{I})_1)^*/\bar{\Gamma}_1^*,
\]
and the logarithmic map

\[
\text{log}: (1 + (\bar{R}/\bar{I})_1)^* \to \text{Nil}(\bar{R}/\bar{I})_1
\]

is an isomorphism.

Lemma 7.2. (1) \( \text{Nil}(\bar{R}/\bar{I})_1 = \text{Nil}(R/I)_1 \otimes \mathcal{O}(\mathbb{C}^m) \).
(2) $\log \bar{\Gamma}_s^\infty = \log \Gamma_s^\infty \otimes \mathcal{O}(\mathbb{C}^m)$.

Proof. (1) Since $R/I$ is commutative and $\mathcal{O}(\mathbb{C}^m)$ is a polynomial ring in $m$ variables, it follows from a theorem of E. Snapper (see p.70 in [9]) and Lemma 7.1 that

$\text{Nil}(\bar{\mathbb{R}}/\bar{I}) = \text{Nil}(R/I) \otimes \mathcal{O}(\mathbb{C}^m)$.

Here elements in $\mathcal{O}(\mathbb{C}^m)$ have degree zero with respect to our $\mathbb{C}^*$-action, so the identity in the lemma follows by taking elements whose degree zero terms vanish in (7.3).

(2) Through the projection from $B \oplus \mathbb{C}^m$ on $B$, one can think of $\Gamma_s^\infty$ as a subgroup of $\bar{\Gamma}_s^\infty$, hence $\log \bar{\Gamma}_s^\infty \supset \log \Gamma_s^\infty$. By Lemma 6.2 (applied with $B \oplus \mathbb{C}^m$ in place of $B$), $\log \bar{\Gamma}_s^\infty$ is a module over $\mathcal{O}(B \oplus \mathbb{C}^m)^G = \mathcal{O}(B)^G \otimes \mathcal{O}(\mathbb{C}^m)$. It follows that $\log \bar{\Gamma}_s^\infty \supset \log \Gamma_s^\infty \otimes \mathcal{O}(\mathbb{C}^m)$.

We shall prove the converse inclusion relation. By definition, an element in $\Gamma_s^\infty$ is represented by the $(S, S)$-component of a $G$-vector bundle automorphism $\bar{A}$ of the trivial bundle $(B \oplus \mathbb{C}^m) \times (F \oplus S)$ over $B \oplus \mathbb{C}^m$ such that $\bar{A}$ restricted to $\{0\} \oplus \mathbb{C}^m$ is the identity. Since $\log[\bar{A}(S, S)]$ is contained in $\text{Nil}(\bar{\mathbb{R}}/\bar{I})_1 = \text{Nil}(R/I)_1 \otimes \mathcal{O}(\mathbb{C}^m)$, one can express

$$\log[\bar{A}(S, S)] = \sum_{i=1}^q r_i p_i$$

with $r_i \in \text{Nil}(R/I)_1$ and $p_i \in \mathcal{O}(\mathbb{C}^m)$. We may assume that the polynomials $p_i$’s are linearly independent over $\mathbb{C}$. Then there are points $x_1, \ldots, x_q$ in $\mathbb{C}^m$ such that $q$ vectors $(p_1(x_j), \ldots, p_q(x_j))$ for $j = 1, \ldots, q$ are linearly independent. We consider the restriction of $\bar{A}$ to $B \times \{x_j\}$, denoted by $A_j$, and think of $A_j$ as a $G$-vector bundle automorphism of $B \times (F \oplus S)$. We have that $\log[A_j(S, S)] = \sum_{i=1}^q p_i(x_j) r_i$ and $\log[A_j(S, S)]$ is an element of $\log \Gamma_s^\infty$ for each $j$. It follows that $r_i$ is an element of $\log \Gamma_s^\infty$ for each $i$ because the $q$ vectors $(p_1(x_j), \ldots, p_q(x_j))$ for $j = 1, \ldots, q$ are linearly independent and $\log \Gamma_s^\infty$ is a vector space over $\mathbb{C}$. Therefore, $\log[\bar{A}(S, S)]$ is an element of $\log \Gamma_s^\infty \otimes \mathcal{O}(\mathbb{C}^m)$. Since $A$ is arbitrary, this proves the desired converse inclusion relation. □

Proof of Theorem 1.3. The theorem follows from Theorem 3.1 and Lemma 7.2. □

References


Stable class of equivariant algebraic vector bundles

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