MAPPING PROPERTIES OF THE MARCINKIEWICZ INTEGRALS ON HOMOGENEOUS GROUPS

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Abstract. Under the cancellation property and the Lipschitz condition on kernels, we prove that the Marcinkiewicz integrals defined on a homogeneous group $H$ are bounded from $H^1(H)$ to $L^1(H)$, from $L^\infty_c(H)$ to $BMO(H)$, and from $L^p(H)$ to $L^p(H)$ for $1 < p < \infty$ assuming the $L^q$-boundedness for some $q > 1$.

1. Introduction

Stein [8] defined a higher dimensional analogue of the Marcinkiewicz integral by

$$
\mu_{\Omega} f(x) = \left( \int_0^\infty |F_t(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},
$$

where

$$
F_t(x) = \int_{|y-x|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) \, dy,
$$

$\Omega$ is a homogeneous function of degree zero whose restriction to $S^{n-1}$ belongs to $\Lambda^\alpha (S^{n-1})$ and satisfies the cancellation property,

$$
\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0.
$$

Here, $S^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$ and $\Lambda^\alpha (S^{n-1})$ denotes the Lipschitz space of order $\alpha$ on $S^{n-1}$. The continuity of Marcinkiewicz

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integrals is very useful in harmonic analysis \[9, 10, 15\]. Stein \[8\] proved that if \(\Omega\) is in \(\Lambda^\alpha\left(S^{n-1}\right)\) with \(0 < \alpha \leq 1\), then
\[
\left| \{x \in \mathbb{R}^n : \mu_\Omega f(x) > \lambda \} \right| \leq \frac{C}{\lambda} \|f\|_{L^1}
\]
and
\[
\|\mu_\Omega f\|_{L^p} \leq C_p \|f\|_{L^p},
\]
where \(1 < p \leq 2\), and if \(\Omega\) is an integrable odd function, then
\[
\|\mu_\Omega f\|_{L^p} \leq C_p \|f\|_{L^p}
\]
for \(2 < p < \infty\).

The problem most immediately suggested by Marcinkiewicz \[7\], who conjectured the \(L^p\)-boundedness of (1.1) for \(n = 1\) and for \(\Omega(t) = \text{sign } t\) until Zygmund \[14\] proved that the conjecture holds for \(1 < p < \infty\), has been extensively studied beginning with the 1958's article of Stein \[8\]. Benedek, Calderon and Panzone \[1\] proved that if \(\Omega \in C^1(\mathbb{R}^n \setminus \{0\})\) is a homogeneous function of degree zero satisfying the cancellation property, then \(\mu_\Omega\) is bounded on \(L^p(\mathbb{H})\) for \(1 < p < \infty\). Torchinsky and Wang considered the weighted \(L^p\)-boundedness of \(\mu_\Omega\), and showed that if \(\Omega\) is in \(\Lambda^\alpha\left(S^{Q-1}\right)\) and \(\mu_\Omega\) is bounded on \(L^p(\mathbb{R}^n)\) for \(1 < p < \infty\), then for \(\omega\) satisfying an \(A_p\) condition, \(\mu_\Omega\) is bounded on \(L^p(\omega)\) \[13\]. Further results on (1.1) were obtained when \(\Omega\) satisfies some smoothness conditions \[1\], \[8\] and \[13\].

In this paper, we prove the \(H^1-L^1, L^\infty-BMO\) and \(L^p-L^p\) (\(1 < p < \infty\)) boundedness of Marcinkiewicz integrals defined on homogeneous groups under the cancellation property and the Lipschitz condition on \(\Omega\) and under the \(L^q\)-boundedness of \(\mu_\Omega\).

This paper is organized as follows: in the next section, some preliminary materials are introduced. The main theorem is stated in Section 3. End-point results appear in Sections 4 and 5. Combined with an interpolation argument, the \(L^p\) boundedness for \(1 < p < \infty\) will be shown in Section 6.

2. Preliminaries and notations

In this section, we introduce notations related to homogeneous groups along with some preliminary materials. Mainly, we follow \[6\].
2.1. Homogeneous groups

A nilpotent Lie group $\mathbb{H}$ with a dilation group $\{\delta_r\}_{r>0}$ is said to be a homogeneous group. The dilation group is given by

$$\delta_r = \exp (A \ln r)$$

with a suitable matrix $A$ having positive eigenvalues.

$\mathbb{H}$ has a natural homogeneous norm $| \cdot |$, and the homogeneous dimension $Q$. Abusing the notation, the bi-invariant measure on $\mathbb{H}$ will be denoted by $| \cdot |$.

Remark 2.1. Let $\mathbb{H}$, $\{\delta_r\}$, $| \cdot |$ and $Q$ be as above. The eigenvalues of the matrix defining $\{\delta_r\}$ are listed as $1 = d_1 \leq d_2 \leq \cdots \leq d_n$ and we let $\bar{d} = \max \{d_i : i = 1, \cdots, n\}$.

1. $|\delta_r x| = r|x|$ for each $x \in \mathbb{H}$, $r > 0$.
2. There exist $C_1, C_2 > 0$ such that

$$C_1 \|x\| \leq |x| \leq C_2 \|x\|^{\bar{d}}$$

whenever $|x| \leq 1$.

Here, $\|\cdot\|$ denotes the euclidean norm.

3. There exists a constant $\gamma > 0$ such that for all $x, y \in \mathbb{H}$,

$$|x \circ y| \leq \gamma(|x| + |y|) \text{ for all } x \in \mathbb{H}$$

(2.1)

$$|x \circ y| - |x| \leq \gamma|y| \text{ for all } x, y \in \mathbb{H} \text{ such that } |y| \leq |x|/2.$$  

(2.2)

4. $|\delta_r E| = r^Q|E|$.

5. We let

$$S = \{x \in \mathbb{H} : |x| = 1\}.$$  

There is a unique Radon measure $\sigma$ on $S$ such that for all $f \in L^1(\mathbb{H})$,

$$\int_{\mathbb{H}} f(x) \, dx = \int_{0}^{\infty} \int_{S} f(\delta_r y) r^{Q-1} \, d\sigma(y) \, dr.$$  

2.2. The Hardy space $H^1(\mathbb{H})$

For the definition of the Hardy space $H^1(\mathbb{H})$, we refer the interested readers to [6].
2.2.1. $H_{q,0}^1$-atoms. Let $q \in (1, \infty]$. A function $a(x)$ on $\mathbb{H}$ is said to be an $H_{q,0}^1$-atom (associated to a ball $B$) if it satisfies the following conditions:

(a) $a(x)$ is supported in $\bar{B}$;
(b) $|a(x)| \leq |B|^\frac{1}{q} - 1$ almost everywhere; and
(c) $\int_{\mathbb{H}} a(x) \, dx = 0$.

Remark 2.2. Let $a(x)$ be an $H_{\infty,0}^1$-atom. Then we have

(2.3) \[ \|a\|_{L^1(\mathbb{H})} \leq 1. \]

2.3. Atomic decomposition

An equivalent way of looking at $H^1(\mathbb{H})$ is the decomposition of elements in $H^1(\mathbb{H})$ into $H_{q,0}^1$-atoms.

Theorem 2.1 (Decomposition Theorem). Let $q \in (1, \infty]$. For $f \in H^1(\mathbb{H})$ there exist a collection of $H_{q,0}^1$ atoms $\{a_k\}_{k \in \mathbb{N}}$ and a sequence of nonnegative real numbers $\{\lambda_k\}_{k \in \mathbb{N}}$ with $\sum_{k=1}^{\infty} \lambda_k < \infty$ so that

\[ f = \sum_{k=1}^{\infty} \lambda_k a_k \]

in the sense of distributions, and we have

\[ \|f\|_{H^1} \approx \sum_{k=1}^{\infty} \lambda_k. \]

2.4. BMO

Definition 2.3. A locally integrable function $f : \mathbb{H} \to \mathbb{C}$ is said to be in $BMO$ if there exists a constant $C$ such that for each ball $B$

\[ \frac{1}{|B|} \int_{B} |f(x) - f_B| \, dx \leq C, \]

holds, where

\[ f_B = \frac{1}{|B|} \int_{B} f(x) \, dx. \]
3. Marcinkiewicz integrals

Let $\Omega$ be a measurable function on a homogeneous group $\mathbb{H}$, which is homogeneous of degree 0 in the sense that

$$\Omega(\delta_r x) = \Omega(x)$$

holds for a.e. $x \in \mathbb{H} \setminus \{0\}$ and $r > 0$. We define the Marcinkiewicz integral $\mu f$ as follows:

$$(3.1) \quad \mu f(x) = \left( \int_0^\infty |F_t(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where

$$F_t(x) = \int_{B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}| Q_{-1} f(y)} dy.$$

We will study the mapping properties of $\mu f$. To be more specific, we will prove:

**Theorem 3.1.** Let $\mathbb{H}$, $\Omega$ and $\mu f$ be as above. We assume the following:

- $\Omega|_S \in \Lambda^\alpha (S)$;
- $\int_S \Omega(x') \; d\sigma(x') = 0$; and
- $\mu f$ is bounded in $L^q (\mathbb{H})$ for some $q > 1$.

Then the following inequalities hold:

$$\| \mu f \|_{L^1} \leq C_1 \| f \|_{H^1}, \quad f \in H^1 (\mathbb{H})$$

$$\| \mu f \|_{BMO} \leq C_\infty \| f \|_{L^\infty}, \quad f \in L^\infty_\infty (\mathbb{H})$$

and

$$\| \mu f \|_{L^p} \leq C_p \| f \|_{L^p}, \quad f \in L^p (\mathbb{H})$$

for $1 < p < \infty$.

4. $H^1-L^1$ boundedness

In this section, we establish $H^1-L^1$ boundedness of the Marcinkiewicz integral. In view of Theorem 2.1 and the sublinearity of $\mu f$, it suffices to verify the inequality (3.2) when $f$ is an arbitrary $H^1_{\infty,0}$-atom. Let $a(x)$
be an $H^1_{\infty,0}$-atom supported in $B_r(x_0)$. We split the integral into two parts,
\[
\int_{\mathbb{H}} \mu_{\Omega} a(x) \, dx = \int_{B_{2\lambda r}(x_0)} \mu_{\Omega} a(x) \, dx + \int_{\mathbb{H}\setminus B_{2\lambda r}(x_0)} \mu_{\Omega} a(x) \, dx \equiv I + II.
\]

4.1. Estimation on $I$

By hypothesis, we have
\[
(4.1) \int_{B_{2\lambda r}(x_0)} |\mu_{\Omega} a(x)|^q \, dx \lesssim \int_{B_r(x_0)} |a(x)|^q \, dx \lesssim |B_r(x_0)|^{-q+1}.
\]
By Hölder’s inequality and (4.1), we obtain
\[
I \leq \left( \int_{B_{2\lambda r}(x_0)} |\mu_{\Omega} a(x)|^q \, dx \right)^{\frac{1}{q}} \left( |B_{2\lambda r}(x_0)|^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \lesssim 1.
\]

4.2. Estimation on $II$

Before we proceed, we introduce some simple facts on balls in $\mathbb{H}$.

**Definition 4.1.** For $E \subset \mathbb{H}$ and $x \notin E$, we will use the following notation.
\[
d(x,E) = \inf \{|x \circ y^{-1}| : y \in E\}.
\]

**Lemma 4.2.** Let $x \notin B_{2\lambda r}(x_0)$. Then we have
\[
d(x, B_r(x_0)) \geq r.
\]
**Proof.** Suppose
\[
d(x, B_r(x_0)) < r.
\]
Then there exists $y \in B_r(x_0)$ such that $|y \circ x^{-1}| < r$. So we get
\[
|x \circ x_0^{-1}| \leq \lambda \left( |x \circ y^{-1}| + |y \circ x_0^{-1}| \right) < 2\lambda r.
\]
A contradiction to $x \notin B_{2\lambda r}(x_0)$.

**Lemma 4.3.** Let $x \notin B_{2\lambda r}(x_0)$ and $y \in B_r(x_0)$. Then we have
\[
|x \circ y^{-1}| \leq 2\lambda |x \circ x_0^{-1}| \leq 4\lambda^2 |x \circ y^{-1}|.
\]
Proof. Observe that
\[
|x \circ y^{-1}| \leq \lambda (|x \circ x_0^{-1}| + |x_0 \circ y^{-1}|)
\]
\[
\leq 2\lambda |x \circ x_0^{-1}|
\]
\[
\leq 2\lambda^2 (|x \circ y^{-1}| + |y \circ x_0^{-1}|)
\]
\[
\leq 4\lambda^2 |x \circ y^{-1}|
\]
from
\[
|x \circ y^{-1}| \geq d(x, B_r(x_0)) \geq \rho \geq |y \circ x_0^{-1}|
\]

Lemma 4.4. Let \(x \notin B_{2\lambda r}(x_0)\) and \(t < d(x, B_r(x_0))\). Then we have
\[B_r(x_0) \cap B_t(x) = \emptyset.\]

Proof. If \(y \in B_r(x_0)\), then we have
\[
|y \circ x^{-1}| \geq d(x, B_r(x_0)) \geq t.
\]

Also, observe the following fact.

Fact 4.5. There exist constants \(C > 0\), \(\varepsilon \in (0,1)\) and \(\rho > 0\) such that
\[
|\delta_s x \circ x^{-1}| \leq C|1-s|^{\rho}|x|
\]
uniformly in \(x \in \mathbb{H}\) and \(|1-s| < \varepsilon\).

Fix \(x \in \mathbb{H} \setminus B_{2\lambda r}(x_0)\). Then, by Lemma 4.2 we have
\[
d(x, B_r(x_0)) \leq |x \circ x_0^{-1}| \leq 2\lambda d(x, B_r(x_0)).
\]

We have
\[
\mu(a(x))^2 = \int_{d(x, B_r(x_0))}^{\infty} \left( \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} a(y) \, dy \right)^2 \frac{dt}{t^3}
\]
\[
= \int_{d(x, B_r(x_0))}^{\infty} \left( \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega(x \circ y^{-1})}{t |x \circ y^{-1}|^{Q-1}} a(y) \, dy \right) \cdot J_t a(x) \, \frac{dt}{t^2},
\]
where
\[
J_t a(x) = \left( \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} a(y) \, dy \right).
\]
Lemma 4.6. Let $\Omega$, $a$, and $B_r(x_0)$ be as above. Then we have
\[
\left| \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega (x \circ y^{-1})}{t \| x \circ y^{-1} \|^{Q-1}} a(y) \, dy \right| \lesssim Ma(x),
\]
whenever $t > d(x, B_r(x_0))$, $x \in \mathbb{H} \setminus B_{2\lambda r}(x_0)$.

Proof. There are two cases.
Case 1. $t \leq 2d(x, B_r(x_0))$. From
\[
\left| \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega (x \circ y^{-1})}{t \| x \circ y^{-1} \|^{Q-1}} a(y) \, dy \right| \lesssim \frac{\| \Omega \|_{\infty}}{t} \int_{B_t(x)} |a(y)| \, dy
\]
we have
\[
\lesssim \frac{\| \Omega \|_{\infty}}{t^Q} \int_{B_t(x)} |a(y)| \, dy \lesssim Ma(x),
\]
which completes the proof.

Case 2. $t \geq 2(d(x, B_r(x_0)))$. From $B_r(x_0) \cap B_t(x) \subset B_{d(x, B_r(x_0))}(x)$, we have
\[
\left| \int_{B_r(x_0) \cap B_t(x)} \frac{\Omega (x \circ y^{-1})}{t \| x \circ y^{-1} \|^{Q-1}} a(y) \, dy \right| \lesssim \frac{\| \Omega \|_{\infty}}{d(x, B_r(x_0))^Q} \int_{B_{d(x, B_r(x_0))}(x_0)} |a(y)| \, dy
\]
\[
\lesssim \frac{\| \Omega \|_{\infty}}{d(x, B_r(x_0))^Q} \int_{B_{d(x, B_r(x_0))}(x_0)} |a(y)| \, dy \lesssim Ma(x),
\]
which completes the proof.

For $J_t a(x)$ we have the following:

Lemma 4.7. With $\Omega$, $a$, and $B_r(x_0)$ as above,
\[
J_t a(x) \lesssim \begin{cases} 
\quad tMa(x) & \text{if } d(x, B_r(x_0)) \leq t \leq \lambda (d(x, B_r(x_0)) + 2\lambda r) \\
\quad t^\nu |x \circ x_0^{-1}|^{-Q+1-\nu} & \text{if } t \geq \lambda (d(x, B_r(x_0)) + 2\lambda r)
\end{cases}
\]
for any $x \in \mathbb{H} \setminus B_{2\lambda r}(x_0)$, where $\nu = \min \{\alpha, \rho \alpha, 1\}$. 
Marcinkiewicz integrals: homogeneous groups

**Proof.** We have two cases.

**Case 1.** \( t \geq \lambda (d(x, B_r(x_0)) + 2\lambda r) \).

From \( B_r(x_0) \subset B_t(x) \), \( \int_{B_t(x_0)} a(y) \, dy = 0 \) and the Lipschitz condition on \( \Omega \), we get

\[
J_t a(x) = \left| \int_{B_t(x_0) \cap B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} a(y) \, dy \right|
\]

Notice the following:

\[
\left| \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} - \frac{\Omega(x \circ x_0^{-1})}{|x \circ x_0^{-1}|^{Q-1}} \right|
\]

\[
+ \left| \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} - \frac{1}{|x \circ y^{-1}|^{Q-1}} \right| \cdot \left| \frac{1}{|x \circ y^{-1}|^{Q-1}} - \frac{1}{|x \circ x_0^{-1}|^{Q-1}} \right|
\]

\[
\lesssim \left| \frac{\delta_{[|x \circ y^{-1}|^{-1}]} (x \circ y^{-1}) \circ \delta_{[|x \circ x_0^{-1}|^{-1}]} (x \circ x_0^{-1})^{-1}}{|x \circ x_0^{-1}|^{Q-1}} \right|^{\alpha} + \frac{|y \circ x_0^{-1}|}{|x \circ x_0^{-1}|^{Q}}.
\]

From (2.1), we obtain

\[
\lesssim \left| \frac{\delta_{[|x \circ y^{-1}|^{-1}]} (x \circ y^{-1}) \circ \delta_{[|x \circ x_0^{-1}|^{-1}]} (x \circ x_0^{-1})^{-1}}{|x \circ x_0^{-1}|^{Q-1}} \right|^{\alpha} + \frac{|y \circ x_0^{-1}|}{|x \circ x_0^{-1}|^{Q}}
\]

\[
\lesssim \left| \frac{r}{|x \circ y^{-1}|} + \left( 1 - \frac{|x \circ x_0^{-1}|}{|x \circ y^{-1}|} \right)^{\rho} \frac{|x \circ x_0^{-1}|}{|x \circ x_0^{-1}|^{Q}} \right| + \frac{r^\rho}{|x \circ x_0^{-1}|^{\rho}}.
\]
and so,

\[
\left| \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} - \frac{\Omega(x \circ x_0^{-1})}{|x \circ x_0^{-1}|^{Q-1}} \right| \lesssim \frac{r^\alpha}{|x \circ x_0^{-1}|^{Q+\alpha-1}} + \frac{r^{\rho \alpha}}{|x \circ x_0^{-1}|^{Q+\rho \alpha-1}} + \frac{r}{|x \circ x_0^{-1}|^Q}
\]

since \( |x \circ x_0^{-1}| \geq 2\lambda r \). Therefore we obtain

\[
J_t a(x) \lesssim \frac{r^{\nu}}{|x \circ x_0^{-1}|^{Q-1+\nu}}.
\]

Case 2. \( t \leq \lambda(d(x, B_r(x_0)) + 2\lambda r) \).

Since \( t \sim d(x, B_r(x_0)) \), we get

\[
J_t a(x) \leq \left\| \Omega \right\| \int_{B_t(x)} \frac{1}{|x \circ y^{-1}|^{Q-1}} |a(y)| \, dy \\
\lesssim \frac{1}{t^{Q-1}} \int_{B_t(x)} |a(y)| \, dy \\
\lesssim t M a(x).
\]

The proof of Lemma 4.7 is completed. \( \square \)

Thus, for \( x \in \mathbb{H} \setminus B_{2\lambda r}(x_0) \),

\[
\mu_{\Omega} a(x)^2 \lesssim \left( \int_{d(x, B_r(x_0))}^{\infty} J_t a(x) \, dt \right)^2 : Ma(x)
\]

\[
= \left( \int_{d(x, B_r(x_0))}^{\lambda(d(x, B_r(x_0)) + 2\lambda r)} J_t a(x) \, dt \right)^2 \\
+ \int_{\lambda(d(x, B_r(x_0)) + 2\lambda r)}^{\infty} J_t a(x) \, dt \right)^2 : Ma(x)
\]
and so we obtain
\[
\mu_{\Omega a}(x) \lesssim \frac{r^{1/2} \cdot |Ma(x)|}{|x \circ x_0|^{3/4}} + \frac{r^{\nu/2} \cdot |Ma(x)|^{3/4}}{|x \circ x_0|^{Q+\nu/2}}.
\]

Pick \( p_1, p_2, q_1, \) and \( q_2 \) with the following properties:
\begin{itemize}
  \item \( 2Q < p_1 < \infty; \)
  \item \( \frac{2Q}{p_1} < p_2 < 2; \)
  \item \( \frac{1}{p_1} + \frac{1}{q_1} = 1; \) and
  \item \( \frac{1}{p_2} + \frac{1}{q_2} = 1. \)
\end{itemize}

From Hölder’s inequality, the Maximal theorem, and (2.3), we obtain
\[
r^{1/2} \int_{\mathbb{H} \setminus B_{2\lambda r}(x_0)} \frac{|Ma(x)|}{|x \circ x_0|^{1/2}} \, dx
\]
\[
\lesssim r^{1/2} \left( \int_{\mathbb{H} \setminus B_{2\lambda r}(x_0)} \frac{|x \circ x_0|^{-1/4}}{d\rho} \, dx \right)^{1/2} \|Ma\|_{q_1}
\]
\[
\lesssim r^{1/2} \left( \int_{2r}^{\infty} \rho^{-\frac{p_1}{2}+Q-1} \, d\rho \right)^{1/2} \|a\|_{q_1}
\]
\[
\lesssim 1
\]
and
\[
r^{\nu/2} \int_{\mathbb{H} \setminus B_{2\lambda r}(x_0)} \frac{|Ma(x)|^{3/4}}{|x \circ x_0|^{Q+\nu/2}} \, dx
\]
\[
\lesssim r^{\nu/2} \left( \int_{\mathbb{H} \setminus B_{2\lambda r}(x_0)} \frac{|x-x_0|^{-(Q+\nu)p_2/2}}{d\rho} \, dx \right)^{1/2} \|Ma\|_{q_2}
\]
\[
\lesssim r^{\nu/2} \left( \int_{2r}^{\infty} \rho^{-\frac{(Q+\nu)p_2}{2}+Q-1} \, d\rho \right)^{1/2} \|a\|_{q_2}
\]
\[
\lesssim 1.
\]
This shows

$$II \lesssim 1.$$  

Altogether, we obtain

$$\int_H \mu_\Omega a(x) \, dx \lesssim 1$$

and the proof is complete. \hfill \Box

5. \textit{L}^\infty-\textit{BMO boundedness}

In this section, we study the \textit{L}^\infty-\textit{BMO} boundedness of the Marcinkiewicz integrals. Let $f \in L^\infty(\mathbb{H})$ be compactly supported and let $B_r(x_0)$ be any ball. We write

$$f = f \chi_{B_{2\lambda r}(x_0)} + f \chi_{\mathbb{H} \setminus B_{2\lambda r}(x_0)} \equiv f_1 + f_2.$$  

Then $f_1 \in L^2(\mathbb{H})$. By Hölder’s inequality and by hypothesis,

$$\int_{B_r(x_0)} \mu_\Omega f_1(x) \, dx \leq |B_r(x_0)|^{1/3} \left( \int_{B_r(x_0)} [\mu_\Omega f_1(x)]^q \, dx \right)^{1/q} \lesssim |B_r(x_0)|^{1/3} \|f_1\|_{L^q(\mathbb{H})} \lesssim |B_r(x_0)|^{1/3} |B_{2\lambda r}(x_0)|^{1/3} \|f_1\|_{L^\infty} \lesssim |B_r(x_0)| \|f\|_{L^\infty}.$$  

Thus we obtain

$$1 \left| B_r(x_0) \right| \int_{B_r(x_0)} \mu_\Omega f_1(x) \, dx \lesssim \|f\|_{L^\infty}.$$  

For $x \in B_r(x_0)$ we have $|x \circ y^{-1}| > r$ whenever $y \in \mathbb{H} \setminus B_{2\lambda r}(x_0)$. Let $x \in B_r(x_0)$ and let

$$III = \left( \int_0^\infty |F_t(x) - F_t(x_0)|^2 \frac{dt}{t^3} \right)^{1/2}.$$
Then we have

\[ III = \left( \int_0^\infty \left| \int_{B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} - \frac{\Omega(x_0 \circ y^{-1})}{|x_0 \circ y^{-1}|^{Q-1}} \right| f_2(y) \, dy \right)^2 \frac{dt}{t^3} \]

\[ = \left( \int_r^\infty \left| \int_{B_{t}(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} - \frac{\Omega(x_0 \circ y^{-1})}{|x_0 \circ y^{-1}|^{Q-1}} \right| f_2(y) \, dy \right)^2 \frac{dt}{t^3} \]

\[ \lesssim \left( \int_r^\infty \left( \int_{B_{2t\lambda}(x_0)\setminus B_{2\lambda r}(x_0)} \frac{r^\nu}{|y-x_0|^{Q-1+\nu}} \, dy \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}} \|f_2\|_{L^\infty} \]

\[ \lesssim \left( \int_r^\infty \left( \int_{2\lambda r}^{2\lambda t} \frac{r^\nu}{s^{Q-1+\nu}} \, ds \right)^{\frac{2}{3}} \frac{dt}{t} \right)^{\frac{1}{2}} \|f_2\|_{L^\infty} \]

A triangle inequality provides us

\[ |\mu_\Omega f(x) - \mu_\Omega f_2(x_0)| \leq \mu_\Omega f_1(x) + III, \]

which verifies

\[ \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\mu_\Omega f(x) - \mu_\Omega f_2(x_0)| \, dx \lesssim \|f\|_{L^\infty}. \]

This proves the $L^\infty_c$-$BMO$ boundedness.

6. $L^p$-boundedness

From $H^1$-$L^1$ boundedness and the $L^q$-boundedness, it is clear that $\mu_\Omega$ is bounded in $L^p$ for $1 < p \leq q$. To prove the $L^p$-boundedness for $q < p < \infty$, we define Stein’s linearizing function $\varphi(x,t)$ which is a function defined for $x \in \mathbb{H}$, $0 < t < \infty$, so that it satisfies the conditions:

(a) $\varphi$ vanishes if $t$ is small enough, or if $t$ is large enough and is bounded.

(b) For all $x$,

\[ \int_0^\infty |\varphi(x,t)|^2 \frac{dt}{t^3} \leq 1 \]

holds.
Now we define
\[ T\phi f(x) = \int_0^\infty \int_{B_t(x)} \frac{\Omega(x \circ y^{-1})}{|x \circ y^{-1}|^{Q-1}} f(y) \, dy \varphi(x,t) \, \frac{dt}{t^3}. \]

By (a), (b) and by Hölder’s inequality,
\[ |T\phi f(x)| \leq \mu_\Omega f(x) \quad \text{and} \quad \mu_\Omega f(x) = \sup_{\varphi} |T\varphi f(x)| \]
for all \( \varphi \) satisfying (a) and (b) of the above.

It can be shown that \( T\varphi \) is bounded from \( L^\infty_c(\mathbb{H}) \) to \( BMO(\mathbb{H}) \) with uniform bounds for those \( \varphi \) with the above properties. An interpolation yields the \( L^p \) boundedness for \( p \in (q, \infty) \) of \( T\varphi \) uniformly in \( \varphi \) with above properties, which implies the \( L^p \) boundedness of \( \mu_\Omega \).

References
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