A COMPLETENESS ON GENERALIZED FIBONACCI SEQUENCES

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ABSTRACT.

1. Introduction

Let \( V = (v_1, v_2, \cdots) \) be a sequence of positive integers arranged in non-decreasing order. We define \( V \) to be complete if every positive integer \( n \) is the sum of some subsequence of \( V \), that is,

\[
(1.1) \quad n = \sum_{i=1}^{\infty} a_i v_i \quad \text{where} \quad a_i = 0 \text{ or } 1.
\]

For example, let the Fibonacci sequence be denoted by \( F \):

\[
F = (F_1, F_2, F_3, \cdots) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots),
\]

where \( F_0 = 0 \), \( F_1 = F_2 = 1 \), and, for \( n > 2 \), \( F_n = F_{n-1} + F_{n-2} \).

John L. Brown, Jr. gave a criterion for completeness in 1961:

BROWN’S CRITERION. A sequence \( V \) is complete if and only if

(i) \( v_1 = 1 \) and

(ii) for all \( n = 1, 2, \cdots \)

\[
s_{n-1} = v_1 + v_2 + \cdots + v_{n-1} \geq v_n - 1.
\]

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COROLLARY A. If $v_1 = 1$ and $v_{n+1} \leq 2v_n$, then $V$ is complete.

From the above corollary, we can easily show that, the well known fact, the Fibonacci sequence is complete.

A sequence $V$, although unable to produce some numbers at the beginning, might be able to generate all numbers beyond some point $N$. Such sequence, we shall say that they are weakly complete in contrast to the strongly complete sequences which are capable of generating all positive integers.

Now, we define the $k$-generalized Fibonacci sequence $\{g_n^{(k)}\}$ as follows:

\[
g_1^{(k)} = g_2^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = g_k^{(k)} = 1, \quad \text{and for } n > k \geq 2
\]
\[
g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.
\]

The above number $g_n^{(k)}$ is called the $n$th $k$-generalized Fibonacci number. Let $G^{(k)} = (1, 1, 2, 4, 8, 16, 32, \cdots)$ for $k$-generalized Fibonacci sequence $\{g_n^{(k)}\}$. For example, if $k = 7$, then $g_1^{(7)} = \cdots = g_5^{(7)} = 0$, $g_6^{(7)} = g_7^{(7)} = 1$, and then the sequence of 7-generalized Fibonacci numbers is

\[
G^{(7)} = (1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, 15808, \cdots).
\]

The permanent of an $n$-square matrix $A = [a_{ij}]$ is defined by

\[
\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},
\]

where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$. A matrix is said to be a $(0,1)$-matrix if each of its entries is either 0 or 1.

Let $F^{(n,k)} = [f_{ij}]$ be the $n \times n$ $(0,1)$-$(k+1)$st super diagonal matrix defined
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by

$$F^{(n,k)} = \begin{bmatrix}
1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix},$$

(1.4)

where \(f_{11} = \cdots = f_{1k} = 1\) and \(f_{1k+1} = \cdots = f_{1n} = 0\).

In [3], the author determined the relationship between \(g^{(k)}_n\) and the permanent of the \(n \times n\) \((0,1)-(k+1)\)st super diagonal matrix \(F^{(n,k)}\) by using the matrix contraction.

**Theorem 1.1.** [3] Let \(g^{(k)}_{n+1}\) be the \((n+1)\)st \(k\)-generalized Fibonacci number, \(n \geq k\). Then

$$\text{per } F^{(n,k)} = g^{(k)}_{n+k-1}.$$  

(1.5)

And, in [3], the next theorem is a matrix which is not a tridiagonal matrix whose permanent equals to the \((n+1)\)st Fibonacci number.

**Theorem 1.2.** [3] Let

$$U = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 0 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 1
\end{bmatrix}_{n \times n},$$

(1.6)

Then

$$\text{per } P^T U P = F_{n+1},$$

(1.7)

for any permutation matrix \(P\).
2. Completeness on $G^{(k)}$

Now, we consider the completeness on $G^{(k)}$. Let

$$G^{(k)} = (1, 1, 2, 4, 8, \ldots) = (g_1^{(k)}, g_2^{(k)}, g_3^{(k)}, \ldots)$$

be the $k$-generalized Fibonacci sequence, $k \geq 2$. The following theorem shows the completeness on $k$-generalized Fibonacci sequences.

**Theorem 2.1.** The sequence $G^{(k)}$ is complete, $k \geq 2$.

*Proof.* Clearly $g_1^{(k)} = 1$. We proceed by induction on $n$, if $n = 2$, then $g_2^{(k)} = 1 \leq 2g_1^{(k)}$. Assume true for $n - 1$, $g_{n-1}^{(k)} \leq 2g_{n-2}^{(k)}$, and consider for $n$.

$$
\begin{align*}
(g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}) + g_{n-k+1}^{(k)} & \\
\leq g_{n-1}^{(k)} + g_{n-2}^{(k)} + (g_{n-3}^{(k)} + \cdots + g_{n-k-1}^{(k)}) & \\
\iff g_{n}^{(k)} + g_{n-k+1}^{(k)} & \\
= g_{n-1}^{(k)} + g_{n-2}^{(k)} + g_{n-k+1}^{(k)} & \\
= 2g_{n-1}^{(k)} + g_{n-k+1}^{(k)}.
\end{align*}
$$

Since $g_{n-k+1}^{(k)} \geq g_{n-k-2}^{(k)}$, $g_n^{(k)} \leq 2g_{n-1}^{(k)}$. Therefore, by Corollary A, $G^{(k)}$ is complete.

In the Fibonacci sequence $F$, $F - F_r$ is complete. What can we say about the completeness on $G^{(k)} - g_r^{(k)}$? The following theorem is an answer for that.

**Theorem 2.2.** Let $k \geq 3$. $G^{(k)} - g_r^{(k)}$ is complete for $0 < r \leq 2$, and $G^{(k)} - g_r^{(k)}$ is not complete for $r \geq 3$.

*Proof.* Suppose that $0 < r \leq 2$. Since $g_1^{(k)} = g_2^{(k)}$, without loss of generality, let $r = 1$. Then $G^{(k)} - g_r^{(k)} = (g_2^{(k)}, g_3^{(k)}, \ldots)$ and the first term in $G^{(k)} - g_r^{(k)}$ is still 1. If $n \leq k$, then $g_2^{(k)} + g_3^{(k)} + \cdots + g_n^{(k)} = g_{n+1}^{(k)}$. So, $g_2^{(k)} + g_3^{(k)} + \cdots + g_n^{(k)} \geq g_{n+1}^{(k)} - 1$. Thus $G^{(k)} - g_r^{(k)}$ is complete. Now suppose that $n > k$. By induction on $n$,

$$g_2^{(k)} + g_3^{(k)} + \cdots + g_{k+1}^{(k)} = g_{k+2}^{(k)} \geq g_{k+2}^{(k)} - 1.$$
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Assume true for \( n \), i.e., \( g_2^{(k)} + \cdots + g_n^{(k)} \geq g_{n+1}^{(k)} - 1 \). Then

\[
g_2^{(k)} + \cdots + g_n^{(k)} + g_{n+1}^{(k)} \geq 2g_{n+1}^{(k)} - 1
\]

\[
= g_{n+1}^{(k)} + (g_n^{(k)} + g_{n-1}^{(k)} + \cdots + g_{n-k+1}^{(k)}) - 1
\]

\[
= g_{n+2}^{(k)} + g_{n-k+1}^{(k)} - 1
\]

\[
\geq g_{n+2}^{(k)} - 1.
\]

Thus, if \( 0 < r \leq 2 \), then \( G^{(k)} - g_r^{(k)}, k \geq 3 \), is complete.

We now suppose that \( r \geq 3 \). First, we prove that \( g_1^{(k)} + g_2^{(k)} + \cdots + g_{r-1}^{(k)} < g_{r+1}^{(k)} - 1 < g_r^{(k)} \) for all \( r \geq 3 \). Assume that \( 3 \leq r \leq k \). Then, clearly,

\[
g_1^{(k)} + g_2^{(k)} + \cdots + g_{r-1}^{(k)} = g_r^{(k)} < g_{r+1}^{(k)} - 1 < g_r^{(k)}.
\]

Now, assume that \( r > k \). By induction on \( r \), clearly,

\[
g_1^{(k)} + g_2^{(k)} + \cdots + g_k^{(k)} = g_{k+1}^{(k)} < g_{k+2}^{(k)} - 1 < g_{k+2}^{(k)}.
\]

Assume true for \( r \) and consider \( r + 1 \). That is,

\[
g_1^{(k)} + \cdots + g_{r-1}^{(k)} + g_r^{(k)} < g_r^{(k)} + g_{r+1}^{(k)} - 1
\]

\[
= g_{r-1}^{(k)} + g_{r-2}^{(k)} + \cdots + g_{r-k+1}^{(k)} + g_{r+1}^{(k)} - 1
\]

\[
= g_{r+1}^{(k)} + g_r^{(k)} + g_{r-1}^{(k)} + \cdots + g_{r-k}^{(k)} - 1 - g_r^{(k)}
\]

\[
= g_{r+2}^{(k)} - 1 - g_{r-k}^{(k)}
\]

\[
< g_{r+2}^{(k)} - 1.< g_{r+2}^{(k)}.
\]

Thus, in any cases, \( g_1^{(k)} + \cdots + g_{r-1}^{(k)} < g_{r+1}^{(k)} - 1 < g_r^{(k)} \) for all \( r \geq 3 \). This result is same that \( g_{r+1}^{(k)} - 1 \) is unattainable as a sum of terms in a subsequence of \( G^{(k)} - g_r^{(k)}, r \geq 3 \). Therefore, \( G^{(k)} - g_r^{(k)}, r \geq 3 \), is not complete.

In the Fibonacci sequence \( F \), \( F - F_r - F_s \) is not even weakly complete. In fact, \( G^{(k)} - g_r^{(k)} - g_s^{(k)} \), \( s < r \), is not complete. For example, if \( r = 2 \) and \( s = 1 \), then \( G^{(k)} - g_2^{(k)} - g_1^{(k)} \) does not contains 1. Thus \( G^{(k)} - g_r^{(k)} - g_s^{(k)} \) can never be complete. Then will \( G^{(k)} - g_r^{(k)} - g_s^{(k)} \) be good enough to be weakly complete? An answer can be given as following:
THEOREM 2.3. The sequence $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$ is not weakly complete, $k \geq 2$.

Proof. If $k = 2$, then the proof is completed. Now, suppose that $k \geq 3$. We easily see that the number $g_{2k+1}^{(k)} + 1$ is unattainable as a sum of terms in a subsequence of $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$. That is,

$$g_{1}^{(k)} + \cdots + g_{k}^{(k)} + g_{k+2}^{(k)} + \cdots + g_{2k}^{(k)} = g_{k+1}^{(k)} + g_{k+2} + \cdots + g_{2k}^{(k)} < g_{2k+1}^{(k)} + 1 < g_{2k+2}^{(k)}.$$

We will use this result as the basis of an induction that the number $n = g_{2k+1}^{(k)} + g_{k+1}^{(k)} + 1$ is unattainable for all $t = 1, 2, 3, \cdots$. The result is established, then, for $t = 1$. That is, the number $g_{2k+1}^{(k)} + g_{k+1}^{(k)} + 1$ is unattainable as a sum of a subsequence of $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$. Suppose, for some value $t \geq 1$, that the number $n = g_{2k+1}^{(k)} + g_{k+1}^{(k)} + 1$ is unattainable as a sum of a subsequence of $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$. Consider now the number $g_{2k+1}^{(k)} + g_{k(t+1)}^{(k)} + 1$, that is, $n = g_{2k+1}^{(k)} + g_{k(t+1)}^{(k)} + 1$. After the disposal of $g_{k+1}^{(k)}$ and $g_{2k+1}^{(k)}$, the numbers we have display

$$g_{1}^{(k)}, g_{2}^{(k)}, \cdots, g_{k}^{(k)}, g_{k+2}^{(k)}, \cdots, g_{2k}^{(k)}, g_{2k+2}^{(k)}, \cdots, g_{t+1}^{(k)}, g_{t}^{(k)}, \cdots, g_{t+k}^{(k)}.$$

The number $g_{2k+1}^{(k)} + g_{k}^{(k)}$ is attainable and

$$g_{2k+1}^{(k)} + g_{t+k}^{(k)} < g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1 < g_{2k+1}^{(k)} + 1.$$

With the number $g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1$ in hand, we get

$$g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1 - (g_{t+k+1}^{(k)} + \cdots + g_{t+k+1}^{(k)} + \cdots + g_{t+k+1}^{(k)}) = g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1,$$

is unattainable where the $g_{t+k+1}^{(k)} + \cdots + g_{t+k+1}^{(k)}$ is attainable. Therefore, if $g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1$ is unattainable, so is $g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1$. By induction, then, $g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1$ is unattainable for all $t = 1, 2, 3, \cdots$. Since there are numbers $g_{2k+1}^{(k)} + g_{t+k}^{(k)} + 1$ which exceed every choice of positive integer, the sequence $G^{(k)} - g_{k+1}^{(k)} - g_{2k+1}^{(k)}$ is not even weakly complete.
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By the above theorem, \( G^{(k)} - \frac{g_k^{(k)}}{g_2^{(k)}} \) is not even weakly complete, in general. We consider some \( k \)-generalized Fibonacci sequence \( G^{(k)} \):

i) \( G^{(4)} = (1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, 10671, \cdots) \)

ii) \( G^{(5)} = (1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, 13624, \cdots) \)

iii) \( G^{(6)} = (1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, 15109, \cdots) \)

iv) \( G^{(7)} = (1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, 15808, \cdots) \)

v) \( G^{(8)} = (1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, 16128, \cdots) \)

For example, consider

\[
G^{(4)} - 8 - 108 = (1, 1, 2, 4, 15, 29, 56, 108, 401, 773, 1490, 2872, 5536, 10671, \cdots)
\]

882 is unattainable as a sum of terms in a subsequence of \( G^{(4)} - 8 - 108 \), since 882 = \( g_9^{(4)} + g_{12}^{(4)} + 1 \).

Let \( L_n \) be the \( n \)th Lucas number. That is, \( L_n = F_{n+1} + F_{n-1} \) for all \( n = 1, 2, 3, \cdots \). Thus we have \( L_1 = 1, L_2 = 3, L_3 = 4 \), and so on. Since the Fibonacci numbers are connected by the fundamental recursion \( F_n = F_{n-1} + F_{n-2} \), it follows immediately that the Lucas numbers are likewise related:

(2.1) \( L_n = L_{n-1} + L_{n-2} \) for \( n > 2 \).

Let \( G^{(k)}_i = (g_0^{(k)} , g_1^{(k)} , g_2^{(k)} , \cdots) \) be the \( k \)-generalized Fibonacci sequence such that \( g_0^{(k)} = g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, g_{k-1}^{(k)} = g_k^{(k)} = 1 \). Now, we define the \( k \)-generalized Lucas sequence \( \{l_n^{(k)}\} \) by the following as

(2.2) \[ l_{n+1}^{(k)} = g_n^{(k)} + g_{n+k}^{(k)}, \quad n = 0, 1, 2, \cdots. \]

That is, \( l_1^{(k)} = g_0^{(k)} + g_k^{(k)} \), \( l_2^{(k)} = g_1^{(k)} + g_{k+1}^{(k)} \), \cdots, \( l_k^{(k)} = g_{k-1}^{(k)} + g_{2k-1}^{(k)} \). Then, for \( n > k \)

\[
l_{n+1}^{(k)} = g_n^{(k)} + g_{n+k}^{(k)} = (g_{n-1}^{(k)} + \cdots + g_{k-1}^{(k)}) + (g_{n+k-1}^{(k)} + \cdots + g_{n+k-k}^{(k)})
\]

\[
= (g_{n-1}^{(k)} + g_{n+k-1}^{(k)}) + \cdots + (g_{n-k}^{(k)} + g_n^{(k)})
\]

(2.3) \[ l_n^{(k)} + l_{n-1}^{(k)} + \cdots + l_{n-k+1}^{(k)}. \]
THEOREM 2.4. The $k$-generalized Lucas sequence is not weakly complete for $k \geq 2$.

Proof. Let $\mathcal{L}^{(k)} = (l_1^{(k)}, l_2^{(k)}, l_3^{(k)}, \ldots)$ be the $k$-generalized Lucas sequence. Since $l_1^{(k)} + \cdots + l_{k-1}^{(k)} = g_{2k-1}^{(k)} - g_{k-1}^{(k)} < g_{2k-1}^{(k)} < l_k^{(k)}$, $g_{2k-1}^{(k)}$ is unattainable as a sum of a subsequence of $\mathcal{L}^{(k)}$. We will use this result as the basis of an induction that the number $m = l_n^{(k)} + g_{2k-1}^{(k)}$ is unattainable for $n \geq k + 1$. By induction on $n$, if $n = k + 1$, $l_1^{(k)} + \cdots + l_k^{(k)} = l_{k+1}^{(k)} < l_{k+1}^{(k)} + g_{2k-1}^{(k)}$. Since

$$l_{k+2}^{(k)} = l_{k+1}^{(k)} + l_k^{(k)} + \cdots + l_2^{(k)} = l_{k+1}^{(k)} + (g_{k-1}^{(k)} + g_{2k-1}^{(k)}) + l_{k-1}^{(k)} + \cdots + l_2^{(k)},$$

and

$$l_{k+1}^{(k)} + g_{2k-1}^{(k)} < l_{k+2}^{(k)}.$$ So, $l_1^{(k)} + \cdots + l_k^{(k)} = l_{k+1}^{(k)} < l_{k+1}^{(k)} + g_{2k-1}^{(k)} < l_{k+2}^{(k)}$. Thus, $l_{k+1}^{(k)} + g_{2k-1}^{(k)}$ is unattainable as a sum of a subsequence of $\mathcal{L}^{(k)}$. Suppose, for some value $n > k + 1$, that the number $m = l_n^{(k)} + g_{2k-1}^{(k)}$. The number $l_n^{(k)}$ is attainable and

$$l_{n-(k-1)}^{(k)} + \cdots + l_n^{(k)} = l_{n+1}^{(k)} < l_{n+1}^{(k)} + g_{2k-1}^{(k)} < l_{n+2}^{(k)}.$$

With the number $l_n^{(k)} + g_{2k-1}^{(k)}$ in hand, we get

$$l_n^{(k)} + g_{2k-1}^{(k)} = (l_{n+1}^{(k)} + g_{2k-1}^{(k)}) - (l_{n-1}^{(k)} + \cdots + l_{n-(k-1)}^{(k)}),$$

is unattainable where the $l_{n-1}^{(k)} + \cdots + l_{n-(k-1)}^{(k)}$ is attainable. Therefore, if $l_n^{(k)} + g_{2k-1}^{(k)}$ is unattainable, so is $l_{n+1}^{(k)} + g_{2k-1}^{(k)}$. By induction, then, $l_n^{(k)} + g_{2k-1}^{(k)}$ is unattainable for all $n \geq k + 1$. The proof is completed.

3. Other Results

In case $k = 3$, the fundamental recurrence relation $g_{n+1}^{(3)} = g_n^{(3)} + g_{n+1}^{(3)} + g_{n-2}^{(3)}$ can also be defined as the vector recurrence relation

$$\begin{pmatrix}
g_{n+1}^{(3)} \\
g_{n}^{(3)} \\
g_{n+1}^{(3)}
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix} \begin{pmatrix}
g_{n-1}^{(3)} \\
g_{n}^{(3)} \\
g_{n+1}^{(3)}
\end{pmatrix}$$

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which is visibly equivalent. In terms of the $3 \times 3$ matrix

\begin{equation}
Q = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix},
\end{equation}

applying (3.1) $n$ times, we have

\begin{equation}
\begin{pmatrix}
g_{n+1}^{(3)} \\
g_{n+2}^{(3)} \\
g_{n+3}^{(3)}
\end{pmatrix} = Q^n \begin{pmatrix}
g_{1}^{(3)} \\
g_{2}^{(3)} \\
g_{3}^{(3)}
\end{pmatrix}
\end{equation}

Similarly, for the $k$-generalized Fibonacci sequence, the matrix and the vector recurrence relation is;

\begin{equation}
\begin{pmatrix}
g_{n+1}^{(k)} \\
g_{n+2}^{(k)} \\
\vdots \\
g_{n+k}^{(k)}
\end{pmatrix} = Q^n \begin{pmatrix}
g_{1}^{(k)} \\
g_{2}^{(k)} \\
\vdots \\
g_{k}^{(k)}
\end{pmatrix},
\end{equation}

where

\begin{equation}
Q = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1
\end{bmatrix}_{k \times k}
\end{equation}

We have the following theorem by using the above facts.

**THEOREM 3.1.** For any positive integers $n \geq k, m \geq k$,

\begin{equation}
ge_{n+m}^{(k)} = g_{n}^{(k)} e_{m-(k-1)}^{(k)} + (g_{n}^{(k)} + g_{n-1}^{(k)}) e_{m-(k-2)}^{(k)} e_{n-2}^{(k)} e_{m-(k-3)}^{(k)} + (g_{n}^{(k)} + g_{n-1}^{(k)} + \cdots + g_{n+1}^{(k)}) e_{m}^{(k)}.
\end{equation}
Proof. For $G^{(k)}$, $k \geq 2$, since $g_1^{(k)} = 1$ and $g_2^{(k)} = 1$, we can replace the matrix $Q$ in (3.5) with

$$Q = \begin{bmatrix} 0 & g_1^{(k)} & 0 & \cdots & 0 \\ 0 & 0 & g_1^{(k)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & g_1^{(k)} \\ g_1^{(k)} & g_1^{(k)} & \cdots & g_1^{(k)} & g_2^{(k)} \end{bmatrix}_{k \times k}$$

Then

$$Q^n = \begin{bmatrix} g_{n-1}^{(k)} & g_{n-1}^{(k)} & g_{n-1}^{(k)} + g_{n-2}^{(k)} + g_{n-3}^{(k)} + \cdots + g_{n-k}^{(k)} \\ g_{n-2}^{(k)} & g_{n-2}^{(k)} & g_{n-2}^{(k)} + g_{n-3}^{(k)} + g_{n-4}^{(k)} + \cdots + g_{n-k}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n-k}^{(k)} & g_{n-k}^{(k)} & g_{n-k}^{(k)} + \cdots + g_{n-3}^{(k)} + g_{n-2}^{(k)} \\ g_{n}^{(k)} & g_{n}^{(k)} & g_{n}^{(k)} + \cdots + g_{n-2}^{(k)} + g_{n-1}^{(k)} \end{bmatrix}$$

Since $Q^n Q^m = Q^{n+m}$, $g_{n+m}^{(k)} = (Q^{n+m})_{k1}$.

In the Fibonacci numbers, $F_n \mid F_{tn}$ for all $t = 1, 2, 3, \ldots$, since

$$F_{n+m} = F_{n-1} F_m + F_n F_{m+1}$$

What can we think about the divisibility on $k$-generalized Fibonacci numbers? For example, consider $g_6^{(7)}$ and $g_{12}^{(7)}$ in the 7-generalized Fibonacci sequence $\{g_n^{(7)}\}$. 6|12 but 16 \(\not|\) 1004. And, consider $g_4^{(6)}$, $g_8^{(6)}$ and $g_{16}^{(6)}$ in the 6-generalized Fibonacci sequence. 4|8 and 4|16 but 4 \(\not|\) 63 and 4 \(\not|\) 15109. Thus, we have established the following theorem.

**Theorem 3.2.** For the $k$-generalized Fibonacci sequence, there exists positive integer $t$ such that $g_n^{(k)} \not| g_t^{(k)}$.

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References


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