

# Gauge Uzawa methods for Incompressible flows with Variable Density

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## ABSTRACT

Two new Gauge-Uzawa schemes are constructed for incompressible flows with variable density. One is in the conserved form while the other is in the convective form. It is shown that the first-order versions of both schemes, in their semi-discretized form, are unconditionally stable. Numerical experiments indicate that the first-order (resp. second-order) versions of the two schemes lead to first-order (resp. second-order) convergence rate for all variables and that these schemes are suitable for handling problems with large density ratios such as in the situation of air bubble rising in water.

## THE NAVIER-STOKES EQUATIONS WITH VARIABLE DENSITY

We consider in this paper numerical approximations of incompressible viscous flows with variable density governed by the following coupled nonlinear system:

$$\begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho &= 0, \\ \rho (\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} &= \mathbf{f}, \quad \text{in } \Omega \times (0, T], \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \tag{1}$$

where the unknowns are the density  $\rho > 0$ , the velocity field  $\mathbf{u}$  and the pressure  $p$ ;  $\mu$  is the dynamic viscosity coefficient,  $\mathbf{f}$  represents the external force,  $\Omega$  is a bounded domain in  $\mathbf{R}^d$  ( $d = 2$  or  $3$ ) and  $T > 0$  is fixed time. The system (1) is supplemented with initial and boundary conditions for  $\mathbf{u}$  and  $\rho$ :

$$\begin{cases} \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \quad \text{in } \Omega, \quad \text{and} \quad \rho(\mathbf{x}, t)|_{\Gamma_{\mathbf{u}(\mathbf{x}, t)}} = r(\mathbf{x}, t), \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{u}(\mathbf{x}, t)|_{\Gamma} = \mathbf{g}(\mathbf{x}, t), \end{cases} \tag{2}$$

where  $\Gamma = \partial\Omega$ , and for any velocity field  $\mathbf{v}$ ,  $\Gamma_{\mathbf{v}}$  is the inflow boundary defined by

$$\Gamma_{\mathbf{v}} := \{\mathbf{x} \in \Gamma : \mathbf{v}(\mathbf{x}) \cdot \boldsymbol{\nu} < 0\},$$

with  $\boldsymbol{\nu}$  being the outward unit normal vector. We note that no initial and boundary condition is needed for the pressure  $p$  which can be viewed as a Lagrange multiplier whose mathematical role is to enforce the incompressibility condition.

How to construct stable and efficient numerical schemes for the system (1)-(2) is challenging since, in addition to all the difficulties associated with the incompressible flows with constant density, it involves a transport equation for the density  $\rho$  which enforces, in addition to the incompressibility, that the mass density remains unchanged during the fluid motion. It is now well established (see, for instance, the review [2] and the references therein) that the difficulties associated with the incompressibility can be effectively handled by using a suitable projection type scheme originally proposed by Chorin [1] and Temam [11]. This approach has been used in some papers for incompressible flows with variable density. However, the variable density introduces considerable difficulties for the construction and analysis of accurate and stable projection type schemes. For example, it is well-known that the skew-symmetry of the nonlinear term in the Navier-Stokes equations (with constant density  $\rho_o$ ), namely,

$$\int_{\Omega} (\rho_o \mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} dx = 0, \text{ for } \mathbf{u}, \mathbf{v} \text{ smooth enough and } \mathbf{u} \cdot \boldsymbol{\nu}|_{\Gamma} = 0,$$

plays a very important role in the analysis of the Navier-Stokes equations and the corresponding numerical schemes. However, this property no longer holds when  $\rho$  is not a constant.

### GAUGE-UZAWA METHOD IN CONSERVED FORM

To overcome this difficulty, Guermond and Quartapelle [3] considered the following system in conserved form:

$$\begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho + \frac{\rho}{2} \nabla \cdot \mathbf{u} &= 0, \\ \sigma(\sigma \mathbf{u})_t + (\rho \mathbf{u} \cdot \nabla) \mathbf{u} + \frac{\mathbf{u}}{2} \nabla \cdot (\rho \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} &= \mathbf{f}, \text{ in } \Omega \times (0, T], \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \quad (3)$$

where  $\sigma = \sqrt{\rho}$ .

If we apply Gauge-Uzawa method in [6,8] to (3), then we obtain the Gauge-Uzawa method in conserved form

**Algorithm 1**[Gauge-Uzawa method in conserved form] Set  $\rho^0 = \rho_0$ ,  $\mathbf{u}^0 = \mathbf{u}_0$  and  $s^0 = 0$ ; repeat for  $1 \leq n \leq N \leq T/\tau - 1$ :

**Step 1** Find  $\rho^{n+1}$  as the solution of

$$\begin{cases} \frac{\rho^{n+1} - \rho^n}{\tau} + \mathbf{u}^n \cdot \nabla \rho^{n+1} + \frac{\rho^{n+1}}{2} \nabla \cdot \mathbf{u}^n = 0, \\ \rho^{n+1}|_{\Gamma_{\mathbf{u}^n}} = r^{n+1}, \end{cases}$$

**Step 2** Find  $\hat{\mathbf{u}}^{n+1}$  as the solution of

$$\begin{cases} \sigma^{n+1} \frac{\sigma^{n+1} \hat{\mathbf{u}}^{n+1} - \sigma^n \mathbf{u}^n}{\tau} + \rho^{n+1} (\mathbf{u}^n \cdot \nabla) \hat{\mathbf{u}}^{n+1} \\ \quad + \frac{1}{2} (\nabla \cdot (\rho^{n+1} \mathbf{u}^n)) \hat{\mathbf{u}}^{n+1} + \mu \nabla s^n - \mu \Delta \hat{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1}, \\ \hat{\mathbf{u}}^{n+1}|_{\Gamma} = \mathbf{g}^{n+1}, \end{cases}$$

**Step 3** Find  $\phi^{n+1}$  as the solution of

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho^{n+1}} \nabla \phi^{n+1} \right) = \nabla \cdot \hat{\mathbf{u}}^{n+1}, \\ \partial_{\nu} \phi^{n+1}|_{\Gamma} = 0, \end{cases}$$

**Step 4** Update  $\mathbf{u}^{n+1}$  and  $s^{n+1}$  by

$$\begin{aligned} \mathbf{u}^{n+1} &= \hat{\mathbf{u}}^{n+1} + \frac{1}{\rho^{n+1}} \nabla \phi^{n+1}, \\ s^{n+1} &= s^n - \nabla \cdot \hat{\mathbf{u}}^{n+1}. \end{aligned}$$

We proved the **Stability Result** for the Gauge-Uzawa method in conserved form in [9]:

The Gauge-Uzawa method in conserved form is unconditionally stable in the sense that, for all  $\tau > 0$  and  $0 \leq N \leq T/\tau - 1$ , the following *a priori* bounds hold:

$$\|\rho^{N+1}\|_0^2 + \sum_{n=0}^N \|\rho^{n+1} - \rho^n\|_0^2 = \|\rho^0\|_0^2,$$

and

$$\begin{aligned} &\|\sigma^{N+1} \hat{\mathbf{u}}^{N+1}\|_0^2 + \sum_{n=1}^N \left( \|\sigma^{n+1} \hat{\mathbf{u}}^{n+1} - \sigma^n \mathbf{u}^n\|_0^2 + \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|_0^2 \right) \\ &+ \mu\tau \|s^{N+1}\|_0^2 + \frac{\mu}{2}\tau \sum_{n=1}^N \|\nabla \hat{\mathbf{u}}^{n+1}\|_0^2 \leq \|\sigma^0 \hat{\mathbf{u}}^0\|_0^2 + C\mu\tau \sum_{n=1}^N \|\mathbf{f}^{n+1}\|_{-1}^2. \end{aligned}$$

## GAUGE-UZAWA METHOD IN CONVECTIVE FORM

If we apply Gauge-Uzawa method in [6,8] to (1), then we obtain the Gauge-Uzawa method in convective form.

**Algorithm 2**[Gauge-Uzawa method in convective form] Set  $\rho^0 = \rho_0$ ,  $\mathbf{u}^0 = \mathbf{u}_0$  and  $s^0 = 0$ ; repeat for  $1 \leq n \leq N \leq T/\tau - 1$ :

**Step 1** Find  $\rho^{n+1}$  as the solution of

$$\begin{cases} \frac{\rho^{n+1} - \rho^n}{\tau} + \mathbf{u}^n \cdot \nabla \rho^{n+1} = 0, \\ \rho^{n+1}|_{\Gamma_{\mathbf{u}^n}} = r^{n+1}, \end{cases}$$

**Step 2** Find  $\hat{\mathbf{u}}^{n+1}$  as the solution of

$$\begin{cases} \rho^n \frac{\hat{\mathbf{u}}^{n+1} - \mathbf{u}^n}{\tau} + \rho^{n+1} (\mathbf{u}^n \cdot \nabla) \hat{\mathbf{u}}^{n+1} + \mu \nabla s^n - \mu \Delta \hat{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1}, \\ \hat{\mathbf{u}}^{n+1}|_{\Gamma} = \mathbf{0}, \end{cases}$$

**Step 3** Find  $\phi^{n+1}$  as the solution of

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho^{n+1}} \nabla \phi^{n+1} \right) = \nabla \cdot \hat{\mathbf{u}}^{n+1}, \\ \partial_{\nu} \phi^{n+1}|_{\Gamma} = 0, \end{cases}$$

**Step 4** Update  $\mathbf{u}^{n+1}$  and  $s^{n+1}$  by

$$\begin{aligned}\mathbf{u}^{n+1} &= \hat{\mathbf{u}}^{n+1} + \frac{1}{\rho^{n+1}} \nabla \phi^{n+1}, \\ s^{n+1} &= s^n - \nabla \cdot \hat{\mathbf{u}}^{n+1}.\end{aligned}$$

We proved the **Stability Result** for the Gauge-Uzawa method in convective form in [9]: The Gauge-Uzawa method in convective form is unconditionally stable in the sense that, for all  $\tau > 0$  and  $0 \leq N \leq T/\tau - 1$ , the following *a priori* bounds hold:

$$\|\rho^{N+1}\|_0^2 + \sum_{n=0}^N \|\rho^{n+1} - \rho^n\|_0^2 = \|\rho^0\|_0^2$$

and

$$\begin{aligned}& \|\sigma^{N+1} \hat{\mathbf{u}}^{n+1}\|_0^2 + \sum_{n=0}^N \left( \|\sigma^n (\hat{\mathbf{u}}^{n+1} - \mathbf{u}^n)\|_0^2 + \left\| \frac{1}{\sigma^n} \nabla \phi^n \right\|_0^2 \right) \\ & + \mu\tau \|s^{N+1}\|_0^2 + \frac{\mu}{2}\tau \sum_{n=0}^N \|\nabla \hat{\mathbf{u}}^{n+1}\|_0^2 \leq \|\sigma^0 \hat{\mathbf{u}}^0\|_0^2 + C\mu\tau \sum_{n=0}^N \|\mathbf{f}^{n+1}\|_{-1}^2.\end{aligned}$$

## REFERENCES

1. A.J. Chorin, “Numerical solution of the Navier-Stokes equations”, *Math. Comp.*, 22 (1968) 745-762.
2. J. L. Guermond, P. Minev and J. Shen, “An Overview of Projection Methods for Incompressible Flows” *Comput. Methods Appl. Mech. Eng.*, to appear.
3. “A projection FEM for variable density incompressible flows” *J. Comput. Phys.*, vol. 165, 2000, pp. 167–188.
4. R. Nocketto and J.-H. Pyo, “Optimal relaxation parameter for the Uzawa method”, *Numer. Math.*, vol. 98, 2004, pp. 695 – 702.
5. R. Nocketto and J.-H. Pyo, “Error estimates for semi-discrete gauge methods for the evolution Navier-Stokes equations”, *Math. Comp.*, Vol. 74, 2005, pp. 521–542.
6. R. Nocketto and J.-H. Pyo, “A finite element Gauge-Uzawa method. Part I : the Navier-Stokes equations” *SIAM J. Numer. Anal.*, Vol. 43, 2005, pp. 1043–1068.
7. R. Nocketto and J.-H. Pyo, “A finite element Gauge-Uzawa method. Part II : the Boussinesq equations” *to appear Math. Models Methods Appl. Sci (M3AS)*.
8. J.-H. Pyo, “The Gauge-Uzawa and related projection finite element methods for the evolution Navier-Stokes equations”, *Ph.D dissertation, University of Maryland 2002*.
9. J.-H. Pyo and J. Shen, “Gauge Uzawa methods for incompressible flows with variable density”, *submitted to J. Comput. Phys.*
10. J.-H. Pyo and J. Shen, “Normal mode analysis for a class of 2nd-order projection type methods for unsteady Navier-Stokes equations”, *Discrete Contin. Dyn. Syst. Ser. B*, Vol. 5, 2005, pp. 817–840.
11. R. Temam, “Sur l’approximation de la solution des equations de Navier-Stokes par la methode des pas fractionnaires. II”, *Arch. Rational Mech. Anal.*, 33 (1969), 377–385.