FINITE ELEMENT METHODS FOR DEALING WITH POISSON PROBLEM WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT

We study an accurate numerical method on quasi-uniform grids for two-dimensional interface problems. The method makes use of the singular function representation of the solution, the dual singular functions, and the extraction formula for the stress intensity factors. It is shown that our finite element approximation using continuous piecewise linear element on quasi-uniform grids has $O(h)$ accuracy in H^1 norm. This is confirmed by a numerical experiment for an interface problem whose solution is even not in $H^{1.1}$.

MODEL PROBLEM

Let Ω_j ($j = 1, ..., J$) be open, polygonal subdomains of Ω:

$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \bigcup_{j=1}^{J} \bar{\Omega}_j = \bar{\Omega}.$$

The Model interface problem is: find $u \in H_0^1(\Omega)$ such that

$$-\mathrm{a}_j \, \Delta u = f \quad \text{in } \Omega_j \tag{1}$$

for $j = 1, \ldots, J$ with interface conditions

$$\mathrm{a}_i \frac{\partial u}{\partial \mathbf{n}_i}|_{\Gamma_{ij}} + \mathrm{a}_j \frac{\partial u}{\partial \mathbf{n}_j}|_{\Gamma_{ij}} = 0 \tag{2}$$

for $i, j = 1, \ldots, J$ such that $\Gamma_{ij} \neq \emptyset$. Denote by $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ the common edge of Ω_i and Ω_j and let $\mathbf{n}_j$ be the outward unit normal vector to the boundary $\partial\Omega_j$ of Ω_j. Assume that the diffusion coefficient a is piecewise constant with respect to the partition:

$$\mathrm{a}(x) = \mathrm{a}_j > 0 \quad \text{in } \Omega_j \tag{3}$$

for $j = 1, \dots, J$.

PRELIMINARIES

Using a suitable cut-off function η, the solution of (1) has the following singular function representation:

$$u = w + \sum_{l=1}^{L} \kappa_l \eta_\rho s_l, \tag{4}$$

where $w \in H^2(\Omega_j)$ for $1 \le j \le J$. Moreover, the following regularity estimate holds:

$$\sum_{l=1}^{L} |\kappa_l| + \sum_{j=1}^{J} \|w\|_{H^2(\Omega_j)} \le C \|f\|_{L^2(\Omega)}. \tag{5}$$

We have an extraction formula for the stress intensity factor κ;

$$\kappa_l = \frac{1}{2\alpha_l} \sum_{i=1}^{I} \int_{\Omega_{m_i}} [f \eta_2 s_{-l} + \mathrm{a}_{m_i} w \, \Delta(\eta_2 s_{-l})] \, dx, \tag{6}$$

EXPLICIT FORM FOR SINGULAR FUNCTIONS S_L IN (4)

Let $\Omega_{m_1}, \Omega_{m_2}, \dots, \Omega_{m_I}$ be the subdomains sharing p as a common vertex. Let $\delta > 0$ be a small number such that p is the only vertex of the subdomians inside the disc $D(p, \delta)$ centered at p with radius δ. When p belongs to the boundary of the domain Ω, let polar coordinates (r, θ) be chosen so that $D(p, \delta) \cap \Omega_{m_i} = \{(r, \theta) : 0 < r < \delta, \omega_{i-1} < \theta < \omega_i\}$ for $1 \le i \le I$, where $\omega_0 = 0$, and $\omega_I = \omega$ be the angle between the two edges of $\partial\Omega$ emanating from p. When p belongs to the interior of the domain Ω, the subdomians $\{\Omega_{m_i}\}_{i=1}^{I}$ completely surround p. So we may have the polar coordinates such that $\omega_0 = 0$ and $\omega_I = 2\pi$. Let $\lambda_k = (\alpha_k)^2$ and $\Theta_k(\theta)$ for $k \ge 1$ be, respectively, the positive eigenvalues and the corresponding eigenfunctions of the Sturm-Liouville problem at the vertex: in subintervals (ω_{i-1}, ω_i) $(i = 1, \dots, I)$

$$-\Theta''(\theta) = \lambda \Theta(\theta), \tag{7}$$

on interfaces ω_i $(i = 1, \dots, I-1)$

$$\lim_{\theta \to \omega_i^-} \Theta(\theta) = \lim_{\theta \to \omega_i^+} \Theta(\theta) \quad \text{and} \quad \mathrm{a}_{m_i} \lim_{\theta \to \omega_i^-} \Theta'(\theta) = \mathrm{a}_{m_{i+1}} \lim_{\theta \to \omega_i^+} \Theta'(\theta), \tag{8}$$

and on boundaries $\theta = 0$ and $\theta = \omega$ or 2π

$$\lim_{\theta \to 0^+} \Theta(\theta) = \lim_{\theta \to \omega^-} \Theta(\theta) = 0 \quad \text{if } p \in \partial\Omega \tag{9}$$

$$\lim_{\theta \to 0^+} \Theta(\theta) = \lim_{\theta \to (2\pi)^-} \Theta(\theta), \quad \mathrm{a}_{m_1} \lim_{\theta \to 0^+} \Theta'(\theta) = \mathrm{a}_{m_I} \lim_{\theta \to (2\pi)^-} \Theta'(\theta) \quad \text{if } p \in \Omega, \tag{10}$$

where the eigenfunctions are normalized as follows:

$$\sum_{i=1}^{I} \int_{\omega_{i-1}}^{\omega_i} \mathrm{a}_{m_i} \Theta_j(\theta) \Theta_k(\theta) d\,\theta = \begin{cases} 1 \text{ if } j = k, \\ \\ 0 \text{ if } j \neq k. \end{cases} \tag{11}$$

Let $\alpha_1 \leq \ldots \leq \alpha_L$ be all $\alpha_l^{\prime} s$ that satisfy $0 < \alpha_l < 1$, define the *singular functions* and the *dual singular functions* by

$$s_l(r, \theta) = r^{\alpha_l} \Theta_l(\theta) \quad \text{and} \quad s_{-l}(r, \theta) = r^{-\alpha_l} \Theta_l(\theta), \tag{12}$$

respectively. Note that s_l and s_{-l} are harmonic in Ω_j ($\Delta s_l = \Delta s_{-l} = 0$ in Ω_j). It is easy to see that for $i = 1, \ldots, I$

$$s_l \in H^{1+\alpha_l-\varepsilon}(\Omega_{m_i}) \quad \text{and} \quad s_{-l} \in H^{1-\alpha_l-\epsilon}(\Omega_{m_i}) \tag{13}$$

for any $\varepsilon > 0$.

VARIATIONAL PROBLEM

Substituting the singular function representation of u in (4) into (1), using the extraction formulas of κ_l in (6), and regrouping terms, we have

$$\int_\Omega \mathrm{a} \nabla w \cdot \nabla v dx - \sum_{l=1}^{L} \frac{1}{2\alpha_l} \sum_{i=1}^{I} \int_{\Omega_{m_i}} \mathrm{a}_{m_i} w \, \Delta(\eta_2 s_{-l}) \, dx \int_\Omega \mathrm{a} \Delta(\eta_\rho s_l) \cdot v \, dx \tag{14}$$

$$= \int_\Omega f v \, dx + \sum_{l=1}^{L} \frac{1}{2\alpha_l} \sum_{i=1}^{I} \int_{\Omega_{m_i}} f \eta_2 s_{-l} dx \int_\Omega \mathrm{a} \Delta(\eta_\rho s_l) \cdot v \, dx. \tag{15}$$

Define the bilinear and linear forms by

$$a(w, v) = (\mathrm{a} \nabla w, \nabla v) - \sum_{l=1}^{L} \frac{1}{2\alpha_l} (\mathrm{a} w \, , \Delta(\eta_2 s_{-l}))(\mathrm{a} \Delta(\eta_\rho s_l), v) \tag{16}$$

and

$$g(v) = (f, v) + \sum_{l=1}^{L} \frac{1}{2\alpha_l} (f, \eta_2 s_{-l})(\mathrm{a} \Delta(\eta_\rho s_l), v), \tag{17}$$

respectively. Then its variational problem is to find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = g(v), \quad \forall \; v \in H_0^1(\Omega). \tag{18}$$

We will see the error analysis and its computational results.

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