Piecewise bilinear preconditioning on high-order finite element methods using LGL nodes

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ABSTRACT

High-order finite element methods for discretizing a second-order uniformly elliptic partial differential equation lead to a linear equation $\widetilde{L}_{N^2}U = F$ which requires efficient iterative methods such as Schwarz-based methods, preconditioning methods related to multilevel methods, multigrid methods and etc. This is because such linear systems have large condition numbers which depend on the order of the elements used and the mesh spacing. In particular, an algebraic multigrid (AMG) method is useful in the case of irregular grids. However it was reported that a direct application of AMG to $\widetilde{L}_{N^2}U = F$ is not so efficient (see [1]). The convergence factor degrades rapidly as the order of the elements is increased. For the case of Stokes and elasticity equations, the complexity from the high-order finite element discretizations for AMG is even worse than that of a simple elliptic partial differential equation. In [1], a preconditioning was constructed by using the Legendre-Gauss-Lobatto quadrature points in each cell as mesh points for a bilinear discretization. The preconditioning was approximately inverted by one AMG V-cycle. This approach has several advantages, including the possibility to avoid assembly of the high-order stiffness matrix. Numerical results show that this preconditioning was very effective, especially when accelerated by a conjugate gradient method. It has also an advantage of a straightforward matrix-free implementation for the fine grid high-order element matrix. In order to show that such a bilinear preconditioning is effective, we will consider a uniformly elliptic boundary value problem like

$$L_p u := -\nabla \cdot p(x,y)\nabla u + q(x,y)u \quad \text{in} \quad \Omega = (-1,1)\times(-1,1) \tag{0.1}$$

with boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_D, \quad \mathbf{n}\cdot\nabla u = 0 \quad \text{on} \quad \Gamma_N \tag{0.2}$$

where $\Gamma = \Gamma_D \cup \Gamma_N$ with a nonempty Γ_D and $p(x,y)$ and $q(x,y)$ are nonnegative smooth bounded functions on Ω. The piecewise bilinear finite element preconditioner will be constructed by another uniformly elliptic boundary operator B like

$$Bv := -\nabla \cdot \nabla v + 2v \quad \text{in} \quad \Omega = (-1,1)\times(-1,1) \tag{0.3}$$

with the same boundary (0.2). This operator B yields a matrix $\widehat{B}_{h^2}$ to reduce the condition numbers of a matrix $\widetilde{L}_{N^2}$ induced by high-order elements applied to (0.1).

The main object in this article is to prove that the eigenvalues $(\widehat{B}_{h^2})^{-1}\widetilde{L}_{N^2}$ are independent of the degrees of high-order elements and the mesh sizes. As a result the condition numbers of the

preconditioned systems are fixed and small so that the complexity is no longer problem when the AMG algorithm is applied. These make one to employ multigrid algorithms for solving problems like (0.1) with high-order elements discretizations, which guarantee convergence of the strategy of preconditioning the high-order matrix with a bilinear or trilinear matrix based on Legendre-Gauss-Lobatto quadrature nodes well suited to a solution by multigird methods.

With the direction notation $t = x$ or y, we assume that M^t and N_j^t are natural numbers. Let $\{t_k\}_{k=0}^{M^t}$ be the knots in the interval $I = [-1, 1]$ such that $-1 =: t_0 < t_1 < \cdots < t_{M^t-1} < t_{M^t} := 1$. Let $\{\eta_k\}_{k=0}^{N_j^t}$ and $\{\omega_k\}_{k=0}^{N_j^t}$ be the Legendre-Gauss-Lobatto (=:LGL) points in I arranged by

$$-1 =: \eta_0 < \eta_1 < \cdots < \eta_{N_j^t-1} < \eta_{N_j^t} := 1 \tag{0.4}$$

and its corresponding LGL weights respectively. Here M^t denotes the number of subintervals of $I = [-1, 1]$ and N_j^t denotes the number of LGL points on a j^{th} subinterval by a translation of I. By the translation from I to a j^{th} subinterval $I_j^t := [t_{j-1}, t_j]$ we denote $\mathcal{G} := \{\xi_{j,k}^t\}_{j=1,k=0}^{M^t,N_j^t}$ as the $k^{th}-$ LGL points in each subinterval I_j^t for $j = 1, 2, \cdots, M^t$ where $\xi_{j,k}^t = \frac{h_j^t}{2}\eta_k + \frac{1}{2}(t_{j-1} + t_j), \quad h_j^t = t_j - t_{j-1}$ and the corresponding LGL weights $\{\rho_{j,k}^t\}_{k=0}^{N_j^t}$ are given by $\rho_{j,k}^t = \frac{h_j^t}{2}\omega_k, \quad j = 1, 2, \cdots, M^t$.

Let $\mathcal{P}_k$ be the space of all polynomials $p_k(t)$ defined on I whose degrees are less than or equal to k and let $\mathcal{P}_{N_j^t}^h$ be the subspace of $C[-1, 1]$ which consists of piecewise polynomials $p_{N_j^t}(t)$ with support $I_j^t = [t_{j-1}, t_j]$ whose degree is less than or equal to N_j^t. For the space $\mathcal{P}_{N_j^t}^h$, we describe two types of Lagrangian basis functions with respect to $\mathcal{G}$, one of which are *internal-Lagrange basis functions* denoted as $\{\phi_{j,k}^t(t)\}_{j=1,k=1}^{M^t,N_j^t-1}$ supported in I_j^t and the other of which are *knot-Lagrange basis functions* denoted as $\{\phi_{j,N_j^t}^t(t)\}_{j=1}^{M^t-1}$ with support on $[t_{j-1}, t_{j+1}]$, and $\phi_{1,0}^t(t)$ and $\phi_{M^t,N_j^t}^t(t)$ with support on $[t_1, t_2]$ and $[t_{M^t-1}, t_{M^t}]$ respectively. For two dimensional high-order space, let $[\mathcal{P}_N^h]^2 := \mathcal{P}_{N_j^x}^h \otimes \mathcal{P}_{N_j^y}^h$, whose basis functions are given by tensor products of one dimensional piecewise Lagrange polynomials. Let $\mathcal{V}_{N_j^t}$ be the space of all piecewise Lagrange linear functions $\hat{\psi}_k(x)$ with respect to $\{\eta_k\}_{k=0}^{N_j^t}$ on I. Define $\mathcal{V}_{N_j^t}^h$ as the space of all piecewise Lagrange linear functions $\{\psi_{j,k}^t(t)\}_{j=1,k=0}^{M^t,N_j^t}$ with respect to $\mathcal{G}$. For two dimensional piecewise linear space, let $[\mathcal{V}_N^h]^2 := \mathcal{V}_{N_j^x}^h \otimes \mathcal{V}_{N_j^y}^h$, whose basis functions are given by tensor products of one dimensional piecewise Lagrange linear functions. Define an interpolation operator $\mathcal{I}_{N_j^t} : C[-1, 1] \to \mathcal{P}_{N_j^t}(I)$ such that $(\mathcal{I}_{N_j^t}^h v)(\xi_{j,k}^t) = v(\xi_{j,k}^t), \quad v \in C[-1, 1]$. Define a discrete inner product $\langle u, v\rangle_N$ on $C[-1, 1] \times C[-1, 1]$ as $\langle u, v\rangle_N := \sum_{j=1}^{M^t} \sum_{k=0}^{N_j^t-1} u(\xi_{j,k}^t)v(\xi_{j,k}^t)\rho_{j,k}^t + u(\xi_{M^t,N_j^t}^t)v(\xi_{M^t,N_j^t}^t)\rho_{M^t,N_j^t}^t$ and its corresponding norm is given by $\|u\|_N = \langle u, u\rangle_N^{\frac{1}{2}}, \quad$ for $\quad u \in C[-1, 1]$. Finally, the notation $a \sim b$ for any two real quantities a and b is meant by that there are two positive constants which do not depend on mesh sizes and degrees of polynomials such that $0 < c \le \frac{a}{b} < C < \infty$. The notation (U, V) stands for $\sum u_i v_i$ for any two vectors $U = (u_1, \cdots, u_d)^T$ and $V = (v_1, \cdots, v_d)^T$ where the superscript T denotes the transpose of a vector. The standard spaces H^1 and L^2 will be used.

[1] This work was supported by KOSEF C00006(R02-2004-000-10109-0)

1 MAIN RESULTS

Theorem 1 *For all* $u \in \mathcal{V}^h_{N^t_j}$*, we have*

$$|\mathcal{I}^h_{N^t_j} u|_1 \sim |u|_1, \quad \|u\| \sim \|\mathcal{I}^h_{N^t_j} u\|, \quad \textit{and} \quad \|u\| \sim \|\mathcal{I}^h_{N^t_j} u\|_N. \tag{1.5}$$

Consider two uniformly positive definite elliptic operators defined in $I = (-1, 1)$ such that

$$Lu = -(a_1 u')' + a_2\, u, \quad \text{in} \quad I, \quad u(-1) = u'(1) = 0 \tag{1.6}$$

and

$$Bv = -(b_1 v')' + b_2\, v, \quad \text{in} \quad I, \quad v(-1) = v'(1) = 0 \tag{1.7}$$

where a_1, b_1 are positive constants and a_2, b_2 are nonnegative constants, which lead to two bilinear forms on $\mathbf{V} \times \mathbf{V}$ where $\mathbf{V} := \{u \in H^1(I), \quad u(-1) = u'(1) = 0\}$ as

$$l_1(u, v) = \int_{-1}^{1} a_1 u'v' + a_2 uv\, dt \quad \text{and} \quad b_1(u, v) = \int_{-1}^{1} b_1 u'v' + b_2 uv\, dt. \tag{1.8}$$

For the high-order and piecewise linear approximations to (1.6) and (1.7), let

$$\mathcal{P}^{h,m}_{N_j} := \{v \in \mathcal{P}^h_{N_j}, \quad v(-1) = v'(1) = 0\}, \mathcal{V}^{h,m}_{N_j} := \{u \in \mathcal{V}^h_{N_j}, \quad u(-1) = u'(1) = 0\} \tag{1.9}$$

whose suitable basis functions $\{\phi_\mu\}_{\mu=1}^d$ and $\{\psi_\nu\}_{\nu=1}^d$ can be given respectively where

$$d := \text{dim}(\mathcal{P}^{h,m}_{N_j}) = \text{dim}(\mathcal{V}^{h,m}_{N_j}). \tag{1.10}$$

Then the stiffness matrix $\widehat{L}_N$ with high-order elements based on $\mathcal{G}$ of (1.6) is given by

$$\widehat{L}_N(\mu, \nu) = l_1(\phi_\mu, \phi_\nu), \quad \mu, \nu = 1, 2, \cdots, d, \tag{1.11}$$

and the stiffness matrix $\widehat{B}_h$ associated with piecewise linear elements based on $\mathcal{G}$ corresponding to (1.7) is given by

$$\widehat{B}_h(\mu, \nu) = b_1(\psi_\mu, \psi_\nu), \quad \mu, \nu = 1, 2, \cdots, d. \tag{1.12}$$

Denote $\widehat{M}_N$ and $\widehat{M}_h$ by mass matrices with respect to $\{\phi_\mu\}_{\mu=1}^d$ and $\{\psi_\mu\}_{\mu=1}^d$ respectively, that is, $\mu, \nu = 1, 2, \cdots, d$,

$$\widehat{M}_N(\mu, \nu) = (\phi_\mu, \phi_\nu), \quad \widehat{M}_h(\mu, \nu) = (\psi_\mu, \psi_\nu). \tag{1.13}$$

Since all the stiffness and mass matrices are symmetric and positive definite, the preconditioned matrix below also has all positive real eigenvalues.

Theorem 2 *The eigenvalues of the preconditioned matrix* $\widehat{B}_h^{-1}\widehat{L}_N$ *has all positive real eigenvalues* $\{\lambda_\mu\}_{\mu=1}^d$ *independent of mesh sizes* h_j *and degrees* N_j *of polynomials, that is, there is absolute positive constants* c *and* C *such that* $0 < c \leq \lambda_\mu \leq C < \infty$.

For actual computations, the bilinear form $l_1(\mathcal{I}^h_{N_j} u, \mathcal{I}^h_{N_j} v)$ and $(\mathcal{I}^h_{N_j} u, \mathcal{I}^h_{N_j} v)$ will be calculated at LGL points. Define two matrices $\widetilde{L}_N$ and $\widetilde{M}_N$ as

$$\widetilde{L}_N(\mu, \nu) = l_{1,N}(\phi_\mu, \phi_\nu), \quad \widetilde{M}_N(\mu, \nu) = \langle \phi_\mu, \phi_\nu \rangle_N, \tag{1.14}$$

where

$$l_{1,N}(u,v) = a_1\langle u', v'\rangle_N + a_2\langle u, v\rangle_N. \tag{1.15}$$

Note that the equivalence of numerical quadrature leads to

$$(\widehat{L}_N U, U) \sim (\widetilde{L}_N U, U), \quad \text{and} \quad (\widehat{M}_N U, U) \sim (\widetilde{M}_N U, U) \tag{1.16}$$

and these matrices $\widetilde{L}_N$ and $\widetilde{M}_N$ are symmetric and positive definite.

Corollary 3 *The eigenvalues of the preconditioned matrix $\widehat{B}_h^{-1}\widetilde{L}_N$ has all positive real eigenvalues $\{\lambda_\mu\}_{\mu=1}^d$ independent of mesh sizes h_j and degrees N_j of polynomials.*

We now turn to two dimensional case. For this, we consider the model elliptic operator L such that

$$Lu = -[u_{xx} + u_{yy}] + 2u, \quad u = 0 \quad \text{on} \quad \Gamma_D, \quad \mathbf{n}\cdot\nabla u = 0 \quad \text{on} \quad \Gamma_N, \tag{1.17}$$

which leads to the bilinear form $l(u,v) = (\nabla u, \nabla v) + 2(u,v), \quad \text{for} \quad u, v \in H_D^1(\Omega)$, where $H_D^1(\Omega) := \{u \in H^1(\Omega)\,|\, u = 0 \quad \text{on} \quad \Gamma_D\}$. Let $[\mathcal{P}_N^{h,m}]^2 := \mathcal{P}_{N_j^x}^{h,m} \otimes \mathcal{P}_{N_j^y}^{h,m}, \quad [\mathcal{V}_N^{h,m}]^2 := \mathcal{V}_{N_j^x}^{h,m} \otimes \mathcal{V}_{N_j^y}^{h,m}$. Let us order the LGL points by horizontal lines and we list all LGL points $\{\Xi_P\}_{P=1}^{d^2}$ as

$$\Xi_P = (\xi_\mu, \xi_\nu), \quad \text{where} \quad P = \mu + d(\nu - 1), \quad \mu, \nu = 1, 2, \cdots, d,$$

where d is defined in (1.10). Accordingly, we order the basis vectors $\Phi_P(x,y) \in [\mathcal{P}_N^{h,m}]^2$ and $\Psi_P(x,y) \in [\mathcal{V}_N^{h,m}]^2$ in the same order. Let $\widehat{L}_{N^2}^s$ and $\widehat{B}_{h^2}^s$ be the stiffness matrices on the space $[\mathcal{P}_N^{h,m}]^2$ and $[\mathcal{V}_N^{h,m}]^2$ respectively. From now on, assume that $a_i = b_i = 1, \quad i = 1, 2$ in the operators L_1 and B_1 in (1.6) and (1.7). Then using the one dimensional stiffness matrices $\widehat{L}_{N_j^t}$, $\widehat{B}_{h_j^t}$ and mass matrices $\widehat{M}_{N_j^t}$, $\widehat{M}_{h_j^t}$, we have

$$\widehat{L}_{N^2}^s = \widehat{M}_{N_j^y} \otimes \widehat{L}_{N_j^x} + \widehat{L}_{N_j^y} \otimes \widehat{M}_{N_j^x}, \quad \widehat{B}_{h^2}^s = \widehat{M}_{h_j^y} \otimes \widehat{B}_{h_j^x} + \widehat{B}_{h_j^y} \otimes \widehat{M}_{h_j^x}. \tag{1.18}$$

Theorem 4 *The eigenvalues of $(\widehat{B}_{h^2}^s)^{-1}\widehat{L}_{N^2}^s$ are all positive and bounded. The bounds are independent of the mesh sizes h_j^x, h_j^y and the degrees N_j^x, N_j^y of polynomials.*

For actual computations of $\widehat{L}_{N^2}^s$, we use LGL quadrature formula. For this, consider

$$l_N(u,v) = \langle\nabla u, \nabla v\rangle_N + \langle u, v\rangle_N, \tag{1.19}$$

which can be written as, for $u, v \in [\mathcal{P}_N^{h,m}]^2$, $l_N(u,v) = V^T(\widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x} U$ where the vectors $U = (u_1, \cdots, u_{d^2})^T$ and $V = (v_1, \cdots, v_{d^2})^T$ are vector representations of $u(x,y) = \sum_{P=1}^{d^2} u_P \Phi_P(x,y)$ and $v(x,y) = \sum_{P=1}^{d^2} v_P \Phi_P(x,y)$. Now we will use the matrix $\widehat{B}_{h^2}^s$ in (1.18) as the preconditioner for

$$\widetilde{L}_{N^2}^s U = F \tag{1.20}$$

where

$$\widetilde{L}_{N^2}^s := \widetilde{M}_{N_j^y} \otimes \widetilde{L}_{N_j^x} + \widetilde{L}_{N_j^y} \otimes \widetilde{M}_{N_j^x}. \tag{1.21}$$

Theorem 5 *The eigenvalues of $(\hat{B}_{h^2}^s)^{-1}\widetilde{L}_{N^2}^s$ are all positive and bounded. The bounds are independent of the mesh sizes h_j^x, h_j^y and the degrees N_j^x N_j^y.*

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