The cell boundary element methods

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Abstract

We consider the model second-order elliptic problem:

$$\begin{aligned} -\nabla \cdot K \nabla u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$. Assume that Ω is composed of disjoint polygonal subdomains $\Omega_1, \cdots, \Omega_J$ and that K is a function such that $0 < K_* \le K(x) \le K^* < \infty$ and $K(x) = K_j$ in Ω_j for each j.

The localized problem becomes

$$\begin{aligned} -K_T \Delta u &= f \quad \text{in } T, \\ \left[K \frac{\partial u}{\partial \nu} \right] :\equiv K_T \frac{\partial u}{\partial \nu} + K_{T'} \frac{\partial u'}{\partial \nu'} &= 0 \quad \text{on } e_p = \partial T \cap \partial T'. \end{aligned}$$

The continuity of the flux can be weakened as follows:

$$\int_{e_p} \left[K \frac{\partial u}{\partial \nu} \right] = 0$$

and this is the motivation of the CBE method. Introduce F_T, a particular solution of (0.1) so that

$$F_T(x) = \frac{1}{K_T} \int_T \Gamma(x-y) f(y) dy, \quad x \in T,$$

where $\Gamma(x) = -\frac{1}{2\pi} \log |x|$ is the fundamental solution of $-\Delta$. Then u admits the following decomposition:

$$u = v + (F - H(F)) \quad \text{on } T,$$

where $\Delta v = 0$ and $u = v$ on ∂T. $F - H(F)$ is a buuble function.

Our nonconforming CBE method is to find $v_h \in V_{0,h}$ such that

$$\int_{e_p} \left[K \frac{\partial v_h}{\partial \nu} \right] ds = \int_{e_p} \left[K \left(\frac{\partial H_h(F)}{\partial \nu} - \frac{\partial F}{\partial \nu} \right) \right] ds \quad \text{for all } p \in \mathcal{V}_h^i.$$

Here, H_h is the harmonic interpolation. Then $u_h = v_h + (F - H_h(F))$ is the solution we are looking for.

The advantage of the CBE method is that

- The flux conservation holds as the finite volume method

- The cost for mesh generation is the same as that of the finite element method

Numerical Experiments

The computational domain is taken as the unit square $\overline{\Omega} := [0,1] \times [0,1]$ and a quasi-uniform mesh: the vertices are given as

$$x_i = \frac{2t_i}{1+t_i} \quad \text{and} \quad y_j = \frac{1.5t_j}{1+.5t_j}, \quad 0 \le i,j \le n,$$

where $\{t_j = j/n, j = 0, \cdots, n\}$ and the triangular mesh is then generated by bisecting each rectangle by the diagonal line from the top right to the bottom left.

Example 1. *We consider the following Poisson equation:*

$$\begin{aligned} -\Delta u &= -4 - 6x \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where the function g is chosen so that the exact solution is $u(x,y) = e^x \cos(y) + x^2 + y^2 + x^3$. Then the total flux on $D = [0, 2/3] \times [0, 3/5]$ is 2.4.

n^2	$\|u - u_h\|_{0,h}$	α	$\|\nabla u - \nabla u_h\|_{0,h}$	α	flux
5^2	7.0473e-3		6.3160e-1		
10^2	2.0732e-3	1.7652	3.2263e-1	0.9691	2.4
20^2	5.4437e-4	1.9292	1.6256e-1	0.9889	2.4
40^2	1.3817e-4	1.9781	8.1498e-2	0.9961	2.4

Table 1: Numerical results for the P_1 nonconforming method

n^2	$\|u - u_h\|_{0,h}$	α	$\|\nabla u - \nabla u_h\|_{0,h}$	α	flux
5^2	1.0365e-4		8.9443e-3		
10^2	3.4297e-5	1.5956	2.2992e-3	1.9599	2.4 + 1.57e-4
20^2	9.1723e-6	1.9027	5.7823e-4	1.9914	2.4 + 1.32e-8
40^2	2.3202e-6	1.9830	1.4464e-4	1.9992	2.4 $+\epsilon$

Table 2: Numerical results for the $P_{2+1/2}$-method

n^2	$\|u - u_h\|_{0,h}$	α	$\|\nabla u - \nabla u_h\|_{0,h}$	α	flux
5^2	1.5516e-4		9.0081e-3		
10^2	4.5665e-5	1.7646	2.3189e-3	1.9578	2.4
20^2	1.1795e-5	1.9530	5.8307e-4	1.9917	2.4
40^2	2.9673e-6	1.9909	1.4587e-4	1.9990	2.4

Table 3: Numerical results for the modified $P_{2+1/2}$-method

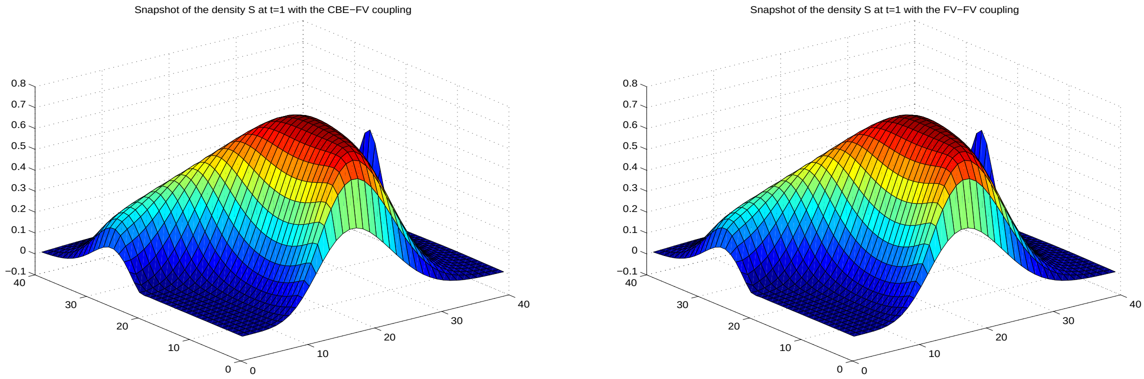

Figure 1: Snapshots of S at $t = 1$ for the P_1 CBE-FV method and FV-FV couplings, respectively

n^2	$\|u - u_h\|_{\infty,h}$	α	$\|\nabla u - \nabla u_h\|_{0,h}$	α
5^2	1.5989e-005		6.8047e-3	
10^2	1.2601e-006	3.6654	1.7463e-3	1.9622
20^2	9.6459e-008	3.7075	4.4266e-4	1.9800
40^2	6.9356e-009	3.7978	1.1146e-4	1.9897

Table 4: Numerical results for the $P_{2+1/2}$-method on the square mesh

Example 2. *We consider a subsurface flow problem.*

$$\begin{aligned} \Delta u &= 1 \quad in\ \Omega, \\ \frac{\partial S}{\partial t} &= -\ \sigma \cdot \nabla S + 0.01 \Delta S \quad in\ \Omega, \end{aligned}$$

where $\sigma = -\nabla u$. *The boundary condition for* u *is given as follows:*

$$\begin{cases} \frac{\partial u}{\partial \nu} = -1 & on\ \{x_1 = 0,\ \ 0 < x_2 < 1/2\}, \\ \frac{\partial u}{\partial \nu} = 2 & on\ \{x_1 = 1,\ \ 1/2 < x_2 < 1\}, \\ u = 0, & elsewhere. \end{cases}$$

The initial and boundary conditions for S *are as follows:*

$$S(x, 0) = \begin{cases} 1, & x \in (0, 1/2) \times (0, 1/2), \\ 0, & elsewhere, \end{cases} \quad and \quad S(x, t) = 0, \quad x \in \partial\Omega.$$

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