Classification of vector bundles over circles and spheres with group actions *

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Today's lecture is based on series of articles jointly prepared or under preparation by the following people.

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Cho, Kim, Masuda, and Suh : "Classification of equivariant complex vector bundles over a circle" (1999) submitted

Cho, Kim, and Suh : "On extensions of representations for compact Lie groups" (1999) submitted

Cho, Kim, Masuda, and Suh : "Classification of equivariant real vector bundles over a circle" (2000) preprint

Cho and Masuda : "Revisit: on the coefficient groups of equivariant K-theory" (2000) preprint

Kim and Suh : "Equivariant K-groups of a real representation 2-sphere" (2000) under preparation

Abstract

Let G be a compact Lie group. In this lecture we give classification results on real and complex G vector bundles over G representation circles and spheres, and some K-theoretic results on them. It requires interplay between topology and group representation theory. In particular we need extensions of representations of a normal subgroup of G to the whole group G. If time permits we also discuss some results on extensions of representations.

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§1. Some Definitions

G: Compact Lie group
X: topological space (usually $\mathcal{C}^\infty$-manifold or algebraic variety)

Definition 1. **topological G action** on X is a continuous map

$$\theta : G \times X \to X$$

such that for $g, h \in G$ and $x \in X$
(1) $\theta(e, x) = x$
(2) $\theta(g, \theta(h, x)) = \theta(gh, x)$
In this case X is called a G **space**.
Smooth action and **algebraic action** are correspondingly defined.

If G acts on X, then the following map is induced.

$$\begin{array}{lcl} \Theta : G \to & \text{Homeom}(X) & \text{top. action} \\ & \text{Diffeom}(X) & \text{smooth action} \\ & \text{Aut}(X) & \text{alg. action} \end{array}$$

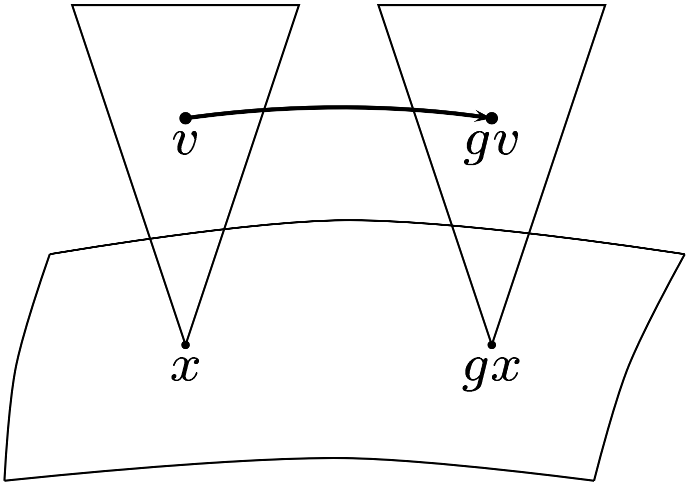
Example 2. Most fundamental example for G space is a G **representation**:
a homomorphism

$$\rho : G \to GL(n, \mathbb{K}), \qquad \mathbb{K} = \mathbb{R}, \mathbb{C}.$$

The vector space $\mathbb{K}^n$ with the linear G action induced from ρ is called the **corresponding** G **module**.

Definition 3. E, X: G space
$p : E \to X$ is a G $\mathbb{K}$-**vector bundle** if
(1) nonequivariantly $p : E \to X$ is a $\mathbb{K}$ vector bundle,
(2) p is a G map, i.e., $p(gv) = gp(v)$ for $g \in G$ $v \in V$,
(3) $\forall g \in G$ and $\forall x \in X$ $g : E_x \to E_{gx}$ is a vector space isomorphism.

G vector bundle $p : E \to X$ is **trivial**
if $\exists V$: a G module and $\exists \tau : E \cong X \times V$ such that $\pi_1 \circ \tau(v) = p(v)$ where $\pi_1 : X \times V \to X$ is the projection.



G action on G vector bundle

$$\begin{tikzcd}
E \arrow[rr, "\tau"] \arrow[dr, "p"'] & \circlearrowleft & X \times V \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

Trivial G vector bundle

§2. Motivation and Problem

♦ Not much is known for G vector bundles: we know we can reduce classification of G vector bundles to that of nonequivariant vector bundles and representation theory if the action on X is either **trivial** or **free**.

♦ Information on G vector bundles (over nonsingular real algebraic G varieties) can 'sometimes' solve the algebraic realization problem and the algebraic approximation problem.

Problem 4 (Algebraic Realization Prob.). For a given closed smooth G manifold M does there always exist a compact nonsingular real algebraic G variety Z such that M and Z are G diffeomorphic?

Problem 5 (Algebraic Approximation Prob.). For a given G-map $f : V \to Z$ between two real algebraic G varieties when can f be approximated by an entire rational G map $\phi : V \to Z$?

◆ To get better information on G vector bundles we need to classify them.

$\text{Vect}_G^{\mathbb{K}}(X) := \{G\ \mathbb{K}\text{-vector bundles over } X\}/\sim$

where $\sim$ is G vector bundle isomorphism.
$\text{Vect}_G^{\mathbb{K}}(X)$ is a semigroup.

Define the the K-group of X by
$K_G(X)$, or $KO_G(X) := K(\text{Vect}_G^{\mathbb{K}}(X))$

Problem 6. Find the semigroup structure of $\text{Vect}_G^{\mathbb{K}}(X)$. Or find the group structure of $K_G(X)$ or $KO_G(X)$.

◆ The above problem is too broad and hopeless for arbitrary space X unless we give some restrictions.

◆ Here we restrict our attention to **circles** and **spheres** with group actions.

§3. Attacking tools for the problem

X: connected G space
$H \triangleleft G$: the kernel of the action
E: G vector bundle over X.
$\Rightarrow$
each fiber E_x determines a unique H module called the **fiber H module** of E.

χ := the character of the fiber H module of E. (`if` $\rho : H \to GL(E_x)$ `is the fiber` H `representation then` $\chi : H \to \mathbb{K}$ `is defined by` $\chi(h) = \mathrm{tr}(h)$ `for` $h \in H$.)

Irr(H) := the set of characters of irreducible H modles
$\Rightarrow$
G acts on Irr(H) as follows:

$$g \cdot \chi(h) = \chi(g^{-1}hg), \qquad g \in G,\ h \in H$$

Definition 7. $\chi \in \mathrm{Irr}(H)$ is G **invariant** if $g \cdot \chi = \chi$ for all $g \in G$.

Proposition 8. *Let E, X, H, and χ be as above. Then χ is G invariant.* $\square$

E: G vector bundle over X
$\chi \in \mathrm{Irr}(H)$

$$\begin{aligned} E(\chi) \ &:= \ \chi\text{-isotypical component of } E \\ &= \ \text{the largest } H \text{ subbundle of } E \text{ with} \\ &\quad\ \text{multiple of } \chi \text{ as the character} \\ &\quad\ \text{of the fiber } H\text{-module.} \end{aligned}$$

$\Rightarrow$ Following is a G vector bundle over X.

$$\bigoplus_{\lambda \in G(\chi)} E(\lambda) = \mathrm{Ind}_{G_\chi}^{G} E(\chi)$$

$\Rightarrow$

$$\begin{aligned} E = \ & \oplus_{\chi \in \mathrm{Irr}(H)/G} \oplus_{\lambda \in G(\chi)} E(\lambda) \\ = \ & \ \oplus_{\chi \in \mathrm{Irr}(H)/G} \mathrm{Ind}_{G_\chi}^{G} E(\chi) \end{aligned}$$

$\mathrm{Vect}_{G_\chi}(X, \chi) :=$ the subset of $\mathrm{Vect}_{G_\chi}(X)$ with multiple of χ as the character of the fiber H module.

Proposition 9. *There exists a semigroup isomorphism*

$$\mathrm{Vect}_G(X) \to \prod_{\chi \in \mathrm{Irr}(H)/G} \mathrm{Vect}_{G_\chi}(X, \chi). \qquad \square$$

♦ This reduces the study of $\mathrm{Vect}_G(X)$ to that of $\mathrm{Vect}_{G_\chi}(X, \chi)$. Here χ is a G_χ invariant H character.

♦ Therefore, for simplicity, it is enough to study $\mathrm{Vect}_G(X, \chi)$ where χ is a G invariant irreducible character of H

♦ On the other hand in **complex category** we have the following further reduction result.

Proposition 10. *For* **complex bundles** *if there exists a G_χ vector bundle over X with χ (not multiple of χ) as the character of the fiber H-module, then we have semigroup isomorphism*

$$\mathrm{Vect}_{G_\chi}(X, \chi) \cong \mathrm{Vect}_{G_\chi/H}(X).$$

Some times $\mathrm{Vect}_{G_\chi/H}(X)$ is easy to compute because G_χ/H acts effectively on X.

§4. Classification results on circles

♦ It is well known that any G circle is equivalent to the unit circle $S(\rho) \subset \mathbb{C}$ of the G module of a representation

$$\rho : G \to O(2).$$

♦ There are four different types of the action on the circle:

(1) $\rho(G) = SO(2)$: transitive action with only rotations

(2) $\rho(G) = O(2)$: transitive action with both rotation and reflections

(3) $\rho(G) = \mathbb{Z}_n$: cyclic action with rotations

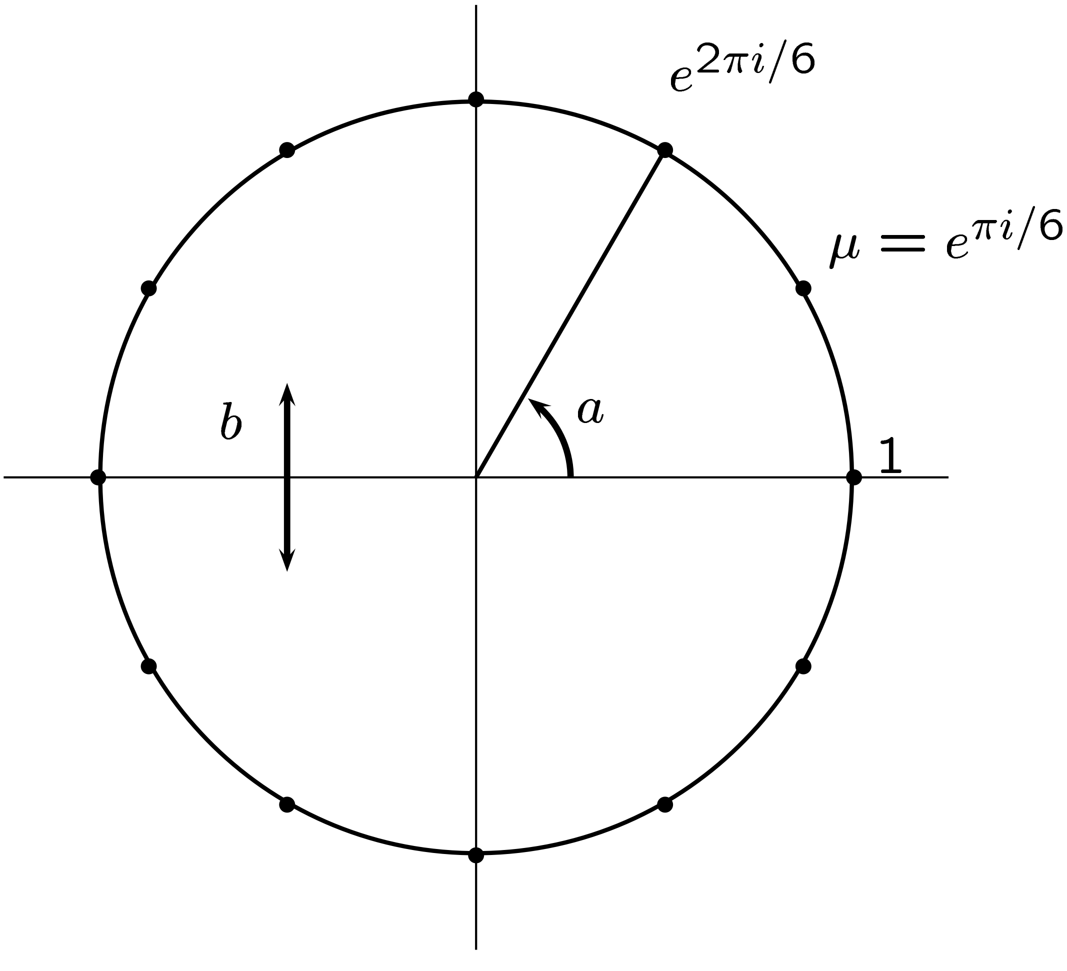
(4) $\rho(G)) = D_n$: dihedral action with rotations and reflections

♦ Even though $D_1 \cong \mathbb{Z}_2$, we consider them differently: $\rho(G) = D_1$ is the reflection action and $\rho(G) = \mathbb{Z}_2$ is the rotation action.

♦ When $\rho(G) = D_n$ there are two points 1 and $\mu := e^{\pi i/n}$ on $S(\rho)$ with different isotropy types: the isotropy groups G_1 and G_μ contain H as an index two subgroup.

For example,

$$\begin{aligned}\rho(G) &= D_6 \\ &= < a, b \mid a^6 = b^2 = aba^{-1} = 1 >\end{aligned}$$



♠ Some known results ♠

Theorem 11 (Kim-Masuda 94).
(1) A real G *line bundle over* $S(\rho)$ *is trivial* $\Leftrightarrow$ *it is nonequivariantly trivial.*
(2) Any nontrivial real G *line bundle over* $S(\rho)$ *is of the form*

$$S(\rho) \times_{\mathbb{Z}_2} V \longrightarrow S(\rho)/\mathbb{Z} = P(\rho).$$

□

This is not true for complex line bundles.

Example 12. $G := \mathbb{Z}_4 =< g >$
$\rho(G) = D_1$
$E = S(\rho) \subset \mathbb{C}$ with $g(z, v) = (\overline{z}, zv)$.
This bundle is nonequivariantly trivial but equivariantly nontrivial: indeed, the G module E_1 and $E_\mu = E_{-1}$ are not isomrophic.(Theorem 28)

Theorem 13 (Yang 95). $G = \mathbb{Z}_m = Z_{2q}$
$\rho(G) = D_1$*: action is only reflection*
$\Rightarrow$

$$\widetilde{K}_{\mathbb{Z}_m}(S(\rho)) \cong R(\mathbb{Z}_m)(\mathbb{C}_+ - \mathbb{C}_-)$$

which is the ideal of $R(\mathbb{Z}_m)$ *generated by one element* $\mathbb{C}_+ - \mathbb{C}_-$*.* □

Here $\mathbb{C}_+$(resp.$\mathbb{C}_-$) is the G module of dimension 1 induced from the trivial (resp. the nontrivial) $D_1 \cong \mathbb{Z}_2$ module of dimension 1 by $G \to G/H = D_1$.

♠ Classification of $\mathbb{C}$ Vector Bundles ♠

Theorem 14 (CKMS99).
Vect$_G(S(\rho), \chi)$ *is generated by*
(1) one element L_χ *if* $\rho(G) \subset SO(2)$
(2) two elements $L_\chi^\pm$ *if* $\rho(G) = O(2)$
(3) four elements $L_\chi^{\pm\pm}$ *with the relation*

$$L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$$

if $\rho(G) = D_n$. *In fact, we have the following semigroup isomorphism, which sends* E *to the fiber* G_z *modules at* $z = 1$ *(and* μ *when* $\rho(G) = D_n$*).*

$$\Gamma : \text{Vect}_G(S(\rho), \chi) \to$$

$$\begin{cases} \text{Rep}(H, \chi) & \text{if } \rho(G) \subset SO(2) \\ \text{Rep}(G_1, \chi) & \text{if } \rho(G) = O(2) \\ \text{Rep}(G_1, G_\mu, \chi) & \text{if } \rho(G) = D_n \end{cases}$$

□

Here

Rep$(H,\chi) = \{H$ modules whose character is $m\chi\}$

Rep$(G_z,\chi) = \{G_z$ modules whose character restricted to H is $m\chi\}$ for $z = 1$ or μ

Rep$(G_1, G_\mu, \chi) =$
$\{(V,W) \in$ Rep$(G_1,\chi) \times$ Rep$(G_1, G_\mu, \chi) \mid$
$\dim V = \dim W\}$

◆ The following theorem shows when the generators of Vect$_{G_\chi}(S(\rho),\chi)$ are trivial

Theorem 15 (CKMS99).
(1) L_χ is trivial.
(2) $L_\chi^{\pm}$ are both trivial or both nontrivial.
(3) If $|\rho(G)|/2$ is odd, then two of $L_\chi^{\pm\pm}$ are trivial and the other two are nontrivial.
If $|\rho(G)|/2$ is even, then all four $L_\chi^{\pm\pm}$ are all trivial or all nontrivial. $\square$

Corollary 16. *If every element of $g \in G$ acts on S^1 with orientation preserving manner, then any complex G vector bundle over S^1 is trivial.* $\square$

◆ As a consequence we can extend Yang's results to any compact Lie group actions with $\rho(G) = D_1$

Theorem 17. *Yang's result Theorem 13 is still true for any compact Lie groups G actions with $\rho(G) = D_1$.* □

◆ We also have complete characterization of G vector bundles over G circle for compact abelian group G. In other words we have exact formula for actions of the generators.

♠ Classification of $\mathbb{R}$ Vector Bundles ♠

♦ Arguments are similar to the complex case, but more complicated. Here are the reasons:

(1) Topology is not as trivial as in the complex case. For example there exists a nontrivial nonequivariant real line bundle over S^1 while nonequivariant complex line bundles over S^1 all trivial.

(2) Real representation theory is more complicated than the complex theory: for real G modules U there are three different types, which are of real, complex, and quaternionic types (that is, $\mathrm{End}_G(U,U) = \mathbb{R}, \mathbb{C},$ and $\mathbb{H}$).

We need the following definition.

Definition 18. Let H be a normal subgroup of K. An H module U **extends** to a K module if $\exists V$: a K module such that $\mathrm{Res}_H V = U$.

There are two cases in which classification works rather exceptionally.

CASE A $\rho(G) \subsetneq SO(2)$, and χ is of real type.

CASE B $\rho(G) = D_n$ and χ is of real type and neither G_1 nor G_μ extendible.

As in the complex case consider the semi-group homomorphism

$$\Gamma : \mathrm{Vect}_G(S(\rho), \chi) \to$$

$$\begin{cases} \mathrm{Rep}(H, \chi) & \text{if } \rho(G) \subset SO(2) \\ \mathrm{Rep}(G_1, \chi) & \text{if } \rho(G) = O(2) \\ \mathrm{Rep}(G_1, G_\mu, \chi) & \text{if } \rho(G) = D_n \end{cases}$$

Theorem 19 (CKMS00).
(1) Except for CASES A and B, Γ is a semi-group isomorphism.
(2) In CASES A and B the map Γ is 2-to-1.

◆ By studying the semi-group structure of the target space of Γ and the above theorem, we can find generators of $\mathrm{Vect}_G(S(\rho), \chi)$. Here comes the complexity of real representations. Namely, to study $\mathrm{Rep}(G_z, \chi)$ we have to know the number of G_z extensions of χ which can be 0, 1, or 2 in real case, while it is always 2 in complex case.

♦ The semigroup structure of $\mathrm{Vect}_G(S(\rho),\chi)$ are of five types depending on the image $\rho(G)$ and the character χ, while there are of three types in complex case depending only on $\rho(G)$.

Notation 20. Let $e_z :=$ the number of G_1 extensions of χ for $z = 1, \mu$. If $\rho(G) \subset SO(2)$ or $\rho(G) = O(2)$, then let $e_\mu = 1$.

Theorem 21 (CKMS00). *Except for CASES A and B,* $\mathrm{Vect}_G(S(\rho),\chi)$ *is generated by*

(1) one element L_χ, *if* $(e_1, e_\mu) = (0,0)$, $(1,0)$, $(0,1)$ *or* $(1,1)$

(2) two elements $L_\chi^\pm$, *if* $(e_1, e_\mu) = (2,1)$ *or* $(1,2)$

(3) three elements $\widetilde{L}_\chi^0, \widetilde{L}_\chi^\pm$ *with relation* $2\widetilde{L}_\chi^0 = \widetilde{L}_\chi^+ + \widetilde{L}_\chi^-$, *if* $(e_1, e_\mu) = (2,0)$ *or* $(0,2)$

(4) four elements $L_\chi^{\pm\pm}$ *with relation* $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$, *if* $(e_1, e_\mu) = (2,2)$

(5) In Case A or B, it is generated by two elements $\tilde{\tilde{L}}_\chi^\pm$ *with relation* $2\tilde{\tilde{L}}_\chi^+ = 2\tilde{\tilde{L}}_\chi^-$.

♦ There is also a similar result on triviality of generators to the complex case.

Theorem 22. *Assume* $\rho(G) = \mathbb{Z}_n$ *or* D_n.
If $n =$ *even, except for* $\widetilde{L}_\chi^0$ *other generators are all trivial or all nontrivial.*
If $n =$*odd, then*
(1) L_χ *is trivial*
(2) $L_\chi^\pm$ *are both trivial*
(3) $\widetilde{L}_\chi^0$ *and* $\widetilde{L}_\chi^\pm$ *are all trivial*
(4) two of $L_\chi^{\pm\pm}$ *are trivial and other two are nontrivial*
(5) one of $\widetilde{\widetilde{L}}_\chi^\pm$ *is trivial and the other is nontrivial*

§5. Classification Results on Spheres

Theorem 23 (Yang 95).
$\rho : G \to O(2)$ with $\rho(G) = D_1$.
$G = \mathbb{Z}_m = Z_{2q}$
$\Rightarrow$

$$\widetilde{K}_{\mathbb{Z}_m}(S(\rho \oplus \mathbb{R})) \cong R(\mathbb{Z}_q)(\chi_2 - 1)$$

◆ Unfortunately, the above Yang's result is not correct . It was first noted and corrected in a generalized setting by Cho-Masuda.

Theorem 24 (Cho-Masuda00).
G: compact abelian Lie group, and ρ as above.
$\Rightarrow$

$$\widetilde{K}_{\mathbb{Z}_m}(S(\rho \oplus \mathbb{R})) = 0. \qquad \square$$

◆ From the isomorphism

$$\mathrm{Vect}_G(X) \to \prod_{\chi \in \mathrm{Irr}(H)/G} \mathrm{Vect}_{G_\chi}(X, \chi)$$

we have a similar isomorphism in its K-group.

$$K_G(X) \cong \prod_{\chi \in \mathrm{Irr}(H)/G} K(\mathrm{Vect}_{G_\chi}(X, \chi))$$

◆ Hence it is enough to study

$$K_G(X,\chi) := K(\mathrm{Vect}_G(X,\chi))$$

where χ is a G invariant irreducible H character.

Theorem 25 (Kim-Suh00).
$\rho : G \to O(2)$ with $\rho(G) = D_n$
(1) Suppose $\exists$ G extension τ of a multiple of χ such that $\mathrm{Res}_{G_Z} \tau$ has a G_z extension of χ for $z = 1$ or μ.
$\Rightarrow$

$$K_G(S(\rho \oplus \mathbb{R}), \chi) \cong K(R(G, G, \chi)).$$

(Note: $(R(G, G, \chi)) =$
$\{(V, W) \in R(G, \chi) \times R(G, \chi) \mid \dim V = \dim W\}$)
(2) Otherwise

$$K_G(S(\rho \oplus \mathbb{R}), \chi) \cong K(R(G, G, \chi)) \times \mathbb{Z}_2.$$

(3) In particular, if χ has a G extension (this is true if $n =$ odd), then

$$K_G(S(\rho \oplus \mathbb{R}), \chi) \cong \begin{cases} R(D_n) \oplus \mathbb{Z}^{(n-1)/2} & n \text{ odd} \\ R(D_n) \oplus \mathbb{Z}^{n/2} & n \text{ even} \end{cases}$$

(4) Cho-Masuda's Theorem 24 follows as a corollary. $\square$

Remark 26. With the previous classification result we are able to partially solve the algebraic realization problem and the algebraic approximation problem. However this is another topic of interest, so we will not mention it here.

§6. Some remarks on proofs of the results and extensions of representations

◆ Some basic ingredients for the proofs of the classification results are as follows.

As before let $\rho : G \to O(2)$.

Theorem 27 (CKMS99 &00).

(1) complex H module is the fiber H module of a complex G vector bundle over $S(\rho)$ $\Leftrightarrow$ its character is G invariant.

(2) For real H module the same result is true with the further condition that the character is G_1 extendible (and G_μ extendible if $\rho(G) = D_n$).

Theorem 28 (CKMS99 &00).
(1) For complex case two bundles $E, E' \in \mathrm{Vect}_G(S(\rho), \chi)$ *are isomorphic* $\Leftrightarrow E_z \cong E'_z$ *as* G_z *modules for* $z = 1, \mu$
(2) Except for CASES A and B the same is true for the real case.

◆ We also need some results on extensions of H representations to G. The phase of the extension problem changes drastically depending on whether G/H is finite or not. For the finite case character theory of finite group theory is used, but for infinite case properties of compact Lie groups are used.

◆ We finish this lecture by stating an extension result when G/H is a infinite compact Lie group, because it has its own interest.

Theorem 29 (CKS00).
$H \lhd G$ *and* G/H *is connect.*
Let $(G/H)' := [G/H, G/H]$.
Any H *module extends to* G *if*
H *factorizes* $p^{-1}(G/H)'$ *into a direct product where* $\rho : G \to G/H$ *is the projection.*
In particular, the condition holds if $\pi_1(G/H)$ *is torsion free.*